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Risk-Sensitive Control of Stochastic Hybrid Systems on Infinite Time Horizon

THORDUR RUNOLFSSON*

Department of Engineering, University of Iceland, Hjardarhagi 2-6, IS 107 Reykjavik, Iceland; EECS Department, The University of Michigan, Ann Arbor, MI 48109-2122, USA

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A risk-sensitive optimal control problem is considered for a hybrid system that consists of continuous time diffusion process that depends on a discrete valued mode variable that is modeled as a Markov chain. Optimality conditions are presented and conditions for the existence of optimal controls are derived. It is shown that the optimal risk-sensitive control problem is equivalent to the upper value of an associated stochastic differential game, and insight into the contributions of the noise input and mode variable to the risk sensitivity of the cost functional is given. Furthermore, it is shown that due to the mode variable risk sensitivity, the equivalence relationship that has been observed between risk-sensitive and H_∞ control in the nonhybrid case does not hold for stochastic hybrid systems.

Keywords: Risk sensitive control; Hybrid systems; Stochastic control; Large deviations

1. INTRODUCTION

Consider the stochastic hybrid system

$$\mathrm{d}x_t = b(x_t, r_t, u_t)\,\mathrm{d}t + \sigma(x_t, r_t)\,\mathrm{d}w_t \tag{1}$$

where $x_t \in \mathbf{R}^n$ is the state, $u_t \in \mathbf{R}^m$ is the control and w_t is a standard n-dimensional Brownian motion. The variable r_t is called the mode of

* Department of Engineering, University of Iceland, Hjardarhagi 2-6, 107 Reykjavik, Iceland. Tel.: 354 525 5271. Fax: 354 525 4632.
E-mail: thordrun@verk.hi.is; thordur@eecs.umich.edu.

the system and is modeled as finite state continuous time Markov chain taking values in the state space S. Systems of this form arise in various applications and system formulations, such as power systems [1], target tracking [2] and fault-tolerant control system [3,2].

The most common approach in the design of control systems for stochastic hybrid systems is to use optimal control methods. In [4] a theory for linear hybrid systems with a Markovian jump parameter (mode variable) is developed and it is shown that the optimal state feedback control law for linear hybrid system with a quadratic cost functional is given by a coupled system of Riccati equations. In [2] the theory for a linear hybrid systems with a quadratic cost functional is developed further and the theory for such systems is quite complete. In [5,6] a detailed treatment of nonlinear stochastic hybrid systems is given. In particular, in [5] conditions for the existence and uniqueness of solutions of such systems are formulated, and in [6] a general theory for the ergodic properties of solutions is developed. Optimal discounted control of a stochastic hybrid system arising in manufacturing systems is considered in [5], and in [6] optimal ergodic (pathwise average) control for a general class of nonlinear stochastic hybrid systems is considered.

In recent years it has been shown that for linear systems there is a close relation between robust control (i.e. H_∞ control), linear differential games and risk-sensitive control. In particular, it was shown in [7] that for linear discrete systems the maximum entropy formulation of the problem and risk-sensitive control are equivalent problems. A relationship between risk-sensitive control of nonlinear systems and stochastic differential games is developed in [8] and the relationship between nonlinear risk-sensitive control and nonlinear H_∞ control is considered in [9]. In the paper [10] the theory of H_∞ control is extended to hybrid linear systems with Markovian jump parameters and it is shown that the optimal state feedback control law is given by a system of coupled Riccati equations which are of the same structural type as the Riccati equation that arises in H_∞ control and linear quadratic differential games.

In this paper we study risk-sensitive control of nonlinear stochastic hybrid systems. In particular, the objective of the control is to minimize the infinite-horizon risk-sensitive cost functional

$$J(u) = \limsup_{T\to\infty} \frac{1}{T} \log E\left[\exp\left\{\int_0^T c(x_t, r_t, u_t)\,\mathrm{d}t\right\}\right] \tag{2}$$

over an appropriate set $\mathcal{U}$ of control policies. Here $c:\mathbf{R}^n \times S \times \mathbf{R}^m \to \mathbf{R}$ is the running cost.

A related problem is studied in [11]. In particular, they consider a risk sensitive control formulation of a manufacturing system whose supply-demand dynamics are described by a first order linear system. The supply rate (control input) is constrained by an upper bound which is described by a continuous time Markov chain. This problem is similar to the one we consider in that the system is hybrid. However, the continuous dynamics of the system they consider is linear and deterministic, and the stochastic aspect of the problem comes through the stochastic upper bound of the supply rate.

It is shown in this paper that the relationship between H_∞ control, linear differential games and risk-sensitive control does not hold any more. The reason is that, unlike in the H_∞ control formulation of hybrid systems in [10], the risk-sensitive cost functional measures the risk sensitivity of the system to transitions caused by the random jump parameter (mode variable) as well as the noise input. The risk sensitivity of the cost functional to transitions induced by the mode may be of a great value in the design of systems where it is desired to make the system performance as insensitive to the value of the mode variable as possible.

The paper is organized as follows. In Section 2 we formulate the problem and state some basic results about the existence of solutions and admissible controls. In Section 3 we state sufficient conditions for optimality for the infinite horizon risk-sensitive control problem. In Section 4 we analyze the risk-sensitive cost functional and show that it can be represented as the optimal value of an auxiliary optimal control problem. Finally, in Section 5 we formulate conditions for the existence of optimal controls for the risk-sensitive optimal control problem. Conclusions and future research directions are summarized in Section 6.

2. PROBLEM FORMULATION

We begin by stating the mathematical assumptions under which we are working. The process w_t in (1) is a standard n-dimensional Brownian Motion on a probability space $(\Omega, \mathcal{F}, P)$. The variable r_t, the mode of the system, is modeled as a continuous time Markov chain on $(\Omega, \mathcal{F}, P)$ taking values in the finite state space $S=\{1,\ldots,N\}$. The dynamics of r_t are described by

$$P(r_{t+\delta t}=j \mid r_t=i, x_t=x, u_t=u)=\begin{cases}\pi_{ij}(x,u)\delta t+o(\delta t), & i\neq j,\\ 1+\pi_{ii}(x,u)\delta t+o(\delta t), & i=j,\end{cases} \tag{3}$$

where the transition rates π_{ij} satisfy $\pi_{ij}(x,u)\geq 0$ and $\pi_{ii}=-\sum_{j\neq i}\pi_{ij}$.

Let $\Pi^u(x)$ be the $N\times N$ matrix with elements $\pi_{ij}(x,u)$, $1\leq i, j\leq N$ and let U be a compact metric space. We make the following assumptions about the controlled system (1) and (3).

Assumption (A0):

(i) $b:\mathbf{R}^n\times S\times U\rightarrow\mathbf{R}^n$, $\sigma:\mathbf{R}^n\times S\rightarrow\mathbf{R}^{n\times n}$ and $\Pi^u:\mathbf{R}^n\times U\rightarrow\mathbf{R}^{N\times N}$ are continuous and globally Lipschitz in x, uniformly in u.

(ii) There exists a constant $\sigma_0>0$ such that $\sigma(x,u)\sigma^T(x,u)\geq\sigma_0 I$ $\forall(x,r)\in\mathbf{R}^n\times S$.

(iii) The matrix Π^u is irreducible for all $(x,u)\in\mathbf{R}^n\times U$, i.e., for all $(x,u)\in\mathbf{R}^n\times U$ there exists a unique $\nu(x,u)\in\mathbf{R}^N$ such that $\nu_i(x,u)>0, i\in S$, $|\nu(x,u)|=1$ and $\nu^T(x,u)\Pi^u(x)=0$.

We will concentrate on control policies that are Markov policies, i.e., $u_t=\tilde{u}(x_t,r_t)$ where $\tilde{u}:\mathbf{R}^n\times S\rightarrow U$ is a measurable map. We denote the set of all Markov policies by $\mathcal{U}$. The class of nonrandomized Markov policies is denoted by $\mathcal{U}_{\mathrm{d}}$ (note that in a nonrandomized policy the map $\tilde{u}$ is a deterministic function).

For a Markov policy $u\in\mathcal{U}$ the hybrid process (x_t,r_t) is Markov process. Furthermore, the following result from [5] establishes that under the above conditions the process defined by (1) and (3) is well defined.

THEOREM 1 *For any $u\in\mathcal{U}$ there exists an almost surely unique solution of* (1) *and* (3) *which is a strong Feller process on $\mathbf{R}^n\times S$.*

Let $P^u(t,(x,i),\Gamma\times\{j\})$ be the transition function for the Markov Process (x_t,r_t) and let P^u_{xi} and E^u_{xi} be the probability distribution and expectation operator corresponding to the initial condition $(x_0,r_0)=(x,i)$ and control $u\in\mathcal{U}$. Let $\mathcal{P}(\mathbf{R}^n\times S)$ denote the set of all probability measures on $\mathbf{R}^n\times S$. A measure $\mu\in\mathcal{P}(\mathbf{R}^n\times S)$ is said to be an invariant measure of the process (x_t, r_t) if for any Borel set $\Gamma \subset \mathbf{R}^n$

$$\sum_{i \in S} \int_{\mathbf{R}^n} P^u(t, (x, i), \Gamma \times \{j\}) \mu^u(\mathrm{d}x, i) = \mu^u(\Gamma \times \{j\}). \tag{4}$$

A control policy $u \in \mathcal{U}$ ($u \in \mathcal{U}_{\mathrm{d}}$) for which the process (x_t, r_t) has a unique invariant measure is said to be stabilizing and the class of all such policies will be denoted by $u \in \mathcal{U}_{\mathrm{s}}(\mathcal{U}_{\mathrm{ds}})$. The class $\mathcal{U}_{\mathrm{ds}}$ is assumed to be nonempty.

Remark Sufficient conditions for the existence of an invariant can be found in the literature [6]. In Theorem 2 below conditions of this sort are formulated.

Let $c: \mathbf{R}^n \times S \times U \to \mathbf{R}$ be a nonnegative function which is continuous in the first and third arguments for each $i \in S$ and which has the property that the set $\{x\colon \sup_{i \in S, u \in U} c(x, i, u) \leq M\}$ is compact for each $M > 0$. The objective of the control is to minimize the infinite horizon risk-sensitive cost functional

$$J(u) = \limsup_{T \to \infty} \frac{1}{T} \log E^u_{xi} \left[\exp\left\{ \int_0^T c(x_t, r_t, u_t)\, \mathrm{d}t \right\} \right] \tag{5}$$

over all control policies $u \in \mathcal{U}_{\mathrm{ds}}$.

For a control policy $u \in \mathcal{U}_{\mathrm{d}}$ the Markov process (x_t, r_t) has infinitesimal generator

$$(A^u f)(x, i) = (L^u_i f)(x, i) + (\Pi^u f)(x, i), \tag{6}$$

where L^u_i is given by

$$(L^u_i f)(x, i) = \sum_{k=1}^n b_k(x, i, u) \frac{\partial f}{\partial x_k}(x, i) + \frac{1}{2} \sum_{k,l=1}^n (\sigma(x, i) \sigma^T(x, i))_{kl} \frac{\partial^2 f}{\partial x_k \partial x_l}(x, i) \tag{7}$$

for any $f \in C^2(\mathbf{R}^n \times S) = \{f: \mathbf{R}^n \times S \to \mathbf{R} \mid f(x, i) \in C^2(\mathbf{R}^n), i \in S\}$ and

$$(\Pi^u f)(x, i) = \sum_{j \in S} \pi_{ij}(x, u) f(x, j). \tag{8}$$

In the following theorem we give a sufficient condition for the existence of a stabilizing control and derive an upper bound on the cost functional $J(u)$. Define $\eta(\rho)=\inf_{|x|>\rho}\inf_{u,i} c(x,i,u)$.

THEOREM 2 *Let $u\in\mathcal{U}_{\mathrm{d}}$ and assume there exists a strictly positive $\psi\in C^2(\mathbf{R}^n\times S)$ and positive constants λ, ρ_0 such that*

(i) $\eta(\rho)>\lambda$ *for all* $\rho>\rho_0$.
(ii) $\lim_{|x|\to\infty}\psi(x,i)=\infty$.
(iii) $\psi(x,i)$ *and* $(\partial\psi/\partial x)(x,i)$ *have polynomial growth in* x *and* $|\partial\psi/\partial x|^2\geq\sigma_0^{-1}$.
(iv) *For all* $(x,i)\in\mathbf{R}^n\times S$, $(A^u\psi)(x,i)+c(x,i)\psi(x,i)\leq\lambda\psi(x,i)$.

Then $u\in\mathcal{U}_{\mathrm{ds}}$ *and* $J(u)\leq\lambda$.

Proof We begin by showing that $u\in\mathcal{U}_{\mathrm{ds}}$. First note that for $|x|>\rho>\rho_0$ we have $\varepsilon=\inf_{|x|>\rho,i\in S}\psi(x,i)(\eta(\rho)-\lambda)>0$ and

$$(A^u\psi)(x,i)\leq(\lambda-c(x,i))\psi(x,i)<-\varepsilon. \tag{9}$$

Therefore, by (9), (ii) and (iii) all the conditions of Theorem 4.4 in [6] are satisfied and thus $u\in\mathcal{U}_{\mathrm{ds}}$. We next show that $J(u)\leq\lambda$. Define a functional

$$(K_t^uf)(x,i)=E_{x,i}^u\left[\mathrm{e}^{\int_0^t c(x_s,r_s,u_s)\,\mathrm{d}s}f(x_t,r_t)\right], \tag{10}$$

where $f:\mathbf{R}^n\times S\to\mathbf{R}$ is a continuous function in x for each $i\in S$. Then it follows from the Markov property that K_t^u is semigroup, i.e. $K_{t+s}^uf=K_t^uK_s^uf$. Note that for $f\in C^2(\mathbf{R}^n\times S)$ the generator of the semigroup K_t^u coincides with A^u+c where A^u is given by (6). It follows from [12, p. 195] that for any $\gamma\in\mathbf{R}$,

$$\mathrm{e}^{-\gamma t}K_t^u\psi=\psi+\int_0^t\mathrm{e}^{-\gamma s}K_s^u(A^u+c-\gamma)\psi\,\mathrm{d}s. \tag{11}$$

Using the inequality in condition (iv) in (11) gives

$$\mathrm{e}^{-\gamma t}K_t^u\psi\leq\psi+(\lambda-\gamma)\int_0^t\mathrm{e}^{-\gamma s}K_s^u\psi\,\mathrm{d}s. \tag{12}$$

Since ψ is positive it follows from the Bellman–Gronwall inequality that

$$e^{-\gamma t}K_t^u\psi \leq e^{(\lambda-\gamma)t}\psi. \tag{13}$$

Define $\tilde{\psi} = \inf_{x\in\mathbf{R}^n, i\in S}\psi(x,i)$. Then it is easy to see that $(K_t^u\psi)(x,i) \geq \tilde{\psi}(K_t^u\mathbf{1})(x,i)$ where $\mathbf{1}(x,i)$ is the function that takes on the value 1 for all $(x,i)\in\mathbf{R}^n\times S$. This and (13) gives

$$K_t^u\mathbf{1} \leq \frac{\psi}{\tilde{\psi}}e^{\lambda t} \tag{14}$$

and noting that

$$J(u) = \limsup_{t\to\infty}\frac{1}{t}\log(K_t^u\mathbf{1})(x,i) \tag{15}$$

gives $J(u) \leq \lambda$.

3. OPTIMALITY CONDITIONS

We begin the analysis of the optimal risk-sensitive control problem with a sufficient condition (verification theorem) for the existence of an optimal stationary policy.

THEOREM 3 *Assume there exists a positive* $\psi\in C^2(\mathbf{R}^n\times S)$ *and a constant* $\lambda > 0$ *such that for all* $(x,i)\in\mathbf{R}^n\times S$

$$\lambda\psi(x,i) = \inf_{u\in U}[(A^u\psi)(x,i) + c(x,i,u)\psi(x,i)]. \tag{16}$$

Then $\lambda \leq J(u)$. *Furthermore, if* $u^*\in\mathcal{U}_{\mathrm{ds}}$ *is a policy for which the minimum in* (16) *is attained for all* $(x,i)\in\mathbf{R}^n\times S$ *and there exists a constant* $\hat{\psi} > 0$ *such that* $\psi(x,i) \geq \hat{\psi}\ \forall(x,i)\in\mathbf{R}^n\times S$, *then* u^* *is optimal and* $\lambda = J(u^*)$.

Proof It follows from (16) that

$$\lambda\psi(x,i) \leq (A^u\psi)(x,i) + c(x,i,u)\psi(x,i). \tag{17}$$

Using (17) in (11) gives

$$e^{-\gamma t} K_t^u \psi \geq \psi + (\lambda - \gamma) \int_0^t e^{-\gamma s} K_s^u \psi \, ds. \tag{18}$$

Using a Bellman–Gronwall type of an inequality we get from (18)

$$K_t^u \psi \geq e^{\lambda t} \psi. \tag{19}$$

Define

$$\psi_R(x, i) = \begin{cases} \psi(x, i), & |x| \leq R, \\ 0, & \text{otherwise.} \end{cases}$$

Then it follows from (19) that

$$K_t^u \psi_R \geq e^{\lambda t} \psi_R. \tag{20}$$

Note that

$$\begin{aligned} (K_t^u \psi_R)(x, i) &\leq \left(\sup_{x \in \mathbf{R}^n} \psi_R(x, i) \right) (K_t^u \mathbf{1})(x, i) \\ &= \|\psi_R(\cdot, i)\|_\infty (K_t^u \mathbf{1})(x, i). \end{aligned} \tag{21}$$

Combining (20) and (21) gives

$$(K_t^u \mathbf{1})(x, i) \geq e^{\lambda t} \frac{\psi_R(x, i)}{\|\psi_R(\cdot, i)\|_\infty} \tag{22}$$

and this gives for $u \in \mathcal{U}$

$$\begin{aligned} J(u) &= \limsup_{t \to \infty} \frac{1}{t} \log (K_t^u \mathbf{1})(x, i) \\ &\geq \lambda + \limsup_{t \to \infty} \frac{1}{t} \log \frac{\psi_R(x, i)}{\|\psi_R(\cdot, i)\|_\infty} \\ &= \lambda. \end{aligned} \tag{23}$$

This proves the first part of the theorem. Assume now that the minimum in (16) is attained at $u^* \in \mathcal{U}_{\rm ds}$. Then

$$\lambda\psi(x,i) = (A^{u^*}\psi)(x,i) + c(x,i,u^*)\psi(x,i) \tag{24}$$

and, consequently, from (11) we have

$$K_t^{u^*}\psi = \mathrm{e}^{\lambda t}\psi. \tag{25}$$

Note that

$$(K_t^{u^*}\psi)(x,i) \geq \hat{\psi}(K_t^{u^*}\mathbf{1})(x,i) > 0 \tag{26}$$

and, therefore

$$(K_t^{u^*}\mathbf{1})(x,i) \leq \frac{(K_t^{u^*}\psi)(x,i)}{\hat{\psi}} = \mathrm{e}^{\lambda t}\frac{\psi(x,i)}{\hat{\psi}}. \tag{27}$$

It follows from (27) that $J(u^*) \leq \lambda$ which when combined with (23) completes the proof.

4. ANALYSIS OF THE COST FUNCTIONAL

In this section we show that for a fixed $u \in \mathcal{U}_{\rm ds}$ the cost functional in (5) can be represented as the optimal value of an auxiliary optimal control problem. This representation of the cost provides an insight into the effect of the noise input as well as the mode variable on the risk sensitivity of the cost. Furthermore, the representation is critical in establishing conditions for the existence of optimal controls. We begin by stating an additional technical assumption that is needed for the subsequent analysis.

(A1) For each $u \in \mathcal{U}_{\rm ds}$ there exists a $\tau^u > 0$, a σ-finite measure η^u on $\mathbf{R}^n \times S$ and a function $q^u(x,i,y,j)$ such that

(a) $q^u(x,i,y,j) > 0$ for η^u – almost all $(x,i) \in \mathbf{R}^n \times S$,
(b) $P^u(\tau^u,(x,i),(\mathrm{d}y,j)) = q^u(x,i,y,j)\eta^u(\mathrm{d}y,j)$,
(c) for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x - x'| < \delta$ then $\sum_{j\in S}\int_{\mathbf{R}^n}|q^u(x,i,y,j) - q^u(x',i,y,j)|\eta^u(\mathrm{d}y,j) < \varepsilon$.

Define the auxiliary system

$$dz_t = (b(z_t, s_t, u_t) + \sigma(z_t, s_t)v_{1t})\,dt + \sigma(z_t, s_t)\,dw_t, \tag{28}$$

where s_t is the continuous time Markov chain on S with generator matrix $\Pi^{u,v_2}(z)$ with entries

$$\pi_{ij}(z, u, v_2) = \pi_{ij}(z, u)\frac{v_2(z,j)}{v_2(z,i)}, i \neq j, \quad \pi_{ii}(z, u, v_2) = -\sum_{j \neq i} \pi_{ij}(z, u, v_2) \tag{29}$$

where $v_2 : \mathbf{R}^n \times S \to \mathbf{R}$ is measurable map satisfying $v_2(z, i) > 0$, $(z, i) \in \mathbf{R}^n \times S$. The variable $v = (v_1, v_2)$ is a control input in (28) and (29). Clearly, due to its form $v_2(z, i)$ is a Markov policy. We assume also that v_{1t} is Markov, i.e., there exists a measurable map $\tilde{v}_1 : \mathbf{R}^n \times S \to \mathbf{R}^n$ such that $v_{1t} = \tilde{v}_1(z_t, s_t)$. We denote the class of all such Markov polices by $\mathcal{V}$. Similarly, if v is nonrandomized then we write $v \in \mathcal{V}_d$.

LEMMA 1 *Let $u \in \mathcal{U}_{ds}$ and $v \in \mathcal{V}$. Assume that $v_2 : \mathbf{R}^n \times S \to [r, R]$ for some $R > r > 0$ and that v_1 is bounded. Then here exists an almost surely unique solution of* (28) *and* (29) *which is a strong Feller process on* $\mathbf{R}^n \times S$.

Proof We have to verify that system (28) and (29) satisfies assumption (A0). Let k be the Lipschitz constant for $b(x, i, u)$, $\sigma(x, i)$ and $\pi_{ij}(x, u)$ and let $0 < M < \infty$ be the bound for v_1. Then

$$\tilde{b}(x, i, u, v_1) = b(x, i, u) + \sigma(x, i)v_1 \tag{30}$$

satisfies

$$\begin{aligned}
&|\tilde{b}(x, i, u, v_1) - \tilde{b}(\hat{x}, i, u, v_1)| \\
&\leq |b(x, i, u) - b(\hat{x}, i, u)| + |(\sigma(x, i) - \sigma(\hat{x}, i))v_1| \\
&\leq k|x - \hat{x}| + k|x - \hat{x}|M \\
&= k(1 + M)|x - \hat{x}|.
\end{aligned} \tag{31}$$

Next note that

$$\begin{aligned}\left|\pi_{ij}(x,u,v_2)-\pi_{ij}(\hat{x},u,v_2)\right| &= \left|\left(\pi_{ij}(x,u)-\pi_{ij}(\hat{x},u)\right)\frac{v_2(\cdot,j)}{v_2(\cdot,i)}\right| \\ &\le k|x-\hat{x}|\sup_{v_2}\left|\frac{v_2(\cdot,j)}{v_2(\cdot,i)}\right| \\ &\le k\frac{R}{r}|x-\hat{x}|. \end{aligned} \tag{32}$$

Therefore, system (28) and (29) satisfies condition (i) of assumption (A0). Condition (ii) is trivially satisfied. Finally, the irreducibility of $\Pi^u(z)$ implies the irreducibility of matrix $\Pi^{u,v_2}(z)$ as well [11, Lemma 4.3] and, therefore, condition (iii) of assumption (A0) is satisfied.

Let $u \in \mathcal{U}_{\rm ds}$, $v \in \mathcal{V}_{\rm d}$ and let $A^{u,v}$ be the infinitesimal generator of (28) and (29), i.e., for any $f \in C^2(\mathbf{R}^n \times S)$,

$$(A^{u,v}f)(z,i) = (L_i^u f)(z,i) + v_1^T\sigma^T(z,i)\frac{\partial f}{\partial x}(z,i) + (\Pi^{u,v_2}f)(z,i). \tag{33}$$

We make the following assumption about the controlled system (28) and (29).

(A2) For each $u \in \mathcal{U}_{\rm ds}$ there exists a nonnegative function $\phi^u \in C^2(\mathbf{R}^n \times S)$ such that:

(i) $\lim_{|z|\to\infty}\phi^u(z,i) = \infty$,
(ii) There exist $\rho > 0$, $\varepsilon > 0$ such that $(A^{u,v}\phi)(z,i) < -\varepsilon$ for $|z| > \rho$ and $i \in S$, and $\left|\frac{\partial\phi^u}{\partial z}(z,i)\right|^2 > \sigma_0^{-1}$.
(iii) $\phi^u(z,i)$ and $\left|\frac{\partial\phi^u}{\partial z}(z,i)\right|^2$ have polynomial growth in z.

Remark Assumption (A2) implies that any Markov policy v for (28) and (29) is stabilizing. Thus, for any $v \in \mathcal{V}$, system (28) and (29) has an unique invariant measure $\mu^{u,v}$, i.e., $\mathcal{V} = \mathcal{V}_{\rm s}$ and $\mathcal{U}_{\rm d} = \mathcal{V}_{\rm ds}$.

For $v_2(z,i) > 0$ define

$$k(z,i,u,v_2) = (\Pi^{u,v_2}\log v_2)(z,i) - \frac{(\Pi^u v_2)(z,i)}{v_2(z,i)} \tag{34}$$

and

$$\bar{c}(z,i,u,v) = c(z,i,u) - \tfrac{1}{2}|v_1|^2 - k(z,i,u,v_2). \tag{35}$$

For system (28) and (29) consider the cost functional

$$\bar{K}(u,v) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \bar{c}(z_t,s_t,u_t,v_t)\,\mathrm{d}t. \tag{36}$$

Then since all Markov control policies v are stabilizing

$$\bar{K}(u,v) = \sum_{i\in S}\int_{\mathbf{R}^n} \bar{c}(z,i,u,v)\mu^{u,v}(\mathrm{d}z,i). \tag{37}$$

THEOREM 4 *Let* $u \in \mathcal{U}_{\mathrm{ds}}$ *and assume* (A0)–(A2). *Then*

$$J(u) = \sup_{v\in\mathcal{V}_{\mathrm{ds}}} \bar{K}(u,v). \tag{38}$$

Before we prove Theorem 4 we need the following lemmas. Let $f\in C_b(\mathbf{R}^n\times S) = \{f:\mathbf{R}^n\times S\to\mathbf{R}\,|\,f(x,i)\in C_b(\mathbf{R}^n),\, i\in S\}$ and define

$$\lambda^u(f) = \lim_{T\to\infty}\frac{1}{T}\log E^u_{xi}\left[\exp\left\{\int_0^T f(x_t,r_t)\,\mathrm{d}t\right\}\right], \tag{39}$$

where (x_t, r_t) is the solution of (1) and (3) corresponding to $u \in \mathcal{U}_{\mathrm{ds}}$.

LEMMA 2 *Assume* (A0)–(A2). *Then the limit* (39) *exists and* $\lambda^u(f)$ *is the unique principal eigenvalue of the operator* $A^u + f$. *Furthermore, corresponding to* $\lambda^u(f)$ *is a uniformly positive eigenfunciton* $h^u_f \in C_b(\mathbf{R}^n \times S)$.

Proof The proof is similar to the proof of Lemma 2.1 in [8]. We sketch the main idea. On $C_b(\mathbf{R}^n \times S)$ define the norm

$$\|f\| = \sup_{i\in S}\sup_{x\in\mathbf{R}^n}|f(x,i)|. \tag{40}$$

For $\beta\in\mathbf{R}$ and $h\in C_b(\mathbf{R}^n\times S)$ define the operator

$$(K^{u,f-\beta}_t h)(x,i) = E^u_{xi}\left[\exp\left\{\int_0^T (f(x_s,r_s)-\beta)\,\mathrm{d}s\right\}h(x_t,r_t)\right]. \tag{41}$$

Then $\{K_t^{u,f-\beta}, t \geq 0\}$ is a positive semigroup of operators on $C_b(\mathbf{R}^n \times S)$. Furthermore, since

$$\left\|K_t^{u,f-\beta} h\right\| \leq \mathrm{e}^{(\|f\|-\beta)t} \|T_t^u h\| \\ \leq \mathrm{e}^{(\|f\|-\beta)t} \|h\|, \tag{42}$$

$\{K_t^{u,f-\beta}, t \geq 0\}$ is a strongly continuous semigroup on $C_b(\mathbf{R}^n \times S)$. Here we have used the easily checked fact that $(T_t^u h)(x,i) = E_{xi}^u[h(x_t, r_t)]$ is a contraction semigroup on $C_b(\mathbf{R}^n \times S)$. Let η^u be the measure in assumption (A1) and let $L_2(\eta^u)$ be the Hilbert space with norm

$$\|g\|_2 = \left[\sum_{i=1}^{N} \int_{\mathbf{R}^n} |g(x,i)|^2 \eta^u(\mathrm{d}x, i)\right]^{1/2}. \tag{43}$$

Then $\{K_t^{f-\beta}, t \geq 0\}$ has a unique extension to $L_2(\eta^u)$ [13]. The remainder of the proof is similar to the proof in [8] and is based on the spectral theory for positive semigroups on $L_2(\eta^u)$ and $C_b(\mathbf{R}^n \times S)$ (see [12,14]). In particular, it is shown that $\mathrm{e}^{(\lambda^u(f)-\beta)t}$ and $(\lambda^u(f) - \beta)$ are the principal eigenvalues of $K_t^{u,f-\beta}$ and $A^u + f - \beta$, respectively, with common eigenfunction h_f^u, and consequently that $A^u + f$ has a principal eigenvalue $\lambda^u(f)$ and eigenfunction h_f^u. We omit the details.

For $\mu \in \mathcal{P}(\mathbf{R}^n \times S)$ and $\lambda^u(f)$ defined by (39) define

$$I^u(\mu) = \sup_{f \in C_b(\mathbf{R}^n \times S)} [\langle \mu, f \rangle - \lambda^u(f)], \tag{44}$$

where

$$\langle \mu, f \rangle = \sum_{i \in S} \int_{\mathbf{R}^n} f(x,i) \mu(\mathrm{d}x, i). \tag{45}$$

LEMMA 3 *Assume* (A0)–(A2). *Then for any* $u \in \mathcal{U}_{\mathrm{ds}}$

$$J(u) = \sup_{\mu \in \mathcal{P}(\mathbf{R}^n \times S)} \left[\sum_{i \in S} \int_{\mathbf{R}^n} c(x, i, u(x,i)) \mu(\mathrm{d}x, i) - I^u(\mu)\right]. \tag{46}$$

Proof The proof is based on the large deviations theory in [13] and is similar to the proof of Theorem 2.1 in [8]. We omit the details.

Proof of Theorem 4 Let $\bar{K}^*(u) = \sup_{v \in \mathcal{V}_{\text{ds}}} \bar{K}(u, v)$. We first show that $J(u) \geq \bar{K}^*(u)$. For $M > 0$ define $G_M^u = \{x \in \mathbf{R}^n\text{: } \max_{i \in S} c(x, i, u(x, i)) \leq M\}$ and

$$c_M(x, i, u) = \begin{cases} c(x, i, u), & x \in G_M^u \\ M, & x \notin G_M^u. \end{cases} \tag{47}$$

Then $c_M(x, i, u) \in C_b(\mathbf{R}^n \times S)$. Let $\lambda^u(c_M)$ be the principal eigenvalue of $A^u + c_M$ and let $\psi_M^u \in C_b(\mathbf{R}^n \times S)$ be the corresponding strictly positive eigenfunction. Then $\phi_M^u = \log \psi_M^u$ satisfies

$$L_i^u \phi_M^u + \frac{1}{2}\left(\frac{\partial \phi_M^u}{\partial x}\right)^T \sigma\sigma^T \frac{\partial \phi_M^u}{\partial x} + \frac{\Pi^u \mathrm{e}^{\phi_M^u}}{\mathrm{e}^{\phi_M^u}} + c_M = \lambda^u(c_M). \tag{48}$$

Adding and subtracting the term $v_i^T \sigma^T (\partial \phi_M^u / \partial x)$ and using a completion of squares argument and the identity

$$\frac{\Pi^u \mathrm{e}^{\phi_M^u}}{\mathrm{e}^{\phi_M^u}} = \sup_{v_2 > 0}\left[\Pi^{u, v_2} \phi_M^u + \frac{\Pi^u v_2}{v_2} - \Pi^{u, v_2} \log v_2\right]$$

we get

$$A^{u,v} \phi_M^u - \tfrac{1}{2}|v_1|^2 - k(z, i, u, v_2) + c_M \leq \lambda^u(c_M). \tag{49}$$

Since $c_M \leq c$ we have $\lambda^u(c_M) \leq J(u)$. Next note that it follows by a straightforward approximation argument from Lemma 5.2 in [6] that for any control policy $v \in \mathcal{V}_{\text{ds}}$

$$\sum_{i \in S} \int_{\mathbf{R}^n} (A^{u,v} \phi_M^u)(z, i) \mu^{u,v}(\mathrm{d}z, i) = 0. \tag{50}$$

Integrating both sides of (49) with respect to $\mu^{u,v}$ and using (50) gives

$$\sum_{i \in S} \int_{\mathbf{R}^n} \Big(c_M(z, i, u(z, i)) - \tfrac{1}{2}|v_1(z, i)|^2 - k(z, i, u(z, i), v_2(z, i))\Big) \mu^{u,v}(\mathrm{d}z, i)$$
$$\leq \lambda^u(c_M) \leq J(u). \tag{51}$$

Finally, by Fatou's lemma

$$\begin{aligned}\bar{K}(u,v) &= \sum_{i\in S}\int_{\mathbf{R}^n}\Big(c(z,i,u(z,i)) - \tfrac{1}{2}|v_1(z,i)|^2 \\ &\quad - k(z,i,u(z,i),v_2(z,i))\Big)\mu^{u,v}(\mathrm{d}z,i) \\ &\le \liminf_{M\to\infty}\sum_{i\in S}\int_{\mathbf{R}^n}\Big(c_M(z,i,u(z,i)) - \tfrac{1}{2}|v_1(z,i)|^2 \\ &\quad - k(z,i,u(z,i),v_2(z,i))\Big)\mu^{u,v}(\mathrm{d}z,i) \\ &\le J(u). \end{aligned} \tag{52}$$

We now prove the opposite inequality $J(u) \le \bar{K}^*(u)$. Let $v_1^M = \sigma^T(\partial\phi_M^u/\partial x)$ and $v_2^M = \mathrm{e}^{\phi_M^u}$ where ϕ_M^u satisfies (48). Then (48) can be rewritten as

$$A^{u,v^M}\phi_M^u + c_M - \tfrac{1}{2}|v_1^M|^2 - k(z,i,u,v_1^M) = \lambda^u(c_M) \tag{53}$$

where A^{u,v^M} is the infinitesimal generator of (28) and (29) with control v^M. Let (z_t^M, s_t^M) be the solution of (28) and (29) with control v^M and initial condition $(z_0^M, s_0^M) = (z,i)$. Pick $R > 0$ and let $\tau_R = \inf\{t \ge 0: |z_t^M| = R\}$. Then by Ito's formula

$$\begin{aligned} E\Big[\phi_M^u(z_{t\wedge\tau_R}^M, s_{t\wedge\tau_R}^M)\Big] - \phi_M^u(z,i) &= E\Big[\int_0^{t\wedge\tau_R}(A^{u,v}\phi_M^u)(z_s^M, s_\sigma^M)\,\mathrm{d}\sigma\Big] \\ &= E\Big[\int_0^{t\wedge\tau_R}\Big(\lambda^u(c_M) - c_M(z_\sigma^M, s_\sigma^M, u_\sigma) \\ &\qquad + \tfrac{1}{2}|v_{1\sigma}^M|^2 + k(z_\sigma^M, s_\sigma^M, u_\sigma, v_{2\sigma}^M)\Big)\,\mathrm{d}\sigma\Big]. \end{aligned} \tag{54}$$

Letting $R \to \infty$ gives

$$\begin{aligned} E[\phi_M^u(z_t^M, s_t^M)] - \phi_M^u(z,i) &= t\lambda^u(c_M) - E\Big[\int_0^t\Big(c_M(z_\sigma^M, s_\sigma^M, u_\sigma) \\ &\qquad - \tfrac{1}{2}|v_{1\sigma}^M|^2 - k(z_\sigma^M, s_\sigma^M, u_\sigma, v_{2\sigma}^M)\Big)\,\mathrm{d}\sigma\Big]. \end{aligned} \tag{55}$$

Since ϕ^u_M is bounded we get after dividing by t and letting $t \to \infty$

$$\lambda^u(c_M) = \lim_{t\to\infty} \frac{1}{t} E\left[\int_0^t \Big(c_M(z^M_\sigma, s^M_\sigma, u_\sigma) - \tfrac{1}{2}|v^M_{1\sigma}|^2 - k(z^M_\sigma, s^M_\sigma, u_\sigma, v^M_{2\sigma})\Big) \,\mathrm{d}\sigma \right]. \tag{56}$$

Define

$$\bar{K}^*_M(u) = \sup_{v\in\mathcal{V}_{\mathrm{ds}}} \sum_{i\in S} \int_{\mathbf{R}^n} \Big(c_M(z,i,u(z,i)) - \tfrac{1}{2}|v_1(z,i)|^2 - k(z,i,u,v_2(z,i))\Big) \mu^{u,v}(\mathrm{d}z, i). \tag{57}$$

Then by the proof of Theorem 5.1 in [6]

$$\lambda^u(c_M) \le \bar{K}^*_m(u).$$

Next note that by Lemma 3 and Fatou's lemma

$$\begin{aligned}
\liminf_{M\to\infty} \lambda^u(c_M) &= \liminf_{M\to\infty} \sup_{\mu\in\mathcal{P}(\mathbf{R}^n\times S)} \left[\sum_{i\in S} \int_{\mathbf{R}^n} c_M(x,i,u(x,i))\mu(\mathrm{d}x, i) - I^u(\mu)\right] \\
&\ge \liminf_{M\to\infty} \left[\sum_{i\in S} \int_{\mathbf{R}^n} c_M(x,i,u(x,i))\mu(\mathrm{d}x, i) - I^u(\mu)\right] \\
&\ge \sum_{i\in S} \int_{\mathbf{R}^n} \liminf_{M\to\infty} c_M(x,i,u(x,i))\mu(\mathrm{d}x, i) - I^u(\mu) \\
&= \sum_{i\in S} \int_{\mathbf{R}^n} c(x,i,u(x,i))\mu(\mathrm{d}x, i) - I^u(\mu).
\end{aligned} \tag{58}$$

By Lemma 3,

$$J(u) = \sup_{\mu\in\mathcal{P}(\mathbf{R}^n\times S)} \left[\sum_{i\in S} \int_{\mathbf{R}^n} c(x,i,u(x,i))\mu(\mathrm{d}x, i) - I^u(\mu)\right].$$

Thus since (58) holds for any $\mu \in \mathcal{P}(\mathbf{R}^n \times S)$

$$\liminf_{M\to\infty} \lambda^u(c_M) \ge J(u). \tag{59}$$

Recall that since $c_M \leq c$ we have $\lambda^u(c_M) \leq J(u)$. Therefore, from (59) we conclude that $\lambda^u(c_M) \to J(u)$ as $M \to \infty$. Also, since $c_M \leq c$ we have $\bar{K}^*_M(u) \leq \bar{K}^*(u)$. Thus we get

$$J(u) = \lim_{M\to\infty} \lambda^u(c_M) \leq \limsup_{M\to\infty} \bar{K}^*_M(u) \leq \bar{K}^*(u). \tag{60}$$

The proof is complete.

Remark Consider the original control problem (5) for system (1) and (3). Let $\mu^{u,v}$ be the invariant measure of (28) and (29) for the Markov policies $u \in \mathcal{U}_s$ and $v \in \mathcal{V}_s$. Then by (37) and (38) we have

$$\begin{aligned} J^* &= \inf_{u\in\mathcal{U}_{ds}} J(u) \\ &= \inf_{u\in\mathcal{U}_{ds}} \sup_{v\in\mathcal{V}_{ds}} \bar{K}(u,v) \\ &= \inf_{u\in\mathcal{U}_{ds}} \sup_{v\in\mathcal{V}_{ds}} \sum_{i\in S} \int_{\mathbf{R}^n} (c(z,i,u) - \tfrac{1}{2}|v_1|^2 - k(z,i,u,v_2))\mu^{u,v}(\mathrm{d}z,i). \end{aligned} \tag{61}$$

Equation (61) shows that the optimal value of the original optimal risk-sensitive control problem is equivalent to the upper value of a stochastic differential game for the auxillary system (28) and (29) with cost functional (36).

Remark In the cost functional $\bar{K}(u,v)$ the term $\frac{1}{2}|v_1|^2$ represents the risk sensitivity due to the white noise input in (1). On the other hand, the term $k(z,i,u,v_2)$, and in particular the auxiliary control v_2, represents the risk sensitivity due to the mode variable (jump process) r_t in (1). In the H_∞ formulation of linear hybrid systems in [10] a term corresponding to $\frac{1}{2}|v_1|^2$ appears but not a term involving v_2, i.e. the formulation in [10] does not incorporate robustness to changes in the distribution of the mode variable. Therefore, unlike the standard case (i.e. systems with a single mode), the formulation of H_∞ control of linear hybrid systems in [10] is not equivalent to risk sensitive control of such systems.

5. EXISTENCE OF OPTIMAL POLICY

We now establish the existence of an optimal control for the optimal control problem (5) for system (1) and (3). The results in this section are based on the existence results in [6] where additional technical details can be found. We begin with the following result for the auxiliary optimal control problem (28), (29) and (36).

LEMMA 4 *Assume* (A0)–(A2). *Then for each* $u \in \mathcal{U}_{\mathrm{ds}}$ *there exists a* $v_u^* \in \mathcal{V}_{\mathrm{ds}}$ *such that*

$$J(u) = \bar{K}(u, v_u^*) \quad \text{a.s.} \tag{62}$$

Proof For each fixed $u \in \mathcal{U}_{\mathrm{ds}}$ the optimal control problem (36) for system (28) and (29) is of the form of the problems considered in [6]. First note that, under assumptions (A0) and (A2) it follows from Theorem 5.3 in [6] that there exists a $v_u^* \in \mathcal{V}_{\mathrm{ds}}$ such that $\bar{K}(u, v_u^*) = \sup_{v \in \mathcal{V}_{\mathrm{s}}} \bar{K}(u, v)$ a.s. Noting that by Theorem 4 we have $J(u) = \sup_{v \in \mathcal{V}_{\mathrm{ds}}} \bar{K}(u, v)$ completes the proof.

We now discuss the existence of an optimal control for the risk sensitive control problem, i.e., we consider system (1) and (3) and the optimal control problem (5). Let $v_u^* \in \mathcal{V}_{\mathrm{ds}}$ be the optimal control in Lemma 4 and define

$$\bar{c}(z, i, u) = c(z, i, u) - \tfrac{1}{2}|v_{u1}^*(z, i)|^2 - k(z, i, u, v_{u2}^*(z, i)) \tag{63}$$

We will concentrate on cost functionals that satisfy the so called near-monotonicity condition (see [6] for terminology)

$$\text{(A3)} \qquad \liminf_{|z| \to \infty} \inf_{u,i} \bar{c}(z, i, u) > J^*. \tag{64}$$

Roughly speaking, if condition (A3) is satisfied then large values of the system state are penalized, i.e., the optimal control will tend to push the system state towards some bounded set in the state space.

THEOREM 5 *Assume* (A0)–(A3). *Then there exists a* $u^* \in \mathcal{U}_{\mathrm{ds}}$ *such that*

$$J^* = J(u^*) \quad \text{a.s.}$$

Proof By Lemma 4 we know that $J(u) = \bar{K}(u, v_u^*)$ a.s. If the near monotonicity condition (A3) is satisfied it follows from Theorem 5.2 in [6] that there exists a $u^* \in \mathcal{U}_{\rm ds}$ such that

$$\inf_{u \in \mathcal{U}_{\rm ds}} \sum_{i \in S} \int_{\mathbf{R}^n} \bar{c}(z, i, u) \mu^{u, v_u^*}(\mathrm{d}z, i) = \inf_{u \in \mathcal{U}_{\rm ds}} \bar{K}(u, v_u^*) = \inf_{u \in \mathcal{U}_{\rm ds}} J(u) = J(u^*)$$
$$= J^* \quad \text{a.s.}$$

6. CONCLUSIONS

In this paper we have analyzed the infinite horizon risk-sensitive control problem for a general hybrid system comprised of a diffusion process that depends on a mode variable modeled by a Markov chain. We established optimality conditions, obtained an equivalent differential game representation of the cost, and proved the existence of optimal controls under the appropriate technical conditions. The analysis in the paper has shown that even in the simplest cases it is very difficult to obtain closed form analytical solutions to the optimal risk-sensitive control problem for hybrid systems. Therefore, future research should concentrate on developing suboptimal solution methods as well as numerical techniques. Future research should also address optimal risk sensitive control of partially observed systems.

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References

[1] A.S. Willsky and B.C. Levy, "Stochastic stability research for complex power system," Lab. Inf. Decision Systems, MIT, Report no. ET-76-C-01-2295, 1979.
[2] M. Mariton, *Jump Linear Systems in Automatic Control*, Marcel Dekker, New York, 1990.
[3] D.D. Siljak, "Reliable control using multiple control systems," *Int. J. Control* **31** (1980), 303–313.
[4] D.D. Sworder, "Feedback Control of a Class of Linear Systems with Jump Parameters," *IEEE Transactions on Automatic Control* **14** (1969), pp. 9–14.

[5] M. Ghosh, A. Arapostathis and S.I. Marcus, "Optimal Control of Swithching Diffusions with Application to Flexible Manufacturing systems," *SIAM J. Control Optim.* **31** (1993), pp. 1183–1204.
[6] M. Ghosh, A. Arapostathis and S.I. Marcus, "Ergodic Control of Switching Diffusions," *SIAM J. Control Optim.* **35** (1997), pp. 1952–1988.
[7] K. Glover and J.C. Doyle, "State-space Formulae for All Stabilizing Controllers that Satisfy an -norm bound and Relations to Risk Sensitivity," *Systems & Control Letters* **11** (1988), pp. 167–172.
[8] T. Runolfsson, "The Equivalence between Infinite-horizon Optimal Control of Stochastic Systems with Exponential-of-integral Performance Index and Stochastic Differential Games," *IEEE Transactions on Automatic Control* **39** (1994), pp. 1551–1563.
[9] W.H. Fleming and M.R. James, "The Risk-sensitive Index and H-2 and H-infinity Norms for Nonlinear Systems," *Mathematics of Control, Signals, and Systems* **8** (1996), pp. 199–221.
[10] Z. Pan and T. Basar, "H_∞ -control of Markovian jump systems and solutions to associated piecewise-determinisitic differential games," In G.J. Olsder, Ed., *Annals of Dynamics Games*, Birkhauser, New York, 1995, pp. 61–94.
[11] W.H. Fleming and Q. Zhang, "Risk-sensitive Production Planning of a Stochastic Manufacturing System," *SJAM J. Control Optim.* **36** (1998), pp. 1147–1170.
[12] Ph. Clement *et al.*, *One-Parameter Semigroups*, North Holland, New York, 1987.
[13] J.D. Deuschel and D.W. Stroock, *Large Deviations*, Academic Press, San Diego, 1989.
[14] R. Nagel, Ed., *One-Parameter Semigroups of Positive Operators*, Springer Verlag, New York, 1986.