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# ACCOMMODATION OF ZERO-VALUED CHARACTERISTIC ROOTS IN THE CLASSICAL SOLUTION EXPRESSION FOR LINEAR, HOMOGENEOUS, CONSTANT-COEFFICIENT, DIFFERENCE EQUATIONS

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Linear, constant-coefficient difference equations play a central role in many areas of engineering, where cases involving *repeated zero-valued characteristic roots* are sometimes of particular interest. Unfortunately, the classical solution expression presented in the mathematical literature of difference equations is not valid for this latter case. In this paper we develop a unique generalization of the classical solution expression for linear, constant-coefficient, homogeneous difference equations that accommodates the most general case of repeated zero-valued characteristic roots, thereby "completing" the classical theory. A worked example is presented to illustrate our result.

KEYWORDS: Difference equations, discrete-time models, zero characteristic roots

## 1. INTRODUCTION

In many branches of engineering it is expedient to model a dynamical system/process in terms of its behavior or response at specified discrete-values (isolated point-values) of the relevant independent variable. For instance, the static deflection $y(x)$ of a structure at discrete spatial points $x = x_i$, $i = 1,2,3, \cdots$, along the length of the structure. Or, the "motions" $y(t)$ of a dynamical system evaluated at discrete-values $t = t_1, t_2, \cdots$ of the temporal independent variable $t = time$. Such models are called "discrete independent-variable models" (hereafter called discrete-variable models) and the appropriate mathematical tool for their development and analysis is the theory of *difference equations*. In particular, if the underlying dynamic system/process admits to linearization, and the *spacing* between the discrete-values of the independent variable is *constant*, the classical theory of linear difference equations is directly applicable to the development and analysis of such models.

Practical applications of linear, constant-coefficient difference equations in discrete-variable modeling sometimes lead to situations in which the difference equation model naturally turns-out to have one or more *zero-valued* characteristic roots. In such cases the influence of those zero-roots on the behavior of the model's analytical solution may be of particular interest. For example, in the area of linear discrete-time/digital-control system design, engineers often strive to achieve a "closed-loop" system whose linear, constant-coefficient difference-equation model *has all zero-valued characteristic roots*. This latter case yields a desirably "quick" response called "deadbeat response", [1], [2].

The effectiveness of classical linear difference equation theory in analyzing discrete-variable models in engineering has been hampered by the lack of a theoretical basis for accommodating the case of *repeated zero-valued* characteristic roots in conventional analytical solution expressions. In fact, presentations of classical difference-equation theory [3]–[11] typically ignore cases of zero-valued characteristic roots, or implicitly rule them out, by assuming (without explanation) that certain otherwise "arbitrary" coefficients in the difference equation are non-zero. Some texts define the "order" of a difference equation in such a way that *no* zero-valued characteristic roots can ever occur! (See Section 6 of this paper.)

In this paper we present a novel generalization of the classical solution expression for linear, constant-coefficient, homogeneous difference equations. Our generalized solution expression gracefully accommodates the most general case of repeated, zero-valued characteristic roots and thereby serves to "complete" the classical theory by eliminating the need to avoid or rule-out such cases. A numerical example is worked-out in detail to illustrate the ineffectiveness of the classical solution expression and our proposed method to overcome that defect.

## 2. THE CLASSICAL SOLUTION EXPRESSION FOR A LINEAR, CONSTANT-COEFFICIENT HOMOGENEOUS DIFFERENCE EQUATION

In this paper we are concerned with the analytical solution of arbitrary (real-valued) $n^{th}$-order linear, constant-coefficient, homogeneous difference equations of the general form

$$y((k+n)T) + \tilde{a}_n\, y((k+n-1)T) + \cdots + \tilde{a}_2\, y((k+1)T) + \tilde{a}_1\, y(kT) = 0 \qquad (1)$$

subject to the *n arbitrarily specified* "initial condition" (initial sequence) values $y(0) = y_o$; $y(T) = y_1$; $y(2T) = y_2$; $\cdots$; $y((n-1)T) = y_{n-1}$. This is recognized as the natural, difference-equation counterpart of the "initial-value problem" in differential equation theory. The specified constant $T > 0$ is the (uniform) "spacing" between the discrete-values $t = kT$, $k = 0,1,2, \cdots$, of the independent variable $t$, and the $\{\tilde{a}_i\}_1^n$ are given, arbitrary real-valued constants. In the interest of engineering applications, where the value of $T$ may be an important design parameter, we have elected to *not* invoke in (1) the normalization $T = 1.0$ as is customary in mathematical texts.

It is recalled that the conventional solution expression for (1) has two possible formats, depending on whether or not the characteristic polynomial of (1) has repeated (equal-valued) characteristic roots $\tilde{\lambda}_i$. If the n characteristic roots $\{\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_n\}$ associated with (1) are all distinct (non-repeated) the classical solution expression is [11; p. 153]

$$y(kT) = C_1(\tilde{\lambda}_1)^k + C_2(\tilde{\lambda}_2)^k + \cdots + C_n(\tilde{\lambda}_n)^k; \quad C_i = constant; \quad k = 0,1,2,\cdots \qquad (2)$$

On the other hand, *if*, say, two of the characteristic roots $(\tilde{\lambda}_r, \tilde{\lambda}_s)$ of (1) are each repeated $(m_r, m_s)$-times respectively, $\tilde{\lambda}_r \neq \tilde{\lambda}_s$, $m_r > 1$, $m_s > 1$, $m_r + m_s = m \leq n$, and the remaining roots $(\tilde{\lambda}_{m+1}, \cdots, \tilde{\lambda}_n)$ are distinct (non-repeated), the appropriate solution expression for (1) in that case is [11; p. 154]

$$y(kT) = \sum_{1}^{m_r} C_{ri}\, k^{(i-1)} (\tilde{\lambda}_r)^k + \sum_{1}^{m_s} C_{si}\, k^{(i-1)} (\tilde{\lambda}_s)^k + C_{m+1} (\tilde{\lambda}_{m+1})^k$$

$$+ \cdot \cdot \cdot + C_n (\tilde{\lambda}_n)^k ; \quad k = 0,1,2,\cdot \cdot \cdot \tag{3}$$

The obvious, natural generalization of (3) to include an "arbitrary" number of repeated roots $\tilde{\lambda}_i$, with "arbitrary" multiplicities $m_i$, constitutes the most general solution expression for (1) as traditionally presented in texts, [3]–[11]. It is with this understanding, and the desire to avoid any further notational complexity, that we hereafter treat (3) as representing the conventional general solution expression for (1).

## A Shortcoming in (2)

Expressions (2), (3) are presumably "weighted sums of $n$ linearly-independent particular solutions" as required for the general solution $y(kT)$ of the $n^{th}$-order linear, constant coefficient difference equation (1). The values of the weighting constants $\{C_1, \cdots, C_n\}$ in (2), (3) are to be chosen to satisfy the $n$ arbitrarily specified "initial-condition" values for the initial sequence $\{y(0), y(T), y(2T), \cdots, y((n-1)T)\}$. However, a mathematical shortcoming in (2), and an outright mathematical *defect* in (3), arises when (1) has *zero-valued* characteristic roots, $\tilde{\lambda}_i = 0$. In particular, if $\tilde{\lambda}_i = 0$ is a *distinct* characteristic root appearing in (2), the evaluation of (2) for $k = 0$ leads to the *indeterminate expression* $(0)^o$ appearing on the right side of (2). The "appropriate" value one should assign to the term $(0)^o$ in that case has, to this author's knowledge, never been explicitly addressed in any mathematics or engineering textbook. Of course, for engineers it is quite natural to infer, from the graph of $(\rho)^o$ vs. $\rho$, $\rho = real$, that

$$\lim_{\rho \to 0} (\rho)^o = 1, \quad \rho = real. \tag{4}$$

However, the proper evaluation of $(0)^o$, as it arises in (2) when $\tilde{\lambda}_i = 0$, seems to have been quietly disregarded in difference equation texts. It should be mentioned that Mayhan [12] is one engineering author that has recognized the *presence* of this mathematical shortcoming. But he too avoids explicitly mentioning the evaluation of $(0)^o$, in favor of a special brute-force method he advocates [12, pp. 100, 132].

## A Defect in (3)

When $\tilde{\lambda}_i = 0$ appears as a *repeated* characteristic root in (3), the evaluation of expression (3) for $k = 0$ not only involves the aforementioned indeterminate term $(0)^o$, but also terms like $(0)^o$, $0(0)^o$, $0^2\,(0)^o$, $0^3\,(0)^o$, etc. In that case, *regardless* of what fixed value one assigns to the term $(0)^o$, the corresponding set of particular solutions

$$\{(0)^k, k(0)^k, \cdots, k^{(m_i-1)}(0)^k\} \tag{5}$$

that then appears in (3) can *never* be linearly independent on $k = 0,1,2, \cdots$ . Consequently, expression (3) is an *invalid solution expression for all cases of repeated zero-valued characteristic roots* $\tilde{\lambda}_i$ that may arise in the linear, constant-coefficient difference equation (1). This situation would have little consequence if such cases did not *naturally* arise in applications of difference equations, as they do in fact.

## 3. RESOLUTION OF THE TECHNICAL DIFFICULTIES WITH (2),(3) WHEN $\tilde{\lambda}_i = 0$

Our proposed resolution of the "technical difficulties" with (2), (3), when zero-valued characteristic roots $\tilde{\lambda}_i$ naturally appear, is as follows. First, for the purpose of evaluating the term $(\tilde{\lambda}_i)^o$ in (2), (3) when $\tilde{\lambda}_i = 0$, agree to accept the special value[1]

$$(0)^o = 1 \tag{6}$$

based on the limit (4). Then, replace the highly dependent (hence ineffective) "repeated zero-root basis functions" (5) by the unorthodox, slightly mutated set of basis functions

$$\{(0)^k, (0)^{(k-1)^2}, (0)^{(k-2)^2}, \cdots, (0)^{(k-m)^2}\} \tag{7}$$

It is easily verified that, using the evaluation (6), our proposed new basis set (7), for repeated zero-roots, *is* a linearly independent set for all $k = 0,1,2, \cdots$ . Thus, using (7) our proposed "correction" to the inherent defect in (3), *for the case of repeated zero characteristic roots*, consists of re-writing (3) in the following alternative form (here we let $\tilde{\lambda}_r$ in (3) be the repeated zero-valued root, having $m_r > 1$-fold multiplicity and $\tilde{\lambda}_s \neq 0$ with multiplicity $m_s > 1$).

$$y(kT) = C_1(0)^k + C_2(0)^{(k-1)^2} + C_3(0)^{(k-2)^2} + \cdots + C_{m_r}(0)^{(k-m_r)^2}$$
$$+ \sum_1^{m_s} C_{si}\, k^{(i-1)} (\tilde{\lambda}_s)^k + C_{m+1} (\tilde{\lambda}_{m+1})^k + \cdots + C_n (\tilde{\lambda}_n)^k;$$
$$(m_r > 1, m_s > 1, k = 0,1,2,\cdots). \tag{8}$$

Note that, with the evaluation (6), the terms $(0)^{(k-i)^2}$, $i = 1,2, \cdots, m_r$, in (8) act as the classical Kronecker *delta function* $\delta(k - i)$. Moreover, the terms $(0)^{(k-i)^2}$ in (7) can be replaced by a variety of equivalent expressions such as $(0)^{|(k-i)|}$, $|(\cdot)|$ = *absolute value*, or by $(0)^{(k-i)^{2p}}$, $p = 1,2, \cdots$ , etc. without loss of linear independence.

Our generalized expression (8), for accommodating the case of repeated zero-valued characteristic roots in (3), apparently *cannot* be represented as a special case of the (otherwise correct) classical solution expression (3), using any conventional procedure. Thus the first $m_r$ terms in (8) constitute a "particular solution" that has the nature of a "singular-solution" [13], as sometimes encountered in *differential*-equations. This is a rather novel situation for linear difference equations because, as is well-known, *linear differential* equations *cannot* have singular solutions!

## 4. AN EXAMPLE OF THE ZERO-ROOT DEFECT IN (3) AND THE PROPOSED ACCOMMODATION (8)

To illustrate the defect in (3) and our proposed accommodation (8), when $\tilde{\lambda}= 0$ is a *repeated* characteristic root, consider the following specific 3rd-order linear difference equation

$$y((k+3)T) - 0.5\,y((k+2)T) = 0\,;\; k = 0,1,2,\cdots, \tag{9}$$

with the three arbitrarily *specified* "initial-condition" values

$$y(0) = y_o\,;\; y(T) = y_1\,;\; y(2T) = y_2 \tag{10}$$

The corresponding characteristic polynomial for (9) is

$$\tilde{\lambda}^3 - 0.5\tilde{\lambda}^2 = 0 \tag{11}$$

which clearly yields the three characteristic roots ($\tilde{\lambda} = 0$ is a twice-repeated root),

$$\tilde{\lambda}_1 = 0\,;\; \tilde{\lambda}_2 = 0\,;\; \tilde{\lambda}_3 = 0.5 \tag{12}$$

According to the traditional solution expression (3), the general solution $y(kT)$ of (9) would be written as

$$y(kT) = C_1(0)^k + C_2 k(0)^k + C_3(0.5)^k\,;\; k = 0,1,2,\cdots, \tag{13}$$

where the three constants $\{C_1, C_2, C_3\}$ in (13) are to be chosen to satisfy the three given "initial-condition" values (10). However, this latter step is *not possible* for (13), owing to the linear-dependence of the first two terms on the right-side of (13), and thus (13) *is not* a valid expression for the general (complementary) solution of (9).

According to our generalized solution expression (8), for accommodating repeated zero-valued roots, the correct general solution of (9) is given by the expression

$$y(kT) = C_1(0)^k + C_2(0)^{(k-1)^2} + C_3(0.5)^k,\; k = 0,1,2,\cdots \tag{14}$$

Note that the second-term in (14) is, structurally, quite different from its counterpart in (13).

It is readily verified that the three particular solutions in (14) *are* linearly independent, for all $k = 0,1,2, \cdots$, and that the values of the $C_i$, $i = 1,2,3$ in (14), needed to satisfy the three specified initial-conditions (10), can be easily computed to obtain

$$C_1 = y_o - 4y_2 \,;\, C_2 = y_1 - 2y_2 \,;\, C_3 = 4y_2 \tag{15}$$

## 5. THE REMARKABLE GENERALITY OF THE GENERATING-FUNCTION METHOD

As we have already indicated, the solution expressions (2), (3) constitute the typical results found in the contemporary difference-equation literature [3]–[11], even though (3) fails to give the correct answer when repeated, zero-valued characteristic roots occur. Thus, it is surprising that there exists a very old and well-developed mathematical technique, for deriving analytical solutions of difference-equations, that does *not* experience difficulty when zero-valued characteristic roots occur in (1). In particular, as an alternative to using our generalized solution expression (8), the "correct" solution of (1), for *arbitrary* cases of distinct or repeated, zero-valued characteristic roots, can be obtained by straightforward application of the "Generating-Function Method" [11], [14], introduced over 200 years ago by DeMoivre and Laplace, [15], [16] for solving linear difference equations. In the engineering literature the latter method, with a simple reciprocal change of variable ($\tau \rightarrow 1/z$), is used to derive operator expressions called discrete-time "transfer-functions" and is often credited to more recent researchers (circa 1950) under the name "Z-Transform Method." In light of the genesis remarks in [17] and in the seminal paper [18; p. 226], there is little excuse for continuing the perpetuation of this misplaced credit and alias name for the DeMoivre/Laplace Generating-Function Method, in the engineering literature.

## 6. COMMENT ON THE DEFINITION OF "ORDER" FOR A DIFFERENCE EQUATION

The ineffectiveness of (3), for cases of repeated *zero*-valued characteristic roots, is obscured in some mathematical texts [6], [11] by *defining* the "order" of a linear difference equation of the form (1) as the *difference* between the highest and lowest-order discrete-valued arguments (modulo $T$) appearing in the dependent-variable terms. This ingenious definition makes it *impossible* for any linear difference equation (1) to have a zero-valued characteristic root (distinct or repeated)! For instance, by that definition our example (9) would be *defined* as a *first-order* [$= ((k + 3)T - (k + 2)T)/T$] difference equation with only *one* initial-condition, rather than a third-order equation with *three* independent initial conditions (10). Such artistic license in defining "order" is perhaps appropriate at some levels of mathematical abstraction, but is *totally unacceptable* in engineering applications of difference equations for the purpose of modeling real-life dynamical systems over pre-specified ranges of the independent variable. In such applications the dynamical "order" of the system is a natural, intrinsic property that has important *physical* meaning (number of independent initial-conditions) that *must not be* *altered or obscured* by the difference-equation modeling procedure. For instance, an inherently third-order dynamic system that is modeled and treated as a first-order system can lead to potentially dangerous situations, in terms of stability considerations.

## 7. SUMMARY

Difference equations are the natural tool for developing and analyzing mathematical models of "discrete independent-variable" type systems in engineering. However, in those applications the practical utility of the classical solution expression in linear, constant-coefficient difference equation theory has been hampered by the fact that the classical solution expression *does not* yield correct answers when the difference equation model has *repeated, zero-valued* characteristic roots.

In this paper we have introduced a generalization of the classical solution expression that accommodates the most general case of repeated, zero-valued characteristic roots. This generalization involves the introduction of a "singular-type" particular solution for repeated zero-valued roots, and serves to "complete" the classical theory, in the sense that cases involving *zero-valued, repeated roots* need no longer be implicitly (or explicitly) ruled-out as they traditionally have been. A worked numerical example has been presented to illustrate the results.

Some generalizations of the solution expressions used here are presented in [19].

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## *Endnotes*

1. The fixed evaluation (6), of the normally "indeterminate" expression $(0)^o$, is valid in the present context only because of the unique structure of (4) (fixed power of zero) as it arises in (2), (3) when $\tilde{\lambda}_i \rightarrow 0$, for some $i$. A similar circumstance arises in evaluating particular terms in some compact expressions for power-series.