Mathematical Problems in Engineering
Volume 3, pp. 517–538
Reprints available directly from the publisher
Photocopying permitted by license only

Robust Performance Results for Discrete-Time Systems

MAGDI S. MAHMOUD *

Department of Electrical and Computer Engineering, Kuwait University, P.O. Box 5969, Safat-13060, Kuwait

(Received 24 July 1997)

The problems of robust performance and feedback control synthesis for a class of linear discrete-time systems with time-varying parametric uncertainties are addressed in this paper. The uncertainties are bound and have a linear matrix fractional form. Based on the concept of strongly robust H_∞-performance criterion, results of robust stability and performance are developed and expressed in easily computable linear matrix inequalities. Synthesis of robust feedback controllers is carried out for several system models of interest.

Keywords: Robust performance; Feedback control; Uncertain systems; Linear matrix inequalities

1. INTRODUCTION

Robust control theory provides tractable design tools using the time domain and the frequency domain (Doyle *et al.*, 1989; Petersen *et al.*, 1991) when the plant modeling uncertainty or external disturbance uncertainty is of major concern in control systems. In the time domain, efforts have been centered around the problem of quadratic stabilization of linear uncertain systems (Khargonekar *et al.*, 1990). With focus on time-varying uncertainties, robustness results basedon the concepts of quadratic stability and H_∞-disturbance attenuation have been developed (De Souza *et al.*, 1993; Petersen, 1989; Xie *et al.*,

* Fax: 011-965-4817451. E-mail: magdi@eng.kuniv.edu.kw.

1992). For discrete-time systems, Gu *et al.* (1993) and Gu (1994) have proposed a design method based on the trade-off between the minimum achievable H_∞-norm bound and the size of uncertainties. Recently, it has been pointed out (Zhou *et al.*, 1995) that the robust performance problem for linear uncertain systems with time-varying uncertainties has not been sufficiently examined. As an initial step, they have introduced the concept of strongly robust H_∞-performance criterion and have provided some of its applications. This work deals primarily with discrete-time systems and generalizes the results of (De Souza *et al.*, 1993; Gu *et al.*, 1993; Gu, 1994). It contributes to the further development of robust performance problem by casting the concept of strongly robust H_∞-performance criterion into the robust analysis and feedback synthesis of linear systems with time-varying uncertainties modeled by linear matrix fractional representation. Towards our goal, some earlier results are systematically recovered and subsequently extended to new and easily computable forms involving matrix inequalities. These inequalities include the algebraic Riccati inequality (ARI), the Lyapunov inequality (LI) and a linear matrix inequality (LMI). The approach of using matrix inequalities has the advantage that it can be solved using numerically-stable and efficient methods including the interior-point algorithms (Boyd *et al.*, 1994). For simplicity in exposition, the analytical results are organized into lemmas, theorems and corollaries.

The paper is organized as follows: the problem under consideration and related definitions are stated in Section 2. In Section 3 some basic results are provided using matrix inequalities before formally presenting the concept strongly robust H_∞-performance criterion (Zhou *et al.*, 1995). The structure of uncertainty in the form of linear matrix fractional representation is introduced and its properties are analysed in Section 4. Then the model parameterization follows in Section 5 to allow dealing with uncertainties in matrix inequalities format. The subject of robust control design is then studied in Section 6 and the results of several special models are provided. The paper is concluded with some relevant remarks.

NOTATIONS Let $\Re = (-\infty, -\infty)$, $\Re_+ = [0, 1, 2, \ldots, \infty)$, $\aleph$ be the set of complex numbers, $\Re^n$ be any n-dimensional linear vector space over the reals equipped by the standard Euclidean norm $\|\cdot\|$. Let $\mathbf{e}_j$

be a row vector of appropriate dimension with 1 on the jth entry and 0 elsewhere. The Lebsegue space $L_2[0,\infty)$ consists of square-summable sequences on $\Re_+$. We use $W^{\mathrm{t}}, W^{-1}, \lambda(W), \sigma_{\mathrm{M}}(W)$ to denote, respectively, the transpose of, the inverse of, the eigenvalue of and maximum singular value of any square matrix W. The matrix norm is the corresponding induced one; that is $\|W\| = \sigma_{\mathrm{M}}(W)$. LEt$\Im := \{z \in \aleph : |z < 1\}$. We use $W > 0$ ($W < 0$) to denote a positive-(negative-) definite matrix W. For a stable matrix $G(z)$, $\lambda[G(z)] \in \Im$, the H_∞-norm is defined by

$$\|G(z)\|_\infty := \sup_{\theta \in [0,2\pi]} \|G(\mathrm{e}^{\mathrm{j}\theta})\| = \sup_{0 \neq w \in L_2[0,N)} (\|h\|_2 / \|w\|_2),$$

where $\|\cdot\|_2$ stands for the usual ℓ_2 norm given by $\|p\|_2 := \left(\sum_{k=0}^{N} P^2(k)\right)^{1/2}$.

2. PROBLEM STATEMENT AND DEFINITIONS

A wide class of discrete-time uncertain systems is modeled in state-space by

$$x(k+1) = A(\Delta)x(k) + B(\Delta)w(k), \qquad x(0) = 0, \tag{1a}$$

$$z(k) = C(\Delta)x(k) + D(\Delta)w(k), \tag{1b}$$

where at time $k \in \Re_+$, $x(k) \in \Re^n$ is the state vector; $w(k) \in \Re^q$ is the disturbance input vector; $z(k) \in \Re^r$ is the controlled output and $A(\Delta), B(\Delta), C(\Delta), D(\Delta)$ are bounded matrices and their entries are affine functions of $\Delta(k)$; $\Delta(k) \in \Xi$ is (possibly) a time-varying uncertain matrix. The set Ξ is compact with a particular structure to be specified shortly (see Section 4). Note that system (1) could represents a model of an open-loop system under consideration for control design. It could also represents the model of a controlled system for which some robustness issues are to be examined.

DEFINITION 1 System (1) with $w = 0$ is said to be *quadratically stable*, if there exists a matrix $0 < P = P^{\mathrm{t}} \in \Re^{n \times n}$ such that $V(x) = x^{\mathrm{t}}Px$ is a Lyapunov function for the system, that is $\Delta V(x(k)) := V(x(k+1)) - V(x(k)) < 0 \ \forall x \neq 0)$ and $\Delta \in \Xi$.

Distinct from (1) is the *nominal system*

$$x(k+1) = A_0\,x(k) + B_0 w(k), \qquad x(0) = 0, \tag{2a}$$

$$z(k) = C_0\,x(k) + D_0 w(k), \tag{2b}$$

$$A_0 = A(\Delta = 0);\ B_0 = B(\Delta = 0); \tag{3}$$

$$C_0 = C(\Delta = 0);\ D_0 = D(\Delta = 0), \tag{4}$$

for which the transfer function from w to z is given by

$$T_{zw}(z) = \left[\begin{array}{cc} A_0 & B_0 \\ C_0 & D_0 \end{array}\right] := C_0(zI - A_0)^{-1}B_0 + D_0. \tag{5}$$

DEFINITION 2 Given a scalar $\gamma > 0$, the nominal system (2) is said to be stable with H_∞-disturbance attenuation γ if $\lambda(A_0) \in \mathfrak{F}$ and $\|T_{zw}(z)\|_\infty < \gamma$.

Remark 1 It should be noted that the concept of quadratic stability requires the existence of a fixed (uncertainty-independent) quadratic Lyapunov function $V(x)$ for all possible choices of the uncertainty parameters.

From now onwards, without loss of generality, we take $\gamma = 1$. This can be done with the scaling $T \to \gamma^{-1}T$, $D_0 \to \gamma^{-1}D_0$ and $B_0 \to \gamma^{-1}B_0$. Our objective here is to establish robust performance and stability results for a wide class of discrete-time systems with time-varying uncertainties.

3. PRELIMINARIES

In this section, some results relevant to the subsequent developments are presented. Our starting point is the *Strictly Bounded Real* (*SPR*) *Lemma* (Haddad and Bernstein, 1994):

LEMMA 1 *Let $T(z)$ be a proper, real rational transfer function matrix with minimal realization $\{A_0, B_0, C_0, D_0\}$. Then the following statements are equivalent*:

(1) *A_0 is asymptotically stable and $T(z)$ is SPR.*
(2) *There exists a scalar $\varepsilon > 0$, $0 < X = X^{\mathrm{t}} \in \Re^{n\times n}$ and $0 < Y = Y^{\mathrm{t}} \in \Re^{n\times n}$ such that*

(a) $I - D_0^t D_0 - B_0^t X B_0 > 0$;
(b) $X = A_0^t X A_0 + (B_0^t X A_0 + D_0^t C_0)^t (I - D_0^t D_0 - B_0^t X B_0)^{-1}$
$\times (B_0^t X A_0 + D_0^t C_0) + C_0^t C_0 + \varepsilon Y.$ (6)

LEMMA 2 *For the nominal system* (2)–(5) *satisfying Lemma* 1, *the following statements are equivalent*:

(1) $\lambda(A_0) \in \Im$ *and* $\|T_{zw}(z)\|_\infty < 1$.
(2) *The matrix* $(I - B_0^t X B_0 - D_0^t D_0) > 0$ *is invertible and there exists a matrix* $0 < X = X^t \in \Re^{n \times n}$ *solving the following algebraic Riccati inequality* (*ARI*):

$$A_0^t X A_0 - X + (A_0^t X B_0 + C_0^t D_0)(I - D_0^t D_0 - B_0^t X B_0)^{-1} \times (B_0^t X A_0 + D_0^t C_0) + C_0^t C_0 < 0. \quad (7)$$

(3) *There exists a matrix* $0 < Y = Y^t \in \Re^{n \times n}$ *solving the following LMI*:

$$W_0 = \begin{bmatrix} \begin{bmatrix} -Y^{-1} & 0 \\ 0 & -I \end{bmatrix} & \cdot & \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \\ \cdots\cdots\cdots & \cdot & \cdots\cdots\cdots \\ \begin{bmatrix} A_0^t & C_0^t \\ B_0^t & D_0^t \end{bmatrix} & \cdot & \begin{bmatrix} -Y & 0 \\ 0 & -I \end{bmatrix} \end{bmatrix} < 0. \quad (8)$$

Proof (1) $\Rightarrow$ (2): Follows from (Zhou and Khargonekar, 1988) on noting that $\|T_{zw}(Z)\|_\infty < 1$ corresponds to $I - D_0^t D_0 - B_0^t X B_0 > 0$ and $\|C_f(zI - A_f)^{-1} B_f\|_\infty < 1$ with

$$A_f = A_0 + B_0(I - D_0^t D_0 - B_0^t X B_0)^{-1} D_0^t C_0;$$
$$B_f = B_0(I - D_0^t D_0 - B_0^t X B_0)^{-1/2}; \quad (9)$$
$$C_f = (I - D_0^t D_0 - B_0 X B_0^t)^{-1/2} C_0,$$

(2) $\Rightarrow$ (3): Since any constant block matrix of the type

$$R = \begin{bmatrix} R_1 & R_2 \\ R_2^t & R_4 \end{bmatrix} \quad (10)$$

can be expressed in the form

$$\begin{bmatrix} R_1 & R_2 \\ R_2^{\mathrm{t}} & R_4 \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ R_2^{\mathrm{t}} & I \end{bmatrix} \begin{bmatrix} R_1^{-1} & 0 \\ 0 & R_4 - R_2^{\mathrm{t}} R_1^{-1} R_2 \end{bmatrix} \begin{bmatrix} R_1 & R_2 \\ 0 & I \end{bmatrix}. \quad (11)$$

The condition $R < 0$ corresponds to (1) $R_1 < 0$, and (2) $(R_4 - R_2^{\mathrm{t}} R_1^{-1} R_2) < 0$. By setting

$$R_1 = \begin{bmatrix} -Y^{-1} & 0 \\ 0 & -I \end{bmatrix}, \qquad R_2 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}, \qquad R_4 = \begin{bmatrix} -Y & 0 \\ 0 & -I \end{bmatrix},$$

expanding and rearranging, the result follows.

DEFINITION 3 (Zhou *et al.*, 1995). System (1) is said to satisfy *strongly robust H_∞-performance criterion* if there exists a matrix $0 < Y = Y^{\mathrm{t}} \in \Re^{n \times n}$ solving the Δ-dependent ARI.

$$\begin{aligned} &A^{\mathrm{t}}(\Delta) Y A(\Delta) - Y + C^{\mathrm{t}}(\Delta) C(\Delta) \\ &\quad + [A^{\mathrm{t}}(\Delta) Y B(\Delta) + C^{\mathrm{t}}(\Delta) D(\Delta)] R^{-1}(\Delta) \\ &\qquad \times [A^{\mathrm{t}}(\Delta) Y B(\Delta) + C^{\mathrm{t}}(\Delta) D(\Delta)]^{\mathrm{t}} \leq 0, \end{aligned} \quad (12)$$

where

$$R(\Delta) = [I - D^{\mathrm{t}}(\Delta) D(\Delta) - B^{\mathrm{t}}(\Delta) Y B(\Delta)]. \quad (13)$$

COROLLARY 1 *System* (1) *is said to satisfy* strongly robust H_∞-performance criterion *if there exists a matrix* $0 < Y = Y^{\mathrm{t}} \in \Re^{n \times n}$ *solving the* Δ*-dependent LMI*

$$W(\Delta) = \begin{bmatrix} \begin{bmatrix} -Y^{-1} & 0 \\ 0 & -I \end{bmatrix} & \cdot & \begin{bmatrix} A(\Delta) & B(\Delta) \\ C(\Delta) & D(\Delta) \end{bmatrix} \\ \cdots\cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ \begin{bmatrix} A^{\mathrm{t}}(\Delta) & C^{\mathrm{t}}(\Delta) \\ B^{\mathrm{t}}(\Delta) & D^{\mathrm{t}}(\Delta) \end{bmatrix} & \cdot & \begin{bmatrix} -Y & 0 \\ 0 & -I \end{bmatrix} \end{bmatrix} < 0 \quad (14)$$

for all admissible uncertainties $A(\Delta)$, $B(\Delta)$, $C(\Delta)$ *and* $D(\Delta)$. *This is quite clear from Lemma 2 in view of Definition 3.*

Remark 2 It should be noted that Lemma 2 provides a non-standard form of the link between the concepts of H_∞-disturbance attenuation and quadratic stability.

Remark 3 In the light of Definitions 2 and 3, it is easy to see that when a system satisfies strongly robust H_∞-performance criterion then it is necessarily quadratically stable. In the sequel, more will be said on the properties of this criterion and its use in robust control synthesis.

4. UNCERTAINTY STRUCTURE

In the literature on state-space models containing parametric uncertainties, there have been so far three different methods to characterize the uncertainty. In the first method, the uncertainty is assumed to satisfy the so-called *matching condition* (Bahnasawi and Mahmoud, 1989). Loosely speaking, this condition implies that the uncertainties cannot enter arbitrarily into the system dynamics but are rather restricted to lie in the range space of the input matrix. By the second method, the uncertainty is represented by dyadic (rank-1) decomposition (Schmitendorf, 1988). It is well-known that both methods are quite restrictive in practice. According to the third method, the uncertainty is expressed in terms of a norm-bounded form as (Petersen, 1987). It has been reported recently (Zhou *et al.*, 1995) that a general linear fractional representation would be a reasonable structure of uncertainty. Here we follow this trend and consider the uncertainty matrices to be of the form:

$$\begin{bmatrix} A(\Delta) & B(\Delta) \\ C(\Delta) & D(\Delta) \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \begin{bmatrix} S \\ L \end{bmatrix} \Delta(k) [M \quad N], \tag{15a}$$

$$\Delta(k) = H(k)[I - \Theta H(k)]^{-1}, \tag{15b}$$

where A_0, B_0, C_0, D_0 are constant matrices given by (2a), (2b); $S \in \Re^{n \times \alpha}$, $M \in \Re^{\beta \times n}$, $N \in \Re^{\beta \times q}$, $L \in \Re^{r \times \alpha}$ are constant matrices and $H(k) \in \Re^{\alpha \times \beta}$ is unknown matrix satisfying the bounding condition $H^{\mathrm{t}}(k)\ H(k) \leq I$; $k \in \Re_+$. The constant matrix $\Theta \in \Re^{\beta \times \alpha}$ satisfies the contraction condition $\Theta^{\mathrm{t}}\Theta < I$ in order to guarantee the existence of the inversion in (15b). Now, the set of uncertainties

considered in this paper Ξ is defined by

$$\Xi = \{\Delta(k):\ \Delta(k) = H(k)[I - \Theta H(k)]^{-1},\ k \in \Re_+;\ H^{\mathrm{t}}(k)H(k) \leq I \text{ and } \Theta^{\mathrm{t}}\Theta < I\} \tag{16}$$

and it provides a characterization of the time-behavior of the uncertainties subject to two bounding inequalities, one on Θ and the other on $H(k)$.

Before proceeding further, we recall the following result

LEMMA 3 *Let Σ_1, Σ_2 be real constant matrices of compatible dimensions and $\Gamma(k)$ be a real matrix function satisfying $\Gamma^{\mathrm{t}}(k)\Gamma(k) \leq N_0$, $0 < N_0 = N_0^{\mathrm{t}}$. Then the following inequality holds*

$$\Sigma_1\Gamma(k)\Sigma_2 + \Sigma_2^{\mathrm{t}}\Gamma^{\mathrm{t}}(k)\Sigma_1^{\mathrm{t}} \leq \rho^2\Sigma_1\Sigma_1^{\mathrm{t}} + \rho^{-2}\Sigma_2^{\mathrm{t}}N_0\Sigma_2, \quad \rho > 0. \tag{17}$$

Proof Consider the matrix function

$$[\rho\Sigma_1^{\mathrm{t}} - \rho^{-1}\Gamma(k)\Sigma_2]^{\mathrm{t}}[\rho\Sigma_1^{\mathrm{t}} - \rho^{-1}\Gamma(k)\Sigma_2] \geq 0. \tag{18}$$

On expanding (18), we get $\rho^2\Sigma_1\Sigma_1^{\mathrm{t}} - \Sigma_2^{\mathrm{t}}\Gamma^{\mathrm{t}}(k)\Sigma_1^{\mathrm{t}} - \Sigma_1\Gamma(k)\Sigma_2 + \rho^{-2}\Sigma_2^{\mathrm{t}}\Gamma^{\mathrm{t}}(k)\Gamma(k)\Sigma_2 \geq 0$, and by rearranging the terms using $\Gamma^{\mathrm{t}}(k) \times \Gamma(k) \leq N_0$, the desired inequality (17) results.

COROLLARY 2 *When $\Sigma_1\Gamma(k)\Sigma_2 + \Sigma_2^{\mathrm{t}}\Gamma^{\mathrm{t}}(k)\Sigma_1^{\mathrm{t}} + \Pi < 0$, $\Pi^{\mathrm{t}} = \Pi > 0$, it follows from Lemma 3 that this implies that $\rho^2\Sigma_1\Sigma_1^{\mathrm{t}} + \rho^{-2}\Sigma_2^{\mathrm{t}}N_0\Sigma_2 + \Pi < 0$.*

The next lemma provides an alternative and more convenient version of (16).

LEMMA 4 *Given that $\Theta^{\mathrm{t}}\Theta < \mathrm{I}$, then the set of uncertainties Ξ can be expressed as*

$$\begin{aligned}\Xi_{\mathrm{a}} &= \{\Delta(k): \Delta(k)\Delta^{\mathrm{t}}(k) \leq I + \Delta(k)\Theta + \Theta^{\mathrm{t}}\Delta^{\mathrm{t}}(k) + \Delta(k)\Theta\Theta^{\mathrm{t}}\Delta^{\mathrm{t}}(k)\} \\ &= \{\Delta(k):\ \Delta(k) = \Theta^{\mathrm{t}}[I - \Theta\Theta^{\mathrm{t}}]^{-1} + \Psi^{\mathrm{t}}(k)[I - \Theta\Theta^{\mathrm{t}}]^{-1/2}; \\ &\quad \Psi^{\mathrm{t}}(k)\Psi(k) \leq I + \Theta^{\mathrm{t}}[I - \Theta\Theta^{\mathrm{t}}]^{-1}\Theta;\ \Theta^{\mathrm{t}}\Theta < I; \\ &\quad \Psi(k) = [I - \Theta\Theta^{\mathrm{t}}]^{-1/2}\Delta^{\mathrm{t}}(k) - [I - \Theta\Theta^{\mathrm{t}}]^{-1/2}\Theta\}.\end{aligned} \tag{19}$$

Proof Observe that simple manipulation of (15b) implies $H(k) = [I + \Delta(k)\Theta]^{-1}\Delta(k)$. Hence, it directly follows that $[I + \Delta(k)\Theta] = [I - H(k)\Theta]^{-1}$. Since $H^{t}(k)H(k) \leq I$ by definition, we get:

$$\begin{aligned}
&\Delta^{t}(k)(I + \Delta(k)\Theta)^{-t}(I + \Delta(k)\Theta)^{-1}\Delta(k) \leq I,\\
&\Delta^{t}(k)[I + \Delta(k)\Theta + \Theta^{t}\Delta^{t}(k) + \Delta(k)\Theta\Theta^{t}\Delta^{t}(k)]^{-1}\Delta(k) \leq I,\\
&\Delta^{t}(k)[I + \Omega(k)]^{-1}\Delta(k) \leq I,\\
&\Omega(k) := \Delta(k)\Theta + \Theta^{t}\Delta^{t}(k) + \Delta(k)\Theta\Theta^{t}\Delta^{t}(k).
\end{aligned} \tag{20}$$

It then follows from (20) that

$$\begin{aligned}
&\Delta(k)\Delta^{t}(k)[I + \Omega(k)]^{-1}\Delta(k) \leq \Delta(k),\\
&\Delta(k)\Delta^{t}(k)[I + \Omega(k)]^{-1}\Delta(k) \leq [I + \Omega(k)][I + \Omega(k)]^{-1}\Delta(k),\\
&\{\Delta(k)\Delta^{t}(k) - [I + \Omega(k)]\}[I + \Omega(k)]^{-1}\Delta(k) \leq 0.
\end{aligned} \tag{21}$$

Since the indicated inverse in (21) exists, we obtain

$$\Delta(k)\Delta^{t}(k) \leq I + \Delta(k)\Theta + \Theta^{t}\Delta^{t}(k) + \Delta(k)\Theta\Theta^{t}\Delta^{t}(k). \tag{22}$$

Introducing $\Psi(k) = [I - \Theta\Theta^{t}]^{1/2}$, $\Delta^{t}(k) - [I - \Theta\Theta^{t}]^{-1/2}\Theta$, and completing the squares in (22) we reach $\Psi^{t}(k)\Psi(k) \leq I + \Theta^{t}[I - \Theta\Theta^{t}]^{-1}\Theta$ and the proof is completed.

Remark 4 It is interesting to note from Lemma 4 that $\Xi_a \subset \Xi$. The significance of expression (19) over (16) is quite evident in eliminating the inequality on $H(k)$.

Using the above uncertainty structure, we obtain the following result:

LEMMA 5 *Let* Σ_1, Σ_2, $\Sigma_3 = \Sigma_3^{t}$ *be real constant matrices of compatible dimensions. Then* $\forall \Delta(k) \in \Xi_a$ *of* (19) *and for some* $\rho > 0$ *we have*

$$\Sigma_1\Delta(k)\Sigma_2 + \Sigma_2^{t}\Delta^{t}(k)\Sigma_1^{t} + \Sigma_3 < 0 \tag{23}$$

if and only if

$$\begin{bmatrix} \rho^{-1}\Sigma_2^{t} & \rho\Sigma_1 \end{bmatrix} \begin{bmatrix} I & -\Theta \\ -\Theta^{t} & I \end{bmatrix}^{-1} \begin{bmatrix} \rho^{-1}\Sigma_2 \\ \rho\Sigma_1^{t} \end{bmatrix} + \Sigma_3 < 0. \tag{24}$$

Proof Starting with (23), we use (19) to get

$$\Sigma_1\Theta^t(I-\Theta\Theta^t)^{-1}\Sigma_2+\Sigma_1\Psi^t(I-\Theta\Theta^t)^{-1/2}\Sigma_2 + \Sigma_2^t(I-\Theta\Theta^t)^{-1}\Theta\Sigma_1^t+\Sigma_2^t(I-\Theta\Theta^t)^{-1/2}\Psi\Sigma_1^t+\Sigma_3<0. \quad (25)$$

By Corollary 2, there exists some $\rho>0$ such that

$$\Sigma_1\Theta^t(I-\Theta\Theta^t)^{-1}\Sigma_2+\Sigma_2^t(I-\Theta\Theta^t)^{-1}\Theta\Sigma_1^t+\rho^{-2}\Sigma_2^t(I-\Theta\Theta^t)^{-1}\Sigma_2 + \rho^2\Sigma_1[I+\Theta^t(I-\Theta\Theta^t)^{-1}\Theta]\Sigma_1+\Sigma_3<0. \quad (26)$$

Algebraic manipulation of (26) using the facts $[I-\Theta\Theta^t]^{-1}=[I+\Theta(I-\Theta^t\Theta)^{-1}\Theta^t]$ and $\Theta^t\Theta<I$ leads us to:

$$\Sigma_1^t(I-\Theta^t\Theta)^{-1}\Theta^t\Sigma_2+\Sigma_2^t\Theta(I-\Theta^t\Theta)^{-1}\Sigma_1 + \rho^{-2}\Sigma_2^t[\Theta(I-\Theta^t\Theta)^{-1}\Theta^t]\Sigma_2 + \rho^{-2}\Sigma_2^t\Sigma_2+\rho^2\Sigma_1(I-\Theta^t\Theta)^{-1}\Sigma_1+\Sigma_3<0. \quad (27)$$

Turning to (24) and recalling that $\Theta^t\Theta<I$, then a direct expansion yields:

$$\rho^{-2}\Sigma_2^t[I+\Theta(I-\Theta^t\Theta)^{-1}\Theta^t]\Sigma_2+\Sigma_1[(I-\Theta^t\Theta)^{-1}\Theta^t]\Sigma_2 + \Sigma_2^t[\Theta(I-\Theta^t\Theta)^{-1}]\Sigma_1^t+\rho^2\Sigma_1[I-\Theta^t\Theta)^{-1}\Theta^t]\Sigma_1^t+\Sigma_3<0. \quad (28)$$

By completing the squares in (28), it reduces to

$$[\rho^{-1}\Sigma_2^t\Theta+\rho\Sigma_1](I-\Theta^t\Theta)^{-1}[\rho^{-1}\Sigma_2^t\Theta+\rho\Sigma_1]^t+\rho^{-2}\Sigma_2^t\Sigma_2+\Sigma_3<0, \quad (29)$$

which is equivalent to (27) as required.

We are now in a position to provide the first basic result.

THEOREM 1 *For the uncertain system* (1), *the following statements are equivalent*:

(1) *The system satisfies the strongly robust* H_∞*-performance criterion.*
(2) $\forall\Delta(k)\in\Xi_a$, *there exists a matrix* $0<X=X^t\in\Re^{n\times n}$ *solving the*

Δ-dependent ARI

$$A^{t}(\Delta)XA(\Delta) - X + C^{t}(\Delta)C(\Delta) + [A^{t}(\Delta)XB(\Delta) + C^{t}(\Delta)D(\Delta)] \times R^{-1}(\Delta)[B^{t}(\Delta)XA(\Delta) + D^{t}(\Delta)C(\Delta)] < 0, \quad (30)$$
$$R(\Delta) = [I - D^{t}(\Delta)D(\Delta) - B^{t}(\Delta)XB(\Delta)].$$

(3) $\forall \Delta(k) \in \Xi_a$, *there exists a matrix* $0 < X = X^{t} \in \Re^{n \times n}$ *solving the Δ-dependent Lyapunov inequality (LI)*

$$\begin{bmatrix} A(\Delta) & B(\Delta) \\ C(\Delta) & D(\Delta) \end{bmatrix}^{t} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(\Delta) & B(\Delta) \\ C(\Delta) & D(\Delta) \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0. \quad (31)$$

(4) $\forall \Delta(k) \in \Xi_a$, *there exists a matrix* $0 < X = X^{t} \in \Re^{n \times n}$ *solving the Δ-dependent LMI*

$$W(\Delta) = \begin{bmatrix} \begin{bmatrix} -X^{-1} & 0 \\ 0 & -I \end{bmatrix} & \cdot & \begin{bmatrix} A(\Delta) & B(\Delta) \\ C(\Delta) & D(\Delta) \end{bmatrix} \\ \cdots\cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ \begin{bmatrix} A^{t}(\Delta) & C^{t}(\Delta) \\ B^{t}(\Delta) & D^{t}(\Delta) \end{bmatrix} & \cdot & \begin{bmatrix} -X & 0 \\ 0 & -I \end{bmatrix} \end{bmatrix} < 0. \quad (32)$$

Proof Since $\Xi_a \subset \Xi$, then (1) $\Rightarrow$ (2) follows directly. The implication (2) $\Rightarrow$ (3) can be readily obtained in line of Lemma 1 and Corollary 1. Using Corollary 1, the implication (2) $\Rightarrow$ (4) is easily derived. To show (2) $\Rightarrow$ (1), it suffices to recall that (30) is equivalent to (14) and in this way it is readily seen that $\max_{\Delta \in \Xi} \lambda_M[W(\Delta)] = \max_{\Delta \in \Xi} \lambda_M[W(\Delta)]$. This means that $W(\Delta) < 0$ $\forall \Delta \in \Xi$ *if and only if* $W(\Delta) < \forall \Delta \in \Xi_a$. Finally, by Corollary 2, the proof is completed.

5. MODEL PARAMETRIZATION

To build upon the nominal model and substitute for the effects of uncertainties, we introduce in the sequel a ρ-parameterized model of the form

$$x(k+1) = A_0 x(k) + [B_0 \quad \rho S] w_\rho(k), \quad x(0) = 0, \quad (33a)$$

$$z_\rho(k) = \begin{bmatrix} C_0 \\ \rho^{-1} M \end{bmatrix} x(k) + \begin{bmatrix} D_0 & \rho L \\ \rho^{-1} N & \Theta \end{bmatrix} w_\rho(k), \quad (33b)$$

where $w_\rho(k)$ is an $(q+\alpha)$-disturbance input vector from $\ell_2[0,\infty)$ and $z_\rho(k)$ is $(r+\beta)$-controlled output and ρ is a positive scaling parameter.

The second basic result is now established.

THEOREM 2 *System* (1) *is said to satisfy the* strongly robust H_∞-performance criterion *if and only if for some* $\rho>0$ *the* ρ*-parametrized system* (33) *satisfies the* strongly robust H_∞-performance criterion.

Proof By Lemma 2 and Corollary 1, system (33) satisfies the strongly robust H_∞ performance criterion if there exists a matrix $0<X=X^{\mathrm{t}}\in\Re^{n\times n}$ solving the LMI

$$\begin{bmatrix} -X^{-1} & 0 & 0 & \cdot & A_0 & B_0 & \rho S \\ 0 & -I & 0 & \cdot & C_0 & D_0 & \rho L \\ 0 & 0 & -I & \cdot & \rho^{-1}M & \rho^{-1}N & \Theta \\ \cdots & \cdots & \cdots & \cdot & \cdots & \cdots & \cdots \\ A_0^{\mathrm{t}} & C_0^{\mathrm{t}} & \rho^{-1}M^{\mathrm{t}} & \cdot & -X & 0 & 0 \\ B_0^{\mathrm{t}} & D_0^{\mathrm{t}} & \rho^{-1}N^{\mathrm{t}} & \cdot & 0 & -I & 0 \\ \rho S^{\mathrm{t}} & \rho L^{\mathrm{t}} & \Theta^{\mathrm{t}} & \cdot & 0 & 0 & -I \end{bmatrix} < 0. \tag{34}$$

Introducing the nonsingular permutation matrix $P\in\Re^{6\times 6}$ of the form

$$P=[e_1,e_2,e_5,e_3,e_4,e_6];\quad P^{\mathrm{t}}=P^{-1} \tag{35}$$

such that

$$P^{\mathrm{t}}W_\rho P=\begin{bmatrix} -X^{-1} & 0 & A_0 & \cdot & B_0 & 0 & \rho S \\ 0 & -I & C_0 & \cdot & D_0 & 0 & \rho L \\ A_0^{\mathrm{t}} & C_0^{\mathrm{t}} & -X & \cdot & 0 & \rho^{-1}M^{\mathrm{t}} & 0 \\ \cdots & \cdots & \cdots & \cdot & \cdots & \cdots & \cdots \\ B_0^{\mathrm{t}} & D_0^{\mathrm{t}} & 0 & \cdot & -I & \rho^{-1}N^{\mathrm{t}} & 0 \\ 0 & 0 & \rho^{-1}M & \cdot & \rho^{-1}N & -I & \Theta \\ \rho S^{\mathrm{t}} & \rho L^{\mathrm{t}} & 0 & \cdot & 0 & \Theta^{\mathrm{t}} & -I \end{bmatrix} < 0. \tag{36}$$

Using the block matrices

$$\Sigma_3=\begin{bmatrix} -X^{-1} & 0 & A_0 & B_0 \\ 0 & -I & C_0 & D_0 \\ A_0^{\mathrm{t}} & C_0^{\mathrm{t}} & -X & 0 \\ B_0^{\mathrm{t}} & D_0^{\mathrm{t}} & 0 & -I \end{bmatrix},\quad \Sigma_2=[0\quad 0\quad M\quad N],\quad \Sigma_1=\begin{bmatrix} S \\ L \\ 0 \\ 0 \end{bmatrix} \tag{37}$$

then it is easy to see that inequality (36) has the form

$$\begin{bmatrix} \rho^{-1}\Sigma_2^{\mathrm{t}} & \rho\Sigma_1 \end{bmatrix} \begin{bmatrix} I & -\Theta \\ -\Theta^{\mathrm{t}} & I \end{bmatrix}^{-1} \begin{bmatrix} \rho^{-1}\Sigma_2 \\ \rho\Sigma_1^{\mathrm{t}} \end{bmatrix} + \Sigma_3 < 0. \tag{38}$$

By Lemma 5, expression (38) corresponds to

$$\Sigma_3 + \Sigma_1\Delta(k)\Sigma_2 + \Sigma_2^{\mathrm{t}}\Delta(k)\Sigma_1^{\mathrm{t}} < 0, \tag{39a}$$

or equivalently

$$W(\Delta) = \begin{bmatrix} \begin{bmatrix} -X^{-1} & 0 \\ 0 & -I \end{bmatrix} & \cdot & \begin{bmatrix} A(\Delta) & B(\Delta) \\ C(\Delta) & D(\Delta) \end{bmatrix} \\ \cdots\cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ \begin{bmatrix} A^{\mathrm{t}}(\Delta) & C^{\mathrm{t}}(\Delta) \\ B^{\mathrm{t}}(\Delta) & D^{\mathrm{t}}(\Delta) \end{bmatrix} & \cdot & \begin{bmatrix} -X & 0 \\ 0 & -I \end{bmatrix} \end{bmatrix} < 0 \tag{39b}$$

for all uncertainties satisfying the structure (16) and therefore the proof is completed.

6. ROBUST CONTROL SYNTHESIS

Now we turn attention to the problem of robust feedback synthesis. Extending on system (1), we consider a class of uncertain systems of the form

$$\begin{aligned} x(k+1) &= A(\Delta)x(k) + B(\Delta)w(k) + E(\Delta)u(k), \qquad x(0) = 0, \\ z(k) &= C(\Delta)x(k) + D(\Delta)w(k) + F(\Delta)u(k), \\ y(k) &= Q(\Delta)x(k) + J(\Delta)w(k) + V(\Delta)u(k), \end{aligned} \tag{40}$$

where at time $k \in \Re_+$, $x(k) \in \Re^n$ is the state vector; $w(k) \in \Re^q$ is the disturbance input vector; $u(k) \in \Re^m$ is the control input vector; $z(k) \in \Re^r$ is the controlled output; $y(k) \in \Re^p$ is the measurement vector and $A(\Delta), B(\Delta), C(\Delta), D(\Delta), E(\Delta), F(\Delta), V(\Delta), Q(\Delta), J(\Delta)$ are continuous, bounded matrices of appropriate dimensions and their entries are functions of $\Delta(k)$; $\Delta(k) \in \Xi$ is (possibly) a time-varying uncertain matrix. The set Ξ is compact and the uncertainty matrices

are described by

$$\begin{bmatrix} A & B & E \\ C & D & F \\ Q & J & V \end{bmatrix} = \begin{bmatrix} A_0 & B_0 & E_0 \\ C_0 & D_0 & F_0 \\ Q_0 & J_0 & V_0 \end{bmatrix} + \begin{bmatrix} S \\ L \\ \Phi \end{bmatrix} \Delta(k) [M \quad N \quad \Pi], \quad (41a)$$

$$\Delta(k) = H(k)[I - \Theta H(k)]^{-1}; \quad H^{t}(k)H(k) \leq I, \ \Theta^{t}\Theta < I, \quad (41b)$$

where $A_0, B_0, C_0, D_0, E_0, F_0, Q_0, J_0, V_0$ are constant matrices of appropriate dimensions representing the nominal system and S, M, N, L, Φ, Π are constant matrices and $H(k)$ is unknown matrix . The pairs (A_0, E_0), (A_0, Q_0) are stabilizable and detectable, respectively.

6.1 Dynamic Controller

Now the problem of interest is to design a linear dynamic controller $u(k) = K(k)y(k)$ such that the controlled system satisfies the strongly robust H_∞-performance criterion. Proceeding to solve this problem, we consider the dynamic controller (of order σ and based on output feedback) has the following state-space realization

$$\begin{aligned} \zeta(k+1) &= A_c\zeta(k) + B_c y(k), \\ u(k) &= C_c\zeta(k). \end{aligned} \quad (42)$$

Following the development of Section 5, we introduce an ρ-parameterized model of the form:

$$\begin{aligned} x(k+1) &= A_0 x(k) + [B_0 \quad \rho S] w_\rho(k) + E_0 u(k), \qquad x(0) = 0, \\ z_\rho(k) &= \begin{bmatrix} C_0 \\ \rho^{-1}M \end{bmatrix} x(k) + \begin{bmatrix} D_0 & \rho L \\ \rho^{-1}N & \Theta \end{bmatrix} w_\rho(k) + \begin{bmatrix} F_0 \\ \rho^{-1}\Pi \end{bmatrix} u(k), \quad (43) \\ y(k) &= Q_0 x(k) + [J_0 \quad \rho\Phi] w_\rho(k) + V_0 u(k), \end{aligned}$$

where $w_\rho(k)$ is an $(q+\alpha)$-disturbance input vector from $\ell_2[0, \infty)$ and $z_\rho(k)$ is $(r+\beta)$-controlled output and ρ is a positive scaling parameter. We have the following basic result:

THEOREM 3 *The closed-loop uncertain system* (40)–(42) *satisfies the* strongly robust H_∞-performance criterion *if and only if for some*

$\rho > 0$ the ρ-parametrized system (43) *with the dynamic controller* (42) *satisfies the* strongly robust H_∞-performance criterion.

Proof Combining (42) and (43), we get the ρ-parameterised closed-loop system:

$$\begin{aligned}\begin{bmatrix} x(k+1) \\ \zeta(k+1) \end{bmatrix} &= \begin{bmatrix} A_0 & E_0 C_c \\ B_c Q_0 & A_c + B_c V_0 C_c \end{bmatrix} \begin{bmatrix} x(k) \\ \zeta(k) \end{bmatrix} + \begin{bmatrix} B_0 & \rho S \\ B_c J_0 & \rho B_c \Phi \end{bmatrix} w_\rho(k) \\ &= \widehat{A} \begin{bmatrix} x(k) \\ \zeta(k) \end{bmatrix} + \widehat{B} w_\rho(k), \\ z_\rho(k) &= \begin{bmatrix} C_0 & F_0 C_c \\ \rho^{-1} M & \rho^{-1} \Pi C_c \end{bmatrix} \begin{bmatrix} x(k) \\ \zeta(k) \end{bmatrix} + \begin{bmatrix} D_0 & \rho L \\ \rho^{-1} N & \Theta \end{bmatrix} w_\rho(k) \qquad (44) \\ &= \widehat{C} \begin{bmatrix} x(k) \\ \zeta(k) \end{bmatrix} + \widehat{D} w_\rho(k). \end{aligned}$$

By Lemma 2 and Corollary 1, system (44) satisfies the strongly robust H_∞ performance criterion if there exists a matrix $0 < Y = Y^t \in \Re^{(n+\sigma)\times(n+\sigma)}$ solving the LMI

$$\widehat{W} = \begin{bmatrix} \begin{bmatrix} -Y^{-1} & 0 \\ 0 & -I \end{bmatrix} & \cdot & \begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & \widehat{D} \end{bmatrix} \\ \cdots\cdots\cdots & \cdot & \cdots\cdots\cdots \\ \begin{bmatrix} \widehat{A}^t & \widehat{C}^t \\ \widehat{B}^t & \widehat{D}^t \end{bmatrix} & \cdot & \begin{bmatrix} -Y & 0 \\ 0 & -I \end{bmatrix} \end{bmatrix} < 0. \qquad (45)$$

Using the orthogonal matrix P

$$P = [e_1, e_2, e_3, e_7, e_4, e_5, e_6, e_8]; \quad P^t = P^{-1}, \qquad (46)$$

we get

$$P^t \widehat{W} P = \begin{bmatrix} -Y_1^{-1} & 0 & 0 & A_0 & E_0 C_c & B_0 & 0 & \rho S \\ 0 & -Y_2^{-1} & 0 & B_c Q_0 & A_0 + B_c Q_0 C_c & B_c J_0 & 0 & \rho B_c \Phi \\ 0 & 0 & -I & C_0 & F_0 C_c & D_0 & 0 & \rho L \\ A_0^t & Q_0^t B_0^t & C_0^t & -Y_t & 0 & 0 & \rho^{-1} M^t & 0 \\ C_c^t E_0^t & A_0^t + C_c^t V_0^t B_c^t & C_c^t F_0^t & 0 & -Y_2 & 0 & \rho^{-1} C_c^t \Pi^t & 0 \\ B_0^t & J_0^t B_c^t & D_0^t & 0 & 0 & -I & \rho^{-1} N^t & 0 \\ 0 & 0 & 0 & \rho^{-1} M & \rho^{-1} \Pi C_c & \rho^{-1} N & -I & \Theta \\ \rho S^t & \rho \Phi^t B_c^t & \rho L^t & 0 & 0 & 0 & \Theta^t & -I \end{bmatrix} < 0 \qquad (47)$$

where $Y=$ block diag$[Y_1\ Y_2]$. Using the block matrices

$$\Sigma_3 = \begin{bmatrix} -Y_1^{-1} & 0 & 0 & A_0 & E_0C_c & B_0 \\ 0 & -Y_2^{-1} & 0 & B_cQ_0 & A_0+B_cQ_0C_c & B_cJ_0 \\ 0 & 0 & -I & C_0 & F_0C_c & D_0 \\ A_0^t & Q_0^tB_0^t & C_0^t & -Y_1 & 0 & 0 \\ C_c^tE_0^t & A_0^t+C_c^tV_0^tB_c^t & C_c^tF_0^t & 0 & -Y_2 & 0 \\ B_0^t & J_0^tB_c^t & D_0^t & 0 & 0 & -I \end{bmatrix},$$

$$\Sigma_2^t = \begin{bmatrix} 0 \\ 0 \\ 0 \\ M^t \\ C_c^t\Pi^t \\ N^t \end{bmatrix}, \qquad \Sigma_1 = \begin{bmatrix} S \\ B_c\Phi \\ L \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{48}$$

then inequality (45) assumes the block form (24). By Lemma 5, this corresponds to $\Sigma_3 + \Sigma_1\Delta(t)\Sigma_2 + \Sigma_2^t\Delta^t(t)\Sigma_1^t < 0$, or equivalently

$$\widehat{W}(\Delta) = \begin{bmatrix} \begin{bmatrix} -Y^{-1} & 0 \\ 0 & -I \end{bmatrix} & \cdot & \begin{bmatrix} \widehat{A}(\Delta) & \widehat{B}(\Delta) \\ \widehat{C}(\Delta) & \widehat{D}(\Delta) \end{bmatrix} \\ \cdots\cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ \begin{bmatrix} \widehat{A}^t(\Delta) & \widehat{C}^t(\Delta) \\ \widehat{B}^t(\Delta) & D^t(\Delta) \end{bmatrix} & \cdot & \begin{bmatrix} -Y & 0 \\ 0 & -I \end{bmatrix} \end{bmatrix}$$

$\forall\Delta \in \Psi_a$ where

$$\widehat{A}(\Delta) = \begin{bmatrix} A(\Delta) & E(\Delta)C_c \\ B_cQ(\Delta) & A_c + B_cV(\Delta)C_c \end{bmatrix}, \qquad \widehat{B}(\Delta) = \begin{bmatrix} B(\Delta) \\ B_cJ(\Delta) \end{bmatrix}.$$
$$\widehat{C}(\Delta) = [\,C(\Delta) \quad F(\Delta)C_c\,]. \tag{50}$$

Alternatively, by combining (40)–(42) we obtain:

$$\begin{bmatrix} x(k+1) \\ \zeta(k+1) \end{bmatrix} = \begin{bmatrix} A(\Delta) & E(\Delta)C_c \\ B_cQ(\Delta) & A_c + B_cV(\Delta)C_c \end{bmatrix} \begin{bmatrix} x(k) \\ \zeta(k) \end{bmatrix} + \begin{bmatrix} B(\Delta) \\ B_cJ(\Delta) \end{bmatrix} w(k)$$
$$= \widehat{A}(\Delta) \begin{bmatrix} x(k) \\ \zeta(k) \end{bmatrix} + \widehat{B}(\Delta)w(k),$$

$$z(k) = [\,C(\Delta) \quad F(\Delta)C_c\,]\begin{bmatrix} x(k) \\ \zeta(k) \end{bmatrix} + D(\Delta)w(k)$$
$$= \widehat{C}\begin{bmatrix} x(k) \\ \zeta(k) \end{bmatrix} + D(\Delta)w(k). \tag{51}$$

It is readily evident that system (51) satisfies the strongly robust H_∞-performance criterion if and only if condition (47) is met $\forall \Delta \in \Xi_a$ and therefore the proof is completed.

6.2 Special Cases

Case 1 (Full State Measurement)

One important special case of system (40) is described by

$$\begin{aligned} x(k+1) &= A(\Delta)x(k) + B(\Delta)w(k) + E(\Delta)u(k), \qquad x(0) = 0, \\ z(k) &= C(\Delta)x(k) + D(\Delta)w(k) + F(\Delta)u(k), \\ y(k) &= x(k), \end{aligned} \tag{52}$$

for which we seek to design a state-feedback controller $u(k) = F_s x(k)$ that renders the closed-loop system

$$\begin{aligned} x(k+1) &= [A(\Delta) + E(\Delta)F_s]x(k) + B(\Delta)w(k), \qquad x(0) = 0, \\ z(k) &= [C(\Delta) + F(\Delta)F_s]x(k) + D(\Delta)w(k), \end{aligned} \tag{53}$$

satisfying the strongly robust H_∞-performance criterion. The basic result is summarized by the following theorem.

THEOREM 4 *The closed-loop uncertain system* (53) *satisfies the* strongly robust H_∞-performance criterion *if and only if there exists a state-feeback controller* $u(k) = F_s x(k)$ *and two matrices* $0 < Z = Z^t \in \Re^{n \times n}$, $\Omega \in \Re^{m \times n}$ *solving the* Δ*-dependent LMI*

$$W_s(\Delta) = \begin{bmatrix} \begin{bmatrix} -Z & 0 \\ 0 & -I \end{bmatrix} & \cdot & \begin{bmatrix} A(\Delta)Z + E(\Delta)\Omega & B(\Delta) \\ C(\Delta)Z + F(\Delta)\Omega & D(\Delta) \end{bmatrix} \\ \cdots\cdots\cdots & \cdot & \cdots\cdots\cdots \\ \begin{bmatrix} ZA^t(\Delta) + \Omega^t E^t(\Delta) & ZC^t(\Delta) + \Omega^t F^t(\Delta) \\ B^t(\Delta) & D^t(\Delta) \end{bmatrix} & \cdot & \begin{bmatrix} -Z & 0 \\ 0 & -I \end{bmatrix} \end{bmatrix} < 0 \tag{54}$$

$\forall \Delta \in \Xi_a$ *where* $F_s = \Omega Z^{-1}$.

Proof ($\Rightarrow$) By (3) of Theorem 1, system (53) satisfies the strongly robust H_∞-performance criterion if and only if there exists matrix $0 < X = X^t \in \Re^{n \times n}$ such that

$$\begin{bmatrix} A(\Delta)+E(\Delta)F_s & B(\Delta) \\ C(\Delta)+F(\Delta)F_s & D(\Delta) \end{bmatrix}^t \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(\Delta)+E(\Delta)F_s & B(\Delta) \\ C(\Delta)+F(\Delta)F_s & D(\Delta) \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0 \quad \forall \Delta \in \Xi_a. \tag{55}$$

Then it follows from (4) of Theorem 1 that

$$W_f(\Delta) = \begin{bmatrix} \begin{bmatrix} -X^{-1} & 0 \\ 0 & -I \end{bmatrix} & \cdots & \begin{bmatrix} A(\Delta)+E(\Delta)F_s & B(\Delta) \\ C(\Delta)+F(\Delta)F_s & D(\Delta) \end{bmatrix} \\ \cdots\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots \\ \begin{bmatrix} A(\Delta)+E(\Delta)F_s & B(\Delta) \\ C(\Delta)+F(\Delta)F_s & D(\Delta) \end{bmatrix}^t & \cdots & \begin{bmatrix} -X & 0 \\ 0 & -I \end{bmatrix} \end{bmatrix} < 0. \tag{56}$$

Premultiplying inequality (56) by $P^* \operatorname{diag}[I \quad I \quad X^{-1} \quad I]$ and post-multiplying the result by P^*, we arrive at

$$W_s(\Delta) = P^* W_f(\Delta) P^* = \begin{bmatrix} \begin{bmatrix} -X^{-1} & 0 \\ 0 & -I \end{bmatrix} & \cdots & \begin{bmatrix} A(\Delta)X^{-1}+E(\Delta)F_sX^{-1} & B(\Delta) \\ C(\Delta)X^{-1}+F(\Delta)F_sX^{-1} & D(\Delta) \end{bmatrix} \\ \cdots\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots \\ \begin{bmatrix} X^{-1}A^t(\Delta)+X^{-1}F_s^tE^t(\Delta) & X^{-1}C^t(\Delta)+X^{-1}F_s^tF^t(\Delta) \\ B^t(\Delta) & D^t(\Delta) \end{bmatrix} & \cdots & \begin{bmatrix} -X^{-1} & 0 \\ 0 & -I \end{bmatrix} \end{bmatrix} < 0 \quad \forall \Delta \in \Xi_a. \tag{57}$$

By defining $Z = X^{-1}$, $F_s = \Omega Z^{-1}$, it is clear that (57) corresponds to (54).

($\Leftarrow$) This can be obtained by reversing the foregoing procedure and using the state-feedback gain $F_s = \Omega Z^{-1}$.

Remark 5 It should be stressed that the state-feedback design eventually reduces to the search for two matrices $0 < Z = Z^t \in \Re^{n \times n}$, $\Omega \in \Re^{m \times n}$ such that linear matrix inequality (54) is satisfied. Observe that Δ appears affinely in $W_s(\Delta)$ and that $W_s(\Delta)$ is convex in both Z and Ω.

Remark 6 Note that solution of the LMI (54) can be attained by efficient and numerically-stable techniques including interior-point methods (Boyd *et al.*, 1994).

Remark 7 It is interesting to observe that the result of Theorem 4 could be derived as well by using the conclusion arrived at in Theorem 3 and setting $Q(\Delta)=I$, $J(\Delta)=0$, $V(\Delta)=0$, $A_c=0$, $B_c=0$ and $C_c=F_s$ into condition (49).

Case 2 (System and Input Uncertainties Only)

Another important special case of system (40) is described by

$$\begin{aligned} x(k+1) &= A(\Delta)x(k)+B(\Delta)w(k)+E_0u(k), \qquad x(0)=0,\\ z(k) &= C_0x(k)+F_0u(k),\\ y(k) &= x(k) \end{aligned} \tag{58}$$

for which we have the following result:

THEOREM 5 *For the uncertain system* (58), *there exists a state-feedback controller such that the resulting closed-loop system satisfies the* strongly robust H_∞ performance criterion *if and only if there exists a matrix* $0<Z=Z^t\in\Re^{n\times n}$ *solving the LMI*

$$\begin{bmatrix} -Z & \cdot & A_c^t(\Delta) & C_0^t & ZE_0\\ \cdots & \cdot & \cdots\cdots & \cdots\cdots & \cdots\\ A_c(\Delta) & \cdot & \Gamma^{-1} & 0 & 0\\ C_0 & \cdot & 0 & \Psi^{-1} & 0\\ E_0^tZ & \cdot & 0 & 0 & F_0^tF_0 \end{bmatrix} < 0 \quad \forall\Delta\in\Xi_a \tag{59}$$

where

$$\begin{aligned} A_c(\Delta) &= A(\Delta)-E_0(F_0^tF_0)^{-1}F_0^tC_0-E_0(F_0^tF_0)^{-1}E_0^tZ^{-1};\\ \Gamma &= Z[I-B(\Delta)B^t(\Delta)Z]^{-1}; \qquad \Psi = I-F_0(F_0^tF_0)^{-1}F_0^t. \end{aligned} \tag{60}$$

Moreover, the state-feedback controller has a constant gain of the form

$$F_s=-(F_0^tF_0)^{-1}[F_0^tC_0+E_0^tZ]. \tag{61}$$

Proof ($\Rightarrow$) By similarity to the proof of Theorem 4, there exists matrix $0<Z=Z^t\in\Re^{n\times n}$ such that

$$\begin{aligned} &(A(\Delta)+E_0F_s)^tZ(A(\Delta)+E_0F_s)-Z+(C_0+E_0F_s)^t(C_0+E_0F_s)\\ &\quad+(A(\Delta)+E_0F_s)^tZB(\Delta)[I-B^t(\Delta)ZB(\Delta)]^{-1}B^t(\Delta)Z(A(\Delta)+E_0F_s)\\ &\quad<0 \quad \forall\Delta\in\Xi_a. \end{aligned} \tag{62}$$

Rearranging (62), one obtains

$$A^{t}(\Delta)ZA(\Delta) - Z + C_0^{t}C_0 + A^{t}(\Delta)ZB(\Delta)[I - B^{t}(\Delta)ZB(\Delta)]^{-1}B^{t}(\Delta)ZA(\Delta)$$
$$+ F_s^{t}[E_0^{t}ZE_0 + F_0^{t}F_0 + E_0^{t}ZB(\Delta)[I - B^{t}(\Delta)ZB(\Delta)]^{-1}B^{t}(\Delta)ZE_0]F_s$$
$$+ F_s^{t}[E_0^{t}ZA(\Delta) + F_0^{t}C_0 + E_0^{t}ZB(\Delta)[I - B^{t}(\Delta)ZB(\Delta)]^{-1}B^{t}(\Delta)ZA(\Delta)$$
$$+ [A^{t}(\Delta)ZE_0 + C_0^{t}F_0 + A^{t}(\Delta)ZB(\Delta)[I - B^{t}(\Delta)ZB(\Delta)]^{-1}B^{t}(\Delta)ZE_0]F_s$$
$$< 0 \quad \forall\Delta \in \Xi_a. \tag{63}$$

Using the matrix identity in $Z + ZB(\Delta)[I - B^{t}(\Delta)ZB(\Delta)]B(\Delta)Z = Z[I - B(\Delta)B^{t}(\Delta)Z]^{-1}$ in (63) and manipulating with the aid of (60) and (61), we get:

$$A_c^{t}(\Delta)ZA_c(\Delta) - Z + C_0^{t}\Psi C_0 + ZE_0(F_0^{t}F_0)^{-1}E_0Z < 0; \quad \forall\Delta \in \Xi_a. \tag{64}$$

It is readily seen that (64) can be put into the form (59).

($\Leftarrow$) Follows easily by reversing the above procedure.

Remark 8 It should be emphasized that the controller form (61) could be derived as well from linear-quadratic theory by minimizing $\|z\|_2^2$.

COROLLARY 3 *For the class of systems described by*

$$\begin{aligned} x(k+1) &= A(\Delta)x(k) + B_0w(k) + E_0u(k), \qquad x(0) = 0, \\ z(k) &= C_0x(k), \\ y(k) &= x(k), \end{aligned} \tag{65}$$

it is easy to conclude from Theorem 5 that system (65) *satisfies the strongly robust H_∞-performance criterion if and only if there exists matrix $0 < Z = Z^{t} \in \Re^{n\times n}$ solving the LMI*

$$\begin{bmatrix} -Z & \cdot & C_0^{t} & A^{t}(\Delta) - \mu Z(E_0E_0^{t}) \\ \cdots & \cdot & \cdots & \cdots\cdots\cdots \\ C_0 & \cdot & I & 0 \\ A(\Delta) - \mu(E_0E_0^{t})Z & \cdot & 0 & \Gamma^{-1} \end{bmatrix} < 0 \quad \forall\Delta \in \Xi_a, \tag{66}$$

where the state-feedback controller has a constant gain of the form $-\mu E_0^t Z$.

CONCLUSIONS

In this paper, the problems of robust performance analysis and feedback control synthesis have been considered for a class of discrete-time systems with time-varying parameteric uncertainties. The uncertainties have been represented by linear matrix fractional model. By adopting the recent concept of strongly robust H_∞ performance criterion, several previous results have been systematically recovered. Then, new results have been developed. All of these results have been cast into linear matrix inequality (LMI) formalism. Synthesis of robust feedback controllers are carried out for several system models.

References

[1] Bahnasawi, A.A. and M.S. Mahmoud, (1989) *Control of Partially-Known Dynamical Systems*, Springer-Verlag, Berlin.
[2] Boyd, S., L. El-Ghaoui, E. Feron and V. Balakrishnan (1994) *Linear Matrix Inequalities in Systems and Control Theory*, Vol. 15 (Studies in Applied Mathematics), SIAM, Philadelphia.
[3] De Souza, C.E., M. Fu and L. Xie (1993) H_∞ analysis and synthesis of discrete-time systems with time-varying uncertainty, *IEEE Trans. Automatic Control*, **AC-38**, 459–462.
[4] Doyle, J.C., K. Glover, P.P. Khargonekar and B.C. Francis (1989) State-space solutions to standard H_2 and H_∞ control problems, *IEEE Trans. Automatic Control*, **AC-34**, 831–847.
[5] Galimidi, A.R. and B.R. Barmish (1986) The constrained Lyapunov problem and its application to robust output feedback stabilization, *IEEE Trans. Automatic Control*, **AC-31**, 411–419.
[6] Gu, K., X. Ye and N.K. Loh (1993) Design of H_∞ control of discrete-time systems with uncertainties, *Proc. 32nd IEEE Conference on Decision and Control*, USA, Vol. 1, pp. 39–44.
[7] Gu, K. (1994) H_∞ control of systems under norm bounded uncertainties in all system matrices, *IEEE Trans. Automatic Control*, **AC-39**, 1320–1322.
[8] Khargonekar, P.P., I.R. Petersen and K. Zhou (1990) Robust stabilization of uncertain linear systems: Quadratic stabilizability and H_∞ control theory, *IEEE Trans. Automatic Control*, **AC-35**, 356–361.
[9] Mahmoud, M.S. (1991) Computer Operated Systems Control, Marcel Dekker, New York.
[10] Petersen, I.R. (1989) Complete results for a class of state-feedback disturbance attenuation problems, *IEEE Trans. Automatic Control*, **AC-34**, 1196–1199.
[11] Petersen, I.R., B.D.O. Anderson and E.A. Jonchkeere (1991) A first principles solution to the non-singular H_∞ control problem, *Int. J. Robust and Nonlinear Control*, **1**, 171–185.

[12] Schmitendorf, W.E. (1988) Designing stabilizing controllers for uncertain systems using the Riccati equation approach, *IEEE Trans. Automatic Control*, **AC-33**, 376–379.
[13] Xie, L., M. Fu and C.E. de Souza (1992) H_∞ control and quadratic stability of systems with parameter uncertainty, *IEEE Trans. Automatic Control*, **AC-37**, 1253–1256.
[14] Zhou, K. and P.P. Khargonekar (1988) An algebraic Riccati equation approach to H_∞ optimization, *Systems and Control Letters*, **11**, 85–91.
[15] Zhou, K., P.P. Khargonekar, J. Stoustrup and H.H. Niemann (1995) Robust performance of systems with structured uncertainties in state space, *Automatica*, **31**, 249–255.
[16] Haddad, W.M. and D.S. Bernstein (1994) Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circles, and popov and their application to robust stability. Part II: Discrete-time theory, *Int. J. Robust and Nonlinear Control*, **4**, 249–265.