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Stability and Stabilization of Nonlinear Systems and Takagi-Sugeno's Fuzzy Models

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This paper outlines a methodology to study the stability of Takagi-Sugeno's (TS) fuzzy models. The stability analysis of the TS model is performed using a quadratic Liapunov candidate function. This paper proposes a relaxation of Tanaka's stability condition: unlike related works, the equations to be solved are not Liapunov equations for each rule matrix, but a convex combination of them. The coefficients of this sums depend on the membership functions. This method is applied to the design of continuous controllers for the TS model. Three different control structures are investigated, among which the Parallel Distributed Compensation (PDC). An application to the inverted pendulum is proposed here.

Keywords: Stability; Stabilization; Fuzzy model; Fuzzy regulator; LMI

1. INTRODUCTION

Stability of closed-loop systems with a fuzzy controller has been studied for several years. Recently, an interesting method has been introduced using state space models [9]. The models, called TS fuzzy models, are composed of rules with a conclusion part using a state space representation. They allow consequently the use of the potential of linear theory. The first results of stability, due to Tanaka *et al.* [12],

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are Liapunov based. They resume in a collection of Liapunov equations to be solved simultaneously. Some other stability properties were also investigated, using various techniques: for example, an equivalence with switching systems [13], or more recently robust stability of a family of polynomials [8] providing a systematic stability analysis.

For stabilization, linear state feedbacks were first proposed [6, 7] but their results remained fairly poor since they did not take into account the potential of TS models. A more interesting approach, called Parallel Distributed Compensation (PDC), was introduced in [16]. At this stage, the control law has the same structure as the TS fuzzy model: using the premises and rules of the model, the conclusion part is composed of linear state feedback gains. The PDC synthesis is often performed using a Liapunov approach [10, 11] and the obtained equations are Linear Matrix Inequalities (LMI), for which powerful resolution tools [3] are available. Some other control laws were also proposed, based on the dominant model and a high gain controller [2] or on simultaneous stabilization of a collection of linear models [14].

Previous results did not focus on the particular values of the membership functions used, considering that they remain unknown. Yet, information about their values may be very useful: our goal is here to incorporate our knowledge of these functions with a view to derive new interesting stability results. This paper is organized in five sections. Section 2 introduces background materials about TS models and recalls the major previously written stability conditions. The third section presents a theorem that will be used to provide stability conditions, stabilization theorems using various kinds of controllers are given in Section 4. The obtained results are then applied to a model of inverted pendulum in Section 5; the interest of the proposed analysis method is demonstrated through a comparison between our method and the classical ones. Finally Section 6 serves as a conclusion.

Notations $x(t) \in E = \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are respectively the state and control vectors. For any given integer k, I_k denotes $\{1, \ldots, k\}$ and $\mathbb{R}^+$ is the set of strictly positive real numbers. The symbols < 0 and > 0, applied to square matrices mean respectively positive- and negative-definite. $X > Y$, with X and Y square matrices stands for $X - Y > 0$. $\lambda_{\min}$ and $\lambda_{\max}$, applied to a symmetric positive definite matrix P, denote respectively the minimum and maximum eigenvalues of P.

2. BACKGROUND MATERIALS ABOUT TS SYSTEMS

Tagaki-Sugeno proposed an effective way to model a complex dynamical system: the dynamic of a system model is built as a convex sum of the dynamics of a fixed number of linear subsystems. Consequently, r linear systems S_i are considered here.

In a multimodel approach, each subsystem may be a linearization of the system around M_i, $i \in I_r$, point of the state space and the subsystem S_i is then all the more representative of the global system that the state vector $x(t)$ is close to M_i, $i \in I_r$. In this case, the Tagaki-Sugeno model is used to deal with uncertainty through the use of an approximation of the system. The weighting function of S_i is called $w_i(z(t))$, $i \in I_r$, with $z(t) = [z_1(t) \cdots z_p(t)] \in \mathbb{R}^p$, the premise variable vector which depends either linearly or nonlinearly on the state vector.

Another way to use Tagaki-Sugeno model is to write a given nonlinear model under the Takagi-Sugeno form: all the nonlinearities of the system are rejected in the functions $w_i(z(t))$, $i \in I_r$. Note that the subsystems and the rule plants do not have in general a physical sense. An example of such a transformation is given in the simulation section of this article.

2.1. Continuous-time Fuzzy System

The TS fuzzy model can be seen as represented by r plant rules. The *ith* plant rule is:

$$\text{IF } z_1(t) \text{ is } F_1^i \text{ and} \ldots \text{ and } z_p(t) \text{ is } F_p^i$$
$$\text{THEN } \dot{x}(t) = A_i x(t) + B_i u(t)$$

where $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$.

Using a standard fuzzy inference, the final state of the fuzzy model is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(x)(A_i x(t) + B_i u(t)),$$

where $h_i(x) = h_i(z(t)) = w_i(z(t)) / \sum_{l=1}^{r} w_l(z(t))$ and $w_i(z(t)) = \prod_{j=1}^{p} F_j^i(z_j(t))$.

The functions $h_i(z)$ satisfy the *convex sum property i.e.*, $\sum_{i=1}^{r} h_i(z) = 1$, $0 \le h_i(z) \le 1$, $i \in I_r$.

The continuous open-loop system, called **CFS-OL**, is then

$$\dot{x}(t) = \left(\sum_{i=1}^{r} h_i(z) A_i \right) x(t). \tag{2.1}$$

The stability analysis of the system described by (2.1) is performed in Section 3.

For the closed-loop system, different control strategies may be investigated. It is supposed in Section 4 that the state vector is accessible to measurement. Using a simple linear state feedback,

$$u(t) = -K_0 x(t), \quad K_0 \in \mathbb{R}^{m \times n},$$

the new model is **CFS-LIN**:

$$\dot{x}(t) = \left(\sum_{i=1}^{r} h_i(z)(A_i - B_i K_0) \right) x(t). \tag{2.2}$$

The interest of such a feedback is that it allows us to do a pole placement. The stability analysis of (2.2) can be performed in the same way as for (2.1).

It is also possible to construct a control vector based on Parallel Distributed Compensation (PDC):

$$u(t) = -\left(\sum_{i=1}^{r} h_i(x) K_i \right) x(t), \quad K_i \in \mathbb{R}^{m \times n}, \quad i \in I_r.$$

The idea is here to stabilize each subsystem S_i by a linear state feedback K_i, with a view to stabilize the global system. The basic requirement of a PDC controller is that the pairs (A_i, B_i) are stabilizable. The dynamics of this system is expressed by **CFS-PDC**:

$$\dot{x}(t) = \left(\sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z) h_j(z)(A_i - B_i K_j) \right) x(t). \tag{2.3}$$

An important particular case has to be taken into account: when the input matrices are positively linearly dependent *i.e.*, $\exists B \in \mathbb{R}^{n \times m}$ and $k_i > 0$, $i \in I_r$, $B_i = k_i B$, then the control

$$u(t) = -\frac{\sum_{i=1}^{r} h_i(z) K_i}{\sum_{i=1}^{r} h_i(z) k_i} x(t), \quad K_i \in \mathbb{R}^{m \times n}, \quad i \in I_r$$

can be used as demonstrated in [15]. Consequently, the closed-loop model is **CFS-CDF**:

$$\dot{x}(t) = \left(\sum_{i=1}^{r} h_i(z)(A_i - B_i K_i) \right) x(t). \tag{2.4}$$

2.2. Stability Results

THEOREM 1 [12] *Consider the systems* (2.1). *If there exists a symmetric positive definite matrix* P *of* $\mathbb{R}^{n \times n}$ *satisfying*

$$A_i^T P + P A_i < 0, \quad i \in I_r, \tag{2.5}$$

then TS fuzzy model is globally asymptotically stable.

A TS model sometimes includes a constant part in the conclusion of its rules, *i.e.*, $\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + D_i)$, the stability conditions are derived in the same way.

THEOREM 2 [11] *Consider the system* (2.3). *If there exist two symmetric positive definite matrices* P *and* Q *of* $\mathbb{R}^{n \times n}$ *satisfying*

$$\begin{cases} G_{ii}^T P + P G_{ii} + (r-1)Q < 0, & i \in I_r \\ ((G_{ij} + G_{ji})/2)^T P + P((G_{ij} + G_{ji})/2) - Q < 0, & (i,j) \in I_r^2, \ j > i, \end{cases} \tag{2.6}$$

with $G_{ij} = A_i - B_i K_j$, $(i,j) \in I_r^2$, *then the TS fuzzy model is globally asymptotically stable.*

Remark 1 If, only l fuzzy rules are fired at all time t, it is possible to relax the conditions (2.6) by using l instead of r in the first inequality.

These stability conditions are called "relaxed stability conditions". Notice that for $Q = 0$, the stability conditions given by Theorem 2 are the classical ones. An interesting point is that, these stability conditions can be put in a LMI form, allowing the use of powerful resolution tools [3, 1, 4]. Moreover, some performance properties like bounded control, bounded outputs or exponential stability can be taken into account if considered as constraints [11].

An important particular case has already been studied. If matrices B_i satisfy: $\forall i \in I_r$, $\exists k_i > 0$, such that $B_i = k_i B$, $\forall i \in I_r$, then the coupled terms (model A_i, B_i with control law K_j) can be eliminated [5].

Actually, the use of a CDF (Compensation and Division for Fuzzy models) control law provides interesting and better results. In this case, a quadratic Liapunov function leads to stabilization conditions equivalent to the stability ones, replacing A_i by $A_i - B_i K_i$, $i \in I_r$ (2.5). The conservatism of Theorem 2 is consequently strongly reduced.

3. BASIC THEOREMS AND STABILITY CONDITIONS

The theorems given in this section are rather general and will be used to analyse all the previously introduced models. We will see that the knowledge of the membership functions, and particularly their bounds may give us, in most cases, better results than Theorem 2 even if the same type of Liapunov function is used.

3.1. Choice of a Liapunov Candidate Function

To study the stability of the fuzzy models, the following Liapunov function is defined:

$$V(x(t)) = x^T(t) P x(t), \tag{3.1}$$

with P a symmetric positive definite matrix of $\mathbb{R}^{n \times n}$ to be computed later using a LMI approach.

We can calculate the derivative of V along the trajectories of the system:

$$\dot{V}(x(t)) = \dot{x}^T P x + x^T P \dot{x}$$

According to Liapunov's results, a sufficient condition for the origin to be globally asymptotically stable is

$$\begin{cases} \dot{V} = 0 \Longleftrightarrow x = 0, \\ \dot{V} < 0, \ \forall x \in E. \end{cases}$$

With the above defined models, the expression of $\dot{V}$ is in the form:

$$f(x, z) = x^T \left(\sum_{i=1}^{s} \zeta_i(z) Q_i \right) x, \tag{3.2}$$

with $Q_i = Q_i^T \in \mathbb{R}^{n \times n}$ and $\zeta_i(z)$, $i \in I_s$ satisfying the convex sum property. Note that the integer s is not necessarily equal to the number of models r.

Example 1 Let us consider CFS-LIN and the Liapunov function defined by (3.1). The expression of $\dot{V}$ is:

$$\dot{V} = x^T \left(\sum_{i=1}^{r} h_i(z)((A_i - B_i K_0)^T P + P(A_i - B_i K_0)) \right) x,$$

which can be written as:

$$\dot{V} = x^T \left(\sum_{i=1}^{r} \zeta_i(z) Q_i(P, K_0) \right) x,$$

with $\zeta_i(z) = h_i(z)$ and $Q_i(P, K_0) = (A_i - B_i K_0)^T P + P(A_i - B_i K_0)$, $i \in I_r$.

3.2. Basic Theorems

Using the expression (3.2), our goal is to find conditions on Q_i, $i \in I_s$ that ensures the definite negativeness of $f(x, z)$. Note that the values of all the functions $\zeta_i(z)$ are known and should be helpfully used.

It is obvious that

$$\sum_{i=1}^{s} \zeta_i(z) Q_i < 0, \quad \forall z \in \mathbb{R}^p, \tag{3.3}$$

is a sufficient condition for $f(z)$ to be definite negative.

Remark 2 The condition (3.3) is not strictly equivalent to $f(x, z) < 0$. One may find a certain pair of vectors z_0 and x_0 such that the matrix $\zeta_1(z_0)Q_1 + \cdots + \zeta_s(z_0)Q_s$ has some positive eigenvalues even if the quantity $x_0^T(\zeta_1(z_0)Q_1 + \cdots + \zeta_s(z_0)Q_s)x_0$ is negative. Yet, using the condition (3.3) instead of $f(x) < 0$ will not introduce much conservatism in the choice of the matrices Q_i.

Now, an equivalent to (3.3) is investigated. Consider the assumption

(*A*1) there exist at least s points $z_i \in \mathbb{R}^p$, $i \in I_s$ such that $\zeta_i(z_i) = 1$.

Then,

THEOREM 3 *If* (*A*1) *holds then the condition* (3.3) *is equivalent to* $Q_i < 0,\ \forall i \in I_s$.

Proof Under (A1), (3.3) $\Longrightarrow \sum_{i=1}^{s} \zeta_i(z_j) Q_i = Q_j < 0, \quad \forall j \in I_s$.

($Q_i < 0, \forall i \in I_s$) $\Longrightarrow$ (3.3) is obvious since $\zeta_i(z)$, $i \in I_s$ satisfy the convex sum property.

If the condition (*A*1) is not true, it is possible to find a better condition for the definite negativeness of $f(x)$ than the criterion "$Q_i < 0, \forall i \in I_s$". Let $\zeta(z) \in \mathbb{R}^s : \zeta^T(z) = [\zeta_1(z), \ldots, \zeta_s(z)]$ and $D_\zeta = \{\zeta(z) | z \in \mathbb{R}^p\}$. Since $0 \leq \zeta_i(x) \leq 1,\ \forall i \in I_s$, then $D_\zeta \subset [0,1]^s \subset \mathbb{R}^s$. If we define a second assumption:

(*A*2) there exist q points $\Gamma_j \in D_\zeta,\ j \in \{1, \ldots, q\} = I_q$ with $\Gamma_j^T = [\Gamma_{j1}, \ldots, \Gamma_{js}]$ such that any point of D_ζ can be written as a convex combinaison of these q points:

$$\forall h \in D_\zeta, \exists (\theta_1, \ldots, \theta_q),\ \theta_i \geq 0,\ \theta_1 + \cdots + \theta_q = 1,\ \zeta = \sum_{j=1}^{q} \theta_j \Gamma_j,$$

we have the following theorem

THEOREM 4 *Assume that* (*A*2) *holds for the functions* $\zeta_i(z)$, $i \in I_s$. *Then, the condition* (3.3) *is equivalent to*

$$\sum_{i=1}^{s} \Gamma_{ji} Q_i < 0, \quad \forall j \in I_q. \tag{3.4}$$

Proof (3.3) $\Longrightarrow$ (3.4) If (3.3) holds for all elements of D_ζ, then it is true in particular for the points Γ_j, $j \in I_q$, leading to (3.4).

(3.4) $\Longrightarrow$ (3.3): Let us now consider any point $z \in \mathbb{R}^p$ and suppose that (3.4) holds. According to (*A*2), we can write $\zeta(z)$ as a convex combination of the points $\Gamma_j : \zeta(z)^T = [\sum_{j=1}^{q} \theta_j(x) \Gamma_{j1}, \ldots, \sum_{j=1}^{q} \theta_j(x) \Gamma_{js}]$, with θ_j, $j \in I_q$ satisfying the convex sum property. Then,

$$\sum_{i=1}^{s} \zeta_i(z) Q_i = \sum_{i=1}^{s} \left(\sum_{j=1}^{q} \theta_j(z) \Gamma_{ji} \right) Q_i = \sum_{j=1}^{q} \theta_j(z) \left(\sum_{i=1}^{p} \Gamma_{ji} Q_i \right).$$

Since θ_j, $j \in I_q$ are positive and not all egal to 0, (3.3) holds.

Remark 3 Theorem 4 is also valid when a conic combination is considered: the assumption

(*A2'*) there exist q points $\Gamma_j \in D_\zeta$, $j \in \{1, \ldots, q\} = I_q$ with $\Gamma_j^T = [\Gamma_{j1}, \ldots, \Gamma_{js}]$ such that any point of D_ζ can be written as a conic combination of these q points:

$$\forall h \in D_\zeta, \exists(\theta_1, \ldots, \theta_q), \theta_i \geq 0, \ \theta_1 + \cdots + \theta_q > 0, \quad \zeta = \sum_{j=1}^{q} \theta_j \Gamma_j,$$

leads to the same result as (*A2*). For simplicity, only convex combinations are considered in the following even if both convex or conic combinations are valid.

The assumption (*A2*) may not be true for every function $\zeta(z)$. Yet, it is always possible to find a bounded convex polyhedron containing D_ζ. It is then possible to consider another assumption:

(*A3*) there exist k points, $\Gamma_j \in \mathbb{R}^s$, $j \in \{1, \ldots, k\} = I_k$ with $\Gamma_j^T = [\Gamma_{j1}, \ldots, \Gamma_{js}]$ such that any point of D_ζ can be written as a convex combination of these k points.

Using this assumption, we can write another stability condition:

THEOREM 5 *Assume that* (*A3*) *holds for the functions* $\zeta_i(z)$, $i \in I_s$. *If*

$$\sum_{i=1}^{s} \Gamma_{ji} Q_i < 0, \quad \forall j \in I_k. \tag{3.5}$$

holds then (3.3) *is satisfied.*

Proof Obvious according to the previous results.

Note that assumption (*A3*) is always true:

Example 2 Let us consider the s points $\Gamma_1^T = [1, 0, \ldots, 0]$, $\Gamma_2^T = [0, 1, \ldots, 0], \ldots, \Gamma_s^T = [0, 0, \ldots, 1]$. Any point of D_ζ can be obtained as a convex combination of these points. This simple "polyhedric overvaluation" of D_ζ, is only based on the bounds of the functions $\zeta_i(z)$, $i \in I_s$.

3.3. New Stability Conditions

The above results can be directly applied for the stability analysis of continuous time Takagi-Sugeno models:

THEOREM 6 *Consider the system* (2.1). *Let* Γ_j, $j \in I_k$ *be k points such that the functions* $h_i(z)$, $i \in I_r$ *satisfy* (*A*3). *If there exists a symmetric matrix* $P > 0$, $P \in \mathbb{R}^{n \times n}$ *satisfying*

$$\sum_{i=1}^{s} \Gamma_{ji}(A_i^T P + PA_i) < 0, \quad \forall j \in I_k,$$

then the origin of the system (2.1) *is globally asymptotically stable.*

Proof Obvious using Theorem 5 with $s = r$, $\zeta_i(z) = h_i(z)$, $i \in I_r$ and $Q_i = A_i^T P + PA_i$.

Example 3 Let us consider *a* 2 models *TS system* with the following rule matrices: $A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0.5 \end{bmatrix}$ and $h_1(z)$, $h_2(z) \geq 0.2$.

According to Theorem 1, the problem to solve is: find $P \in \mathbb{R}^{2 \times 2}$, $P > 0$ such that $A_1^T P + PA_1 < 0$ and $A_2^T P + PA_2 < 0$.

According to Theorem 6, the problem to solve can be reduced to: find $P \in \mathbb{R}^{2 \times 2}$, $P > 0$ such that $0.2(A_1^T P + PA_1) + 0.8(A_2^T P + PA_2) < 0$ and $0.8(A_1^T P + PA_1) + 0.2(A_2^T P + PA_2) < 0$. The second system of equation has multiple solutions $P > 0$, for instance $P = I$, whereas the other has not (A_2 is not a Hürwitz matrix). Moreover, even if in some cases, both of the systems may have solutions, the proposed method allows to add stronger constraints on the solution, which is particularly interesting for stabilization.

Note that if the functions $h_i(z)$, $i \in I_r$ satisfy (*A*1), Theorem 1 is equivalent to Theorem 6: the points considered for the polyhedric overvaluation are then $\Gamma_1^T = [1, 0, \ldots, 0]$, $\Gamma_2^T = [0, 1, \ldots, 0], \ldots, \Gamma_s^T = [0, 0, \ldots, 1]$.

4. STABILIZATION OF TS MODELS

4.1. Case of State Feedbacks and CDF Controllers

When a quadratic Liapunov function is used, the stabilization problem of TS systems with a linear state feedback or a CDF controller both

lead to the following kind of matricial inequality:

$$\sum_{i=1}^{s} h_i(z)((A_i - B_iK_i)^T P + P(A_i - B_iK_i)) < 0,$$

Note that with the continuous state feedback, we have $K_i = K_0$, $i \in I_r$. Let us now apply the stability theorem of the previous section to these controlled systems.

THEOREM 7 *Consider the system* (2.2) *and let* Γ_j, $j \in I_k$ *be k points such that the functions* $h_i(z)$, $i \in I_r$ *satisfy* (*A*3). *If there exists a matrix* $P > 0$, $P \in \mathbb{R}^{n \times n}$ *satisfying*

$$\sum_{i=1}^{r} \Gamma_{ji}((A_i - B_iK_0)^T P + P(A_i - B_iK_0)) < 0, \quad \forall j \in I_k,$$

then the origin of the system (2.2) *is globally asymptotically stable.*

Proof Obvious according to the stability results.

THEOREM 8 *Consider the system* (2.4) *and let* Γ_j, $j \in I_k$ *be k points such that the functions* $h_i(z)$, $i \in I_r$ *satisfy* (*A*3). *If there exists a matrix* $P > 0$, $P \in \mathbb{R}^{n \times n}$ *satisfying*

$$\sum_{i=1}^{r} \Gamma_{ji}((A_i - B_iK_i)^T P + P(A_i - B_iK_i)) < 0, \quad \forall j \in I_k, \tag{4.1}$$

then the origin of the system (2.4) *is globally asymptotically stable.*

Proof Obvious according to the stability results.

4.2. Case of PDC Controllers

Using a quadratic Liapunov function $V = x^T Px$, sufficient global asymptotic stability conditions are expressed in the form

$$\sum_{i,j=1}^{r} h_i(z)h_j(z)((A_i - B_iK_j)^T P + P(A_i - B_iK_j)) < 0.$$

With a view to apply the stability criterion of the previous section to the TS system controlled by a PDC regulator, the derivative of the Liapunov function has to be written in the form (3.2). For instance, in the continuous case, denoting $G_{ij}=(A_i-B_iK_j)$ leads to

$$\begin{aligned}
&\sum_{i,j=1}^{r} h_i(z)h_j(z)((A_i - B_iK_j)^TP + P(A_i - B_iK_j)) \\
&= \sum_{i,j=1}^{r} h_i(z)h_j(z)(G_{ij}^TP + PG_{ij}) \\
&= \begin{cases} \sum_{i=1}^{r} h_i^2(z)(G_{ii}^TP + PG_{ii}) + \\ \sum_{i=1,j>i}^{r} 2h_i(z)h_j(z)(((G_{ij} + G_{ji})/2)^TP + P((G_{ij} + G_{ji})/2)) \end{cases}
\end{aligned}$$

Then, it is possible to define $s=r(r+1)/2$ functions $\xi_k(z)$, $k\in I_s$ and s corresponding matrices N_k, $k\in I_s$ such that

$$\dot{V} = x^T\left(\sum_{i=1}^{s} \xi_i(z)N_i\right)x,$$

with ξ_k, $k\in I_s$ satisfying the convex sum property. The r first terms are $\xi_i(z) = h_i^2(z)$, $N_i = G_{ii}^TP + PG_{ii}$, $i\in I_r$ and the other ones are naturally built using the coupled terms, this is $\xi_{r+1}(z)=2h_1(z)h_2(z)$, $N_{r+1}=0.5((G_{12}+G_{21})^TP+P(G_{12}+G_{21}))$ and so on.

An application of this transformation is proposed in the simulation section.

THEOREM 9 *Assume that the above defined transformation is performed on the derivative of the Liapunov function x^TPx along the trajectory of the system* (2.3). *Let Γ_j, $j\in I_k$ be k points such that the functions $\xi_k(z)$, $i\in I_k$ satisfy* (A3).

Consider the system (2.3), *if there exists a matrix $P>0$, $P\in\mathbb{R}^{n\times n}$ satisfying*

$$\sum_{i=1}^{s} \Gamma_{ji}N_i(P) < 0, \quad \forall j\in I_k,$$

then the origin of the system (2.3) *is globally asymptotically stable.*

Proof Obvious according to the stability results.

4.3. LMI Resolution

In the section dealing with stability, the conditions were directly written in terms of LMIs. In the above written equations, since P and K_i, $i \in I_r$ are unknown variables, the terms of the type $K_i^T B_i^T P$ are bilinear. Yet, these expressions can be easily written in a LMI formulation for an easy computation of the Liapunov matrix and the controllers.

In the continuous case, the LMI transformation is performed in two steps. First, the inequalities are left- and right-multiplied by $S = P^{-1}$. Then, defining the intermediate unknown variables $U_i = S$.

K_i, $i \in I_r$ leads to LMIs inequalities in the variables S and K_i, $i \in I_r$:

$$\sum_{i=1}^{s} \Gamma_{ji}(SA_i^T + A_i S - U_i^T B_i - B_i U_i) < 0, \quad \forall j \in I_k.$$

Note that this change of variable is bijective.

The main interest of the obtained LMI is the easy numerical resolution using tools like Matlab's LMI toolbox. Another interest relies on the adjunction of some performance constraints that can be expressed as LMIs. When the stabilization problem is considered, it may be very interesting to add LMI constraints with a view to guarantee some dynamical properties on the state or control vector. Here, two examples are considered: constraints on the input and on the decay rate.

4.3.1. Constraint on the Input

Assuming that the initial state condition is a known vector $x(0)$. If the LMIs

$$\begin{pmatrix} 1 & x(0) \\ x(0)^T & S \end{pmatrix} \geq 0, \quad \begin{pmatrix} S & U_i \\ U_i^T & \gamma^2 I \end{pmatrix} \geq 0,$$

hold with γ a positive real constant, we have $\|u(t)\|_2 \leq \gamma$, $t \geq 0$.

4.3.2. Decay Rate

The speed of convergence towards the origin can be expressed with the exponential stability. If the condition $\dot{V}(x,z) \leq -\alpha V(x,z)$ or $\dot{V}(x,z) + \alpha x^T P x \leq 0$ holds, then the guaranteed speed of response grows with the admissible value of $\alpha > 0$. Since $\sum_{i=1}^{s} \Gamma_{ji} = 1$, the stability/stabilization LMIs are not modified much by the adjunction

of the term αP. In most cases, the constant α cannot be considered as an LMI constraint.

Example 4 Let us consider the stabilization LMIs of TS model with a CDF controller: (4.1). In the continuous case, the conditions become

$$\sum_{i=1}^{r} \Gamma_{ji}((A_i - B_iK_i)^T P + P(A_i - B_iK_i) + \alpha P) < 0, \quad \forall j \in I_k,$$

and the usual transformation is performed in the same way.

5. SIMULATION RESULTS

We consider here the following simplified model of an inverted pendulum:

$$\begin{cases} \ddot{X} = (1/M)(-f\dot{X} + mg \sin\theta \cos\theta + Gu(t)) \\ \ddot{\theta} = (1/LM)(g(M+m)\sin\theta - f\cos\theta\dot{X} + \cos\theta\ Gu(t)) \end{cases} \tag{5.1}$$

with the numerical values

$X(t)$	position of the cart	m
$\theta(t)$	angle of the pendulum	*rad*
$u(t)$	control vector	V
M	total mass of the cart	20 Kg
f	friction	$150\,\mathrm{N \cdot m \cdot rad^{-1}}$
m	mass of the pendulum	0.025 Kg
g	gravity	$9.81\,\mathrm{ms^{-2}}$
G	$F(t) = Gu(t)$	$50\,\mathrm{N \cdot V^{-1}}$
L	half-length of the pendulum	0.1 m

Our goal is to find a control $u(t)$ such that the state vector defined by $x^T = [\theta, X, \dot{X}, \dot{\theta}]$, tends asymptotically towards the origin with an eventually fixed upperbound for $\|u(t)\|_2$ or an exponential convergence rate.

In any domain $D_{\theta_0}[-\theta_0, +\theta_0]$, $0 \leq \theta_0 < \pi/2$, we will show that (5.1) can be written exactly as a 4 models Takagi-Sugeno system:

$$\dot{x}(t) = \sum_{i=1}^{4} h_i(x)(A_i x(t) + B_i u(t)). \tag{5.2}$$

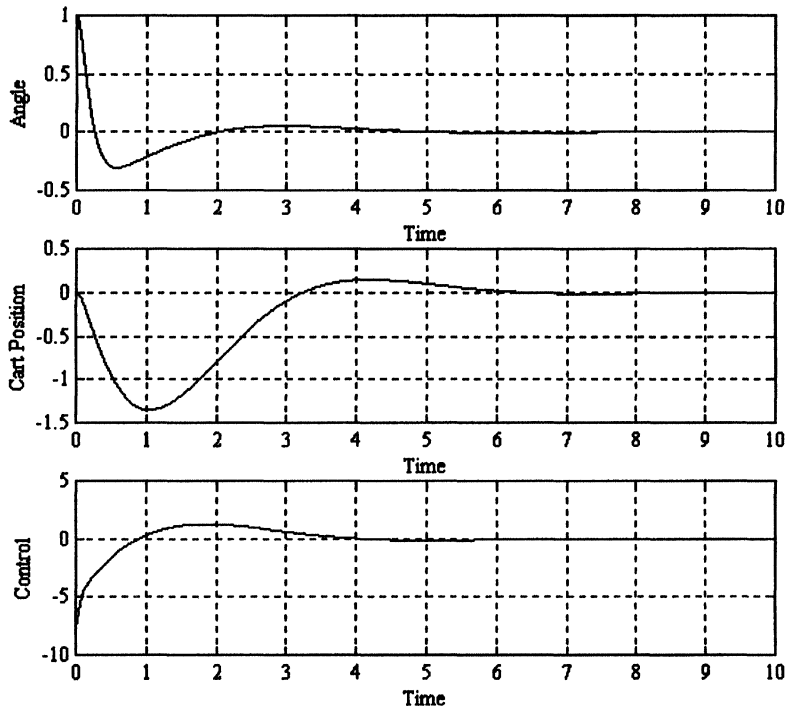

FIGURE 1

It is obvious that (5.1) can be written in the form $\dot{x} = A(x)x(t) + B(x)u(t)$ with

$$A(x) = \begin{bmatrix} \dot{\theta} \\ \dot{X} \\ -(f/M)\dot{X} + ((mg/M)(\sin\theta/\theta)\cos\theta)\theta \\ -f(\cos\theta/LM)\dot{X} + ((g(M+m)/LM)(\sin\theta/\theta))\theta \end{bmatrix},$$

$$B(x) = \begin{bmatrix} 0 \\ 0 \\ G/M \\ G\cos\theta/LM \end{bmatrix}.$$

Defining $\alpha(\theta_0) = (\sin\theta_0/\theta_0)$ and $\beta(\theta_0) = \cos(\theta_0)$, we will now write $\cos\theta$ as a convex sum of β and 1, and $(\sin\theta_0/\theta_0)$ as a convex sum of α

and 1. Then,

$$\begin{cases} (\sin\theta/\theta) = F_{\sin}\cdot 1 + F'_{\sin}\cdot(\sin\theta_0/\theta_0), F'_{\sin} = 1 - F_{\sin}, \\ \cos\theta = F_{\cos}\cdot 1 + F'_{\cos}\cdot\cos\theta_0, \qquad F'_{\cos} = 1 - F_{\cos}, \end{cases}$$

with

$$\begin{cases} F_{\sin} = (1/1-\alpha)((\sin\theta/\theta) - \alpha) \\ F'_{\sin} = (1/1-\alpha)(1 - (\sin\theta/\theta)) \\ F_{\cos} = (1/1-\beta)(\cos\theta - \beta) \\ F'_{\cos} = (1/1-\beta)(1 - \cos\theta). \end{cases}$$

Incorporating these functions in the above written models leads to the T-S model (5.2) with

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ G/M \\ G/LM \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 5 \end{bmatrix}$$

$$B_3 = B_4 = \begin{bmatrix} 0 \\ 0 \\ G/M \\ G\cos\theta_0/LM \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 5\beta \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ (mg/M) & 0 & -(f/M) & 0 \\ (g(M+m)/LM) & 0 & -(f/LM) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0.0123 & 0 & -7.5 & 0 \\ 98.2 & 0 & -75 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ (mg/M)(\sin\theta_0/\theta_0) & 0 & -(f/M) & 0 \\ (g(M+m)/LM)(\sin\theta_0/\theta_0) & 0 & -(f/LM) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0.0123\alpha & 0 & -7.5 & 0 \\ 98.2\alpha & 0 & -75 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ (mg/M)\cos\theta_0 & 0 & -(f/M) & 0 \\ (g(M+m)/LM) & 0 & -(f/LM)\cos\theta_0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0.0123\beta & 0 & -7.5 & 0 \\ 98.2 & 0 & -75\beta & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ (mg/M)(\sin\theta_0/\theta_0)\cos\theta_0 & 0 & -(f/M) & 0 \\ (g(M+m)/LM)(\sin\theta_0/\theta_0) & 0 & -(f/LM)\cos\theta_0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0.0123\alpha\beta & 0 & -7.5 & 0 \\ 98.2\alpha & 0 & -75\beta & 0 \end{bmatrix}$$

The weighting functions are

$$\begin{cases} h_1(\theta) = (1/(1-\beta)(1-\alpha))(\cos\theta - \beta)((\sin\theta/\theta) - \alpha) \\ h_2(\theta) = (1/(1-\beta)(1-\alpha))(\cos\theta - \beta)(1 - (\sin\theta/\theta)) \\ h_3(\theta) = (1/(1-\beta)(1-\alpha))(1-\cos\theta)((\sin\theta/\theta) - \alpha) \\ h_4(\theta) = (1/(1-\beta)(1-\alpha))(1-\cos\theta)(1 - (\sin\theta/\theta)). \end{cases}$$

The CDF control law can not be used to control this system since the matrices B_i, $i \in I_4$ are not linearly dependent. Yet, one may use a PDC control law expressed as $u(t) = -\sum_{i=1}^{4} h_i(\theta)K_i x(t)$. The application of our criterion requires to transform the system into $\dot{x} = \left(\sum_{i=1}^{10} \lambda_i G_i\right)x$, with

$$\begin{cases} \lambda_1 = (h_1(\theta))^2, \ldots, \lambda_4 = (h_4(\theta))^2,\ G_1 = A_1 - B_1K_1, \ldots, G_4 = A_4 - B_4K_4 \\ \lambda_5 = 2h_1h_2,\ G_5 = 0.5((A_1 - B_1K_2) + (A_2 - B_2K_1)) \\ \lambda_6 = 2h_1h_3,\ G_6 = 0.5((A_1 - B_1K_3) + (A_2 - B_3K_1)) \\ \lambda_7 = 2h_1h_4,\ G_7 = 0.5((A_1 - B_1K_4) + (A_4 - B_4K_1)) \\ \lambda_8 = 2h_2h_3,\ G_8 = 0.5((A_2 - B_2K_3) + (A_3 - B_3K_2)) \\ \lambda_9 = 2h_2h_4,\ G_9 = 0.5((A_2 - B_2K_4) + (A_4 - B_4K_2)) \\ \lambda_{10} = 2h_3h_4,\ G_{10} = 0.5((A_3 - B_3K_4) + (A_4 - B_4K_3)). \end{cases}$$

The numerical values of the gains of the PDC depend on the choice of the stabilization technique. The following figure compares numerically two stabilization techniques: classical Tanaka (1) and our method. Note that the results provided by Tanaka's relaxed stability conditions are nearly the same as classical Tanaka. We notice that our results are better than the ones obtained by Tanaka's theorem:

Constraints	*Optimization*	*Our criterion*	*Tanaka's criterion*
$\theta_0 = 1.2,\ \lambda = 0$	$\min(u_{\max})$	7.9	10.2
$u_{\max} = 20,\ \theta_0 = 1.2$	$\max(\lambda)$	6.6	1.8
$u_{\max} = 10,\ \lambda = 0$	$\max(\theta_0)$	1.27	1.19
$\theta_0 = 1.2,\ \lambda = 2$	$\min(u_{\max})$	10.9	21.9

The following simulation curves are obtained with the simulation constraints: optimize $u_{\max}$ with $\theta_0 = 1.2$, $\lambda = 2$, and the initial position $x_0^T = [1, 0, 0, 0]$. The values of the PDC gains are:

$$\begin{cases} K_1 = [4.8 & -0.33 & -1.8 & 0.50] \\ K_2 = [5.6 & -0.27 & -1.9 & 0.59] \\ K_3 = [5.6 & -0.26 & -1.9 & 0.59] \\ K_4 = [9.7 & -0.67 & -2.2 & 1.1], \end{cases}$$

with an optimal value $u_{\max} = 10.9$ and the Liapunov matrix

$$P = \begin{bmatrix} 1 & -0.13 & -0.21 & 0.11 \\ -0.13 & 0.061 & 0.043 & 0.15 \\ -0.21 & 0.043 & 0.061 & -0.023 \\ 0.11 & 0.15 & 0.023 & 0.013 \end{bmatrix}.$$

In this simulation example, the only goal was to stabilize the system with a minimization of the control amplitude under constraints. If it is possible to have a high value for $\|u(t)\|_2$, a higher exponential convergence rate can be used with a view to diminish the time response. Moreover, If a certain type of behavior is required around the origin, it is possible to add a state feedback to the PDC regulator. Since A_1 represents the linearization of the system around the origin, it is possible to simply choose K_1 by a pole placement of $A_1 - B_1 K_1$; consequently, the new stabilization problem will only involve the variables K_2, K_3, K_4, P. Note that, since $\theta_0 = 1.2$ has been chosen, the

regulator given above may not be satisfory if the initial angle is not inside $[-1.2, 1.2]$.

6. CONCLUSION

A new method to find a symmetric positive definite matrix P that guarantees the global asymptotic stability of Takagi-Sugeno's system is investigated here. It has been shown that LMIs in P built as convex or conic combinations of the ones used with Tanaka's condition may be considered. The coefficients of the combinations depend directly on our knowledge of the membership functions, particularly on their bounds. Due to the convexity of this sum, our set of solutions P is equal or larger, in the sense of inclusion, than the one provided by solving the system of Liapunov equations for each rule matrix. The stability conditions thereby given are presented in terms of LMIs, which allows to add easily constraints on the control vector or exponential stability. The proposed simulation of the control of an inverted pendulum proves comparatively the interest of our method. Moreover, this new condition can be further applied to discrete fuzzy system and to the observation problem.

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