

Research Article

Mixed H_2/H_∞ Performance Analysis and State-Feedback Control Design for Networked Systems with Fading Communication Channels

Adrian-Mihail Stoica

Faculty of Aerospace Engineering, University Politehnica of Bucharest, 01106 Bucharest, Romania

Correspondence should be addressed to Adrian-Mihail Stoica, adrian.stoica@upb.ro

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The purpose of the present paper is to provide a performance analysis approach of networked systems with fading communication channels. For a Ricean model of the fading communication channel, it is shown that the resulting system has a hybrid structure including the continuous-time dynamics of the networked systems and a discrete-time dynamics of the communication channels. Moreover, this resulting hybrid system has both multiplicative and additive noise terms. The performance analysis naturally leads to an H_2/H_∞ -type norm evaluation for systems with finite jumps and multiplicative noise. It is proved that this norm depends on the stabilizing solution of a specific system of coupled Riccati's equations with jumps. A state-feedback design problem to accomplish a mixed H_2/H_∞ performance is also considered. A numerical iterative procedure allowing to compute the stabilizing solution of the Riccati-type system with jumps is presented. The theoretical results are illustrated by numerical results concerning the tracking performances of a flight formation with fading communication channel. The paper ends with some concluding remarks.

1. Introduction

The analysis and synthesis of networked control systems have received a major attention over the last decade due to their wide area of applications (see, e.g., [1–3] and their references). These applications include aerial and terrestrial surveillance [3], formation atmospheric flight [4], terrain mapping, and satellites formations for space science missions [5]. In all these applications, the formation members are autonomous vehicles from which the human pilots have been removed in order to avoid their participation at dangerous and repetitive tasks. The specific feature of such networked systems is that the control loop is closed through a communication channel shared by all autonomous vehicles. This communication

network is required since the control law of each formation member usually depends on the measurements from all other vehicles. Even in the distributed control architectures when each vehicle uses only the information about its neighboring vehicles, a communication network is very useful. Indeed, in [6] it is proved that in a *predecessor following* approach, the relative positioning error between vehicles is amplified if the members of the formation have no information about the leader position. This conclusion motivates the use of a *predecessor and leader following* method in which both information about the predecessor and about the leader position are available for each formation member. In [6] it is also proved that in this case the relative spacing errors can be attenuated. These interesting results emphasize the importance of the communication in networked control systems. Most of the communication systems are based on wireless networks which, in contrast with the wired systems, are much more sensitive to information transmission errors.

The main goal of this paper is to analyze the interaction between the control and the communication system with fading. To this end, a model of the fading communication channel is required. There are many such models developed in the recent literature (see, e.g., [7, 8]). Deterministic models of communication networks with time-varying delays are considered in [9–11], in which the maximum admissible delays are determined using the Lyapunov stability theory. In other deterministic models of fading communication channels, the transmission errors are represented as uncertain parameters, and, the control system is designed via specific robust synthesis procedures including linear quadratic Gaussian (LQG) and μ -synthesis. Another class of representations of fading communication channels is based on stochastic models either with Markovian jumps or with white noise [7, 12]. Many useful results concerning the stability, control, and disturbance attenuation of such systems are available in the control literature (see, e.g., [13–15] and their references). In [8], an H_∞ -type design is used to determine a controller for a system with fading communication channel represented as a Markovian system. A Markovian representation of the network status is also used to solve robust fault detection problems by H_∞ techniques for communication systems which may be found in [16]. An extended version for the case of random measurement delays and stochastic data-missing phenomenon is treated in [17]. In the present paper, a discrete-time Ricean model of the communication channel is considered. The stochastic Rice models are often used for models of wireless links [7]. They include both additive and multiplicative white noise terms. The problem analyzed in this paper is the influence of the fading communication channel over the tracking performance of a flight formation. The control system of the flight formation is the one derived in [4]. Since the exogenous inputs in the networked system are both deterministic (the reference signals for the formation control) and stochastic (the white noise terms in the fading communication channel model), a mixed H_2/H_∞ -type approach is appropriate for this analysis. Moreover, the networked system has a hybrid structure due to its continuous-time component represented by the vehicles dynamics and a discrete-time one corresponding to the communication channel. The above mentioned considerations lead to a mixed H_2/H_∞ analysis problem for stochastic systems with finite jumps. The systems with finite jumps are used to represent dynamic systems with continuous-time and discrete-time components. Useful results and developments concerning these systems may be found, for instance, in [18, 19]. The paper provides an analysis and an optimization approach of the mixed H_2/H_∞ performance for a hybrid model of networked systems with fading communication channels. This model is derived in Section 2 of the paper. In Section 3, the expression of the H_2/H_∞ performance is determined in terms of the stabilizing solution of a specific system of coupled Riccati equations with finite jumps. A state-feed-

back design procedure to optimize the mixed performance is presented in Section 4. Numerical aspects concerning the computation of the stabilizing solution are given in Section 5. The theoretical results are illustrated by a numerical example concerning the tracking performance of an aircraft formation. The paper ends with some final remarks and future work.

2. A Model of Networked Systems over Fading Communication Channel

In this section a control problem of a formation of unmanned air vehicles (UAVs) will be briefly presented. Such problems have been intensively analyzed over the last fifteen years (see, e.g., [3]) both for their wide area of applications and for the challenges addressed to the control engineer. In [4], a dynamic inversion-type approach is used to linearize the nonlinear dynamic and kinematic equations of the UAV motion. A simplified linearized model of a flight formation member has the form:

$$\begin{aligned}\dot{\delta} &= Y\dot{\xi}, \\ \dot{\xi} &= -K_d\delta - K_x\xi - \ddot{\bar{x}},\end{aligned}\tag{2.1}$$

where $\delta \in \mathbb{R}^3$ denotes the deviation of the aircraft with respect to its desired position and $\xi \in \mathbb{R}^3$ stands for the deviation of its state $x = [V \ \psi \ \gamma]^T$ (V representing the airspeed, ψ the heading angle, and γ the flight path angle) with respect to some specified value \bar{x} . The input vector $\ddot{\bar{x}}$ includes the desired derivatives of \bar{x} and plays the role of a reference signal in the model (2.1). The constant matrix Y has the diagonal form $Y = \text{diag}(1, V_0, V_0)$, and the state-feedback control gains K_d and K_x are diagonal, too. In the present paper, the case when the reference signal $\ddot{\bar{x}}$ is transmitted from the ground station or from the formation leader using a fading communication channel is considered. The aim is to analyze how the tracking performances of the flight formation are altered due to the communication system. To this end, a model of the fading communication channel is required. In this paper, the L th-order Rice model was adopted. This model is frequently used in wireless mobile links, and it is given by the discrete-time equation:

$$r(i) = \sum_{k=0}^L a_k(i)v(i-k) + n(i),\tag{2.2}$$

where i denotes the moment of time, $v(\cdot)$ denotes the transmitted information, $r(\cdot)$ is the received information, n is a Gaussian white noise with zero mean and unit variance, and $a_k(i)$, $k = 0, \dots, L$ are independent random variables with known mean \bar{a}_k and variance σ_k^2 . In the case of the application considered in this paper, $v(\cdot)$ is just the transmitted reference signal $\ddot{\bar{x}}$. A state-space representation of (2.2) is

$$\begin{aligned}p(i+1) &= Mp(i) + Nv(i), \\ r(i) &= \sum_{k=1}^L a_k(i)P_k p(i) + a_0(i)v(i) + n(i), \quad i = 0, 1, \dots,\end{aligned}\tag{2.3}$$

where, by definition,

$$M = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & & \\ 0 & \cdots & I & 0 \end{bmatrix}, \quad N = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad P_k = [0 \ \cdots \ 0 \ I \ 0 \ \cdots \ 0], \quad (2.4)$$

and $p(i) \in \mathbb{R}^{L \cdot n_v \times 1}$ stands for the state vector of the communication channel, n_v denoting the dimension of the transmitted information vector $v(\cdot)$. In (2.4), the identity and the zero matrices have the size $n_v \times n_v$, and the identity matrix in P_k is on the k th position. The configuration of the communication system (2.3) coupled with the system (2.1) is illustrated in Figure 1. The resulting system from Figure 1 is in fact a hybrid system since the dynamics of (2.1) is a continuous-time one, and (2.3) is a discrete-time system. A state-space realization of such hybrid system can be given using systems with finite jumps of the general form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), & t \neq ih, \\ x(ih^+) &= A_d x(ih) + B_d w_d(i), & i = 0, 1, \dots, \end{aligned} \quad (2.5)$$

in which $h > 0$ denotes the sampling period, the state $x(t)$ is left continuous and right discontinuous at the sampling moments $t = ih$, $i = 0, 1, \dots$, and $w(t)$, $t \neq ih$, and $w_d(i)$, $i = 0, 1, \dots$, are the continuous-time and the discrete-time inputs, respectively, of the system (see, e.g., [19, 20]). Since the received information r is constant between the sampling moments, it can be represented as

$$\begin{aligned} \dot{r}(t) &= 0, & t \neq ih, \\ r(ih^+) &= \sum_{k=1}^L a_k(i) P_k p(ih) + a_0(i) v(i) + n(i), & i = 0, 1, \dots, \end{aligned} \quad (2.6)$$

where the state p of the communication system is given by

$$\begin{aligned} \dot{p}(t) &= 0, & t \neq ih, \\ p(ih^+) &= Mp(ih) + Nv(i), & i = 0, 1, \dots \end{aligned} \quad (2.7)$$

A similar model can be adopted for the continuous-time control system (2.1) which can be represented as

$$\dot{q}(t) = \bar{A}q(t) + \bar{B}r(t), \quad t \geq 0, \quad (2.8)$$

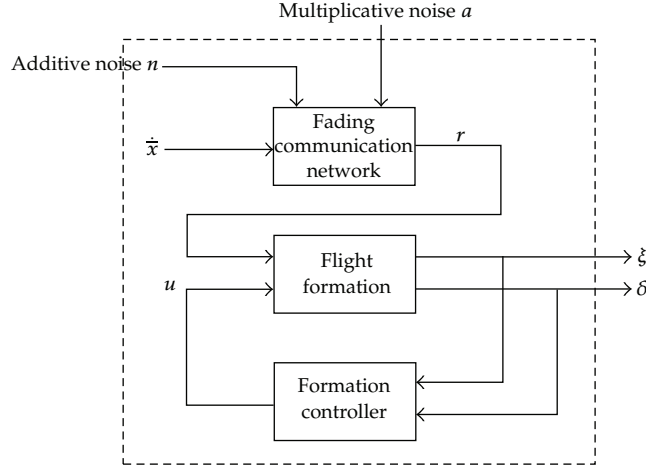


Figure 1: Control configuration of networked systems with fading communication channel.

where

$$q(t) := \begin{bmatrix} \delta^T(t) \xi^T(t) \end{bmatrix}^T, \quad (2.9)$$

$$\bar{A} := \begin{bmatrix} 0 & Y \\ -K_d & -K_x \end{bmatrix}, \quad \bar{B} := \begin{bmatrix} 0 \\ -I \end{bmatrix}.$$

In (2.8) the state $q(t)$ is continuous-time, and therefore,

$$q(ih^+) = q(ih), \quad (2.10)$$

and $r(t)$ is the received perturbed reference signal described by (2.6). From (2.6)–(2.10), it follows that the hybrid networked system with fading communication channels can be represented using a model with finite jumps of the form (2.5) where $x = [q^T \ r^T \ p^T]^T$. Moreover, since in (2.6) the coefficients $a_k(i)$, $k = 0, \dots, L$ and $i = 0, 1, \dots$, are random variables, one may consider the following stochastic version of (2.5) which includes both multiplicative noise components and additive white noise terms:

$$\begin{aligned} dx(t) &= (A_0x(t) + B_0w(t))dt + (A_1x(t) + B_1w(t))dv(t) + Gd\eta(t), & t \neq ih, \\ x(ih^+) &= A_{0d}x(ih) + B_{0d}w_d(i) + (A_{1d}x(ih) + B_{1d}w_d(i))v_d(i) + G_d\eta_d(i), & i = 0, 1, \dots, \\ y(t) &= Cx(t), & t \neq ih, \\ y_d(i) &= C_d x(ih), & i = 0, 1, \dots, \end{aligned} \quad (2.11)$$

where the random variables $\eta(t) \in \mathbb{R}$, $t \geq 0$, and $v(t) \in \mathbb{R}^r$, $t \geq 0$, are such that the pair $(\eta(t), v(t))$ is an $r + 1$ -dimensional standard Wiener process, and $v_d(i) \in \mathbb{R}$ and

$\eta_d(i) \in \mathbb{R}^{r_d}$, $i = 0, 1, \dots$ are sequences of independent random variables on a probability space $(\Omega, \mathcal{P}, \mathcal{F})$. It is assumed that $v(t), \eta(t)$, $t \geq 0$, $v_d(i), \eta_d(i)$, $i = 0, 1, \dots$, are independent stochastic processes with zero mean and unitary second moments. The outputs $y(t)$ and $y_d(i)$ denote the continuous-time and the discrete-time outputs, respectively. By virtue of standard results from the theory of stochastic differential equations (see, e.g., [21]), the system (2.11) has a unique \mathcal{F}_t -adapted solution for any initial condition $x(0)$, \mathcal{F}_t denoting the σ -algebra generated by the random vectors $v(s)$, $\eta(s)$, $v_d(i)$, and $\eta_d(i)$, $0 \leq s \leq t$, $0 \leq ih \leq t$. This solution is almost surely left continuous.

A mixed H_2/H_∞ problem for this class of stochastic systems with jumps will be treated in the next section.

3. Mixed H_2/H_∞ -Type Norm for Systems with Jumps Corrupted with Multiplicative Noise

Before defining and computing the mixed H_2/H_∞ norm for systems of the form (2.11), some useful definitions and preliminary results will be briefly presented.

3.1. Notations, Definitions, and Some Useful Results

Consider the stochastic system with jumps (2.11) in which w and w_d denote continuous-time and discrete-time energy bounded inputs, respectively. It means that $w \in L^2[0, \infty)$, where $L^2[0, \infty)$ denotes the space of the functions $f(t)$, $t \geq 0$ for which $\|f\|_{L^2}^2 := \int_0^\infty |f(t)|^2 dt < \infty$, and $w_d \in \ell^2$ where ℓ^2 is the space of the discrete-time vectors $g(i)$, $i = 0, 1, \dots$ with the property $\|g\|_{\ell^2}^2 := \sum_{i=0}^\infty |g(i)|^2 < \infty$, where $|\cdot|$ stands for the Euclidian norm.

Definition 3.1. The stochastic system with jumps and with multiplicative noise

$$\begin{aligned} dx(t) &= A_0 x(t) dt + A_1 x(t) dv(t), & t \neq ih, \\ x(ih^+) &= A_{0d} x(ih) + A_{1d} x(ih) v_d(i) & i = 0, 1, \dots \end{aligned} \quad (3.1)$$

is *exponentially stable in mean square* (ESMS) if there exist $\alpha > 0$ and $\beta \geq 1$ such that $E[|x(t)|^2] \leq \beta e^{-\alpha t} |x(0)|^2$ for any initial condition $x(0)$ and for all $t \geq 0$, $E[\cdot]$ denoting the mean of the random variable and $x(t)$ representing the solution of (3.1) with the initial condition $x(0)$.

The following result gives necessary and sufficient conditions in which the system with finite jumps (3.1) is ESMS, and its proof may be found in [13].

Proposition 3.2. *The system (3.1) is ESMS if and only if the system of coupled Lyapunov equations*

$$\begin{aligned} -\dot{X}(t) &= A_0^T X(t) + X(t) A_0 + A_1^T X(t) A_1, & t \neq ih, \\ X(ih^-) &= A_{0d}^T X(ih) A_{0d} + A_{1d}^T X(ih) A_{1d}, & i = 0, 1, \dots \end{aligned} \quad (3.2)$$

has a unique symmetric solution $X(t) \geq 0$, $t \geq 0$, right continuous and h -periodic.

Another useful result is the differentiation rule of functions of solutions to stochastic differential equations, well known in the literature as Itô's formula [21].

Proposition 3.3. Let $v(t, x)$ be a continuous function with respect to $(t, x) \in [0, T] \times \mathbb{R}^n$. If $x(t)$ is a solution of the stochastic differential equation

$$dx(t) = a(t)dt + b(t)d\beta(t), \quad (3.3)$$

then

$$\begin{aligned} dv(t, x(t)) = & \left[\frac{\partial v}{\partial t}(t, x(t)) + \left(\frac{\partial v}{\partial x}(t, x(t)) \right)^T a(t) + \frac{1}{2} \text{Tr} b^T(t) \frac{\partial^2 v(t)}{\partial x^2}(t, x(t)) b(t) \right] dt \\ & + \left(\frac{\partial v}{\partial x}(t, x(t)) \right)^T b(t) d\beta(t), \end{aligned} \quad (3.4)$$

where $\text{Tr}(\cdot)$ denotes the trace of the matrix (\cdot) .

The next result will be used in the following sections, and its proof may be found in [13, page 162].

Proposition 3.4. Consider the stochastic system with multiplicative noise

$$dx(t) = (A_0x(t) + B_0u(t))dt + (A_1x(t) + B_1u(t))dv(t) \quad (3.5)$$

and the cost function

$$J(t_0, \tau, u) = E \left[\int_{t_0}^{\tau} \begin{bmatrix} x^T(t) & u^T(t) \end{bmatrix} \begin{bmatrix} M & L \\ L^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \right], \quad (3.6)$$

then

$$\begin{aligned} J(t_0, \tau, u) = & x_0^T X(t_0) x_0 - E \left[x^T(\tau) X(\tau) x(\tau) \right] \\ & + E \left[\int_{t_0}^{\tau} \left(u(t) - \tilde{F}(t)x(t) \right)^T \left(R + B_1^T X(t) B_1 \right) \left(u(t) - \tilde{F}(t)x(t) \right) dt \right], \end{aligned} \quad (3.7)$$

where $X(t)$ verifies the equation

$$\begin{aligned} -\dot{X}(t) = & A_0^T X(t) + X(t) A_0 + A_1^T X(t) A_1 - \left(X(t) B_0 + A_1^T X(t) B_1 \right) \\ & \times \left(R + B_1^T X(t) B_1 \right)^{-1} \left(B_0^T X(t) + B_1^T X(t) A_1 \right) + M \end{aligned} \quad (3.8)$$

and where $\tilde{F}(t) = -(R + B_1^T X(t) B_1)^{-1} (B_0^T X(t) + B_1^T X(t) A_1 + L^T)$.

3.2. The Mixed H_2/H_∞ Norm of the System (2.11)

Assume that the system (2.11) which will be denoted below by \mathcal{G} is ESMS and that $x(0) = 0$. As in the deterministic case, H_2 and H_∞ norms can be defined as follows.

(i) For $w(t) \equiv 0$ and $w_d(i) \equiv 0$, the *impulse-to-energy gain* induced from (η, η_d) to (y, y_d) stands for the H_2 -type norm of the system (2.11). The H_2 -type norm of (2.11) denoted by $\|\mathcal{G}\|_2$ can be determined as

$$\|\mathcal{G}\|_2^2 = \text{Tr}\left(G_d^T Q(ih) G_d\right) + \frac{1}{h} \int_0^h \text{Tr}\left(G^T Q(t) G\right) dt, \quad (3.9)$$

where $Q(t)$, $t \geq 0$ is the solution of the Lyapunov-type system

$$\begin{aligned} -\dot{Q}(t) &= A_0^T Q(t) + Q(t) A_0 + A_1^T Q(t) A_1 + C^T C, \quad t \neq ih, \\ Q(ih^-) &= A_{0d}^T Q(ih) A_{0d} + A_{1d}^T Q(ih) A_{1d} + C_d^T C_d, \quad i = 0, 1, \dots, \end{aligned} \quad (3.10)$$

(see also [18]).

(ii) For $\eta(t) \equiv 0$ and $\eta_d(i) \equiv 0$, the *energy-to-energy gain* induced from (w, w_d) to (y, y_d) stands for the H_∞ -type norm of the system (2.11), denoted by $\|\mathcal{G}\|_\infty$. It represents the smallest $\gamma > 0$ for which the following system of coupled Riccati equations

$$\begin{aligned} -\dot{X}(t) &= A_0^T X(t) + X(t) A_0 + A_1^T X(t) A_1 + C^T C + \left(X(t) B_0 + A_1^T X(t) B_1\right) \\ &\quad \times \left(\gamma^2 I - B_1^T X(t) B_1\right)^{-1} \left(B_0^T X(t) + B_1^T X(t) A_1\right), \quad t \neq ih, \\ X(ih^-) &= A_{0d}^T X(ih) A_{0d} + A_{1d}^T X(ih) A_{1d} + C_d^T C_d + \left(A_{0d}^T X(ih) B_{0d} + A_{1d}^T X(ih) B_{1d}\right) \\ &\quad \times \left(\gamma^2 I - B_{1d}^T X(ih) B_{1d}\right)^{-1} \left(B_{0d}^T X(ih) A_{0d} + B_{1d}^T X(ih) A_{1d}\right), \quad i = 0, 1, \dots \end{aligned} \quad (3.11)$$

has a *stabilizing solution* $X(t) \geq 0$, $t \geq 0$. Recall that a symmetric right continuous, h -periodic function $X(t)$ verifying (3.11) is called a stabilizing solution of (3.11) if

$$\begin{aligned} \gamma^2 I - B_1^T X(t) B_1 &> 0, \quad t \neq ih, \\ \gamma^2 I - B_{1d}^T X(ih) B_{1d} &> 0, \quad i = 0, 1, \dots, \end{aligned} \quad (3.12)$$

and the system with jumps

$$\begin{aligned} dx(t) &= (A_0 + B_0 F(t))x(t)dt + (A_1 + B_1 F(t))x(t)dv(t), \quad t \neq ih, \\ x(ih^+) &= (A_{0d}x(ih) + B_{0d}F(ih))x(ih) + (A_{1d} + B_{1d}F(ih))x(ih)v_d(i), \quad i = 0, 1, \dots \end{aligned} \quad (3.13)$$

is ESMS, where, by definition,

$$\begin{aligned} F(t) &= \left(\gamma^2 I - B_1^T X(t) B_1 \right)^{-1} \left(B_0^T X(t) + B_1^T X(t) A_1 \right), \quad t \neq ih, \\ F(ih) &= \left(\gamma^2 I - B_{1d}^T X(ih) B_{1d} \right)^{-1} \left(B_{0d}^T X(ih) A_{0d} + B_{1d}^T X(ih) A_{1d} \right), \quad i = 0, 1, \dots \end{aligned} \quad (3.14)$$

Similarly with the deterministic case (see, e.g., [22, 23]), a mixed H_2/H_∞ -type norm of (2.11) can be defined solving the optimization problem:

$$J_0 = \sup_{(w, w_d, \eta, \eta_d)} E \left[\|y\|_{L^2}^2 + \|y_d\|_{\ell^2}^2 - \gamma^2 \left(\|w\|_{L^2}^2 + \|w_d\|_{\ell^2}^2 \right) \right], \quad (3.15)$$

where $(w, w_d) \in L^2[0, \infty) \times \ell^2$, the white-noise-type random inputs η and η_d are as in previous subsection and $\gamma > \|G\|_\infty$ with $\|G\|_\infty$. Notice that, if w and w_d are null in (3.15), then J_0 gives the square of the H_2 -type norm induced by the random inputs η and η_d . The main result of this subsection is the following theorem.

Theorem 3.5. *The optimum J_0 defined in (3.15) is given by*

$$J_0 = \text{Tr} \left(G_d^T X(h) G_d \right) + \frac{1}{h} \int_0^h \text{Tr} \left(G^T X(t) G \right) dt, \quad (3.16)$$

where $X(t)$ is the stabilizing solution of the system of coupled Riccati equations (3.11).

Proof. The proof follows applying Itô's formula (Proposition 3.3) for the function $v(t, x) = x^T(t) X(t) x(t)$ with $x(t)$ being the solution of (2.11) and with $X(t)$ the stabilizing solution to the system (3.11). Thus, by direct computations, one obtains

$$\begin{aligned} d \left(x(t)^T X(t) x(t) \right) &= \left[-\rho_c(t) - y^T(t) y(t) + \gamma^2 w^T(t) w(t) + \text{Tr} \left(G^T X(t) G \right) \right] dt \\ &\quad + 2x^T(t) X(t) G d\eta(t) + 2x^T(t) X(t) (A_1 x(t) + B_1 w(t)) d\nu(t), \end{aligned} \quad (3.17)$$

where, by definition,

$$\begin{aligned} \rho_c(t) &:= \left[x^T(t) \left(X(t) B_0 + A_1^T X(t) B_1 \right) - w^T(t) \left(\gamma^2 I - B_1^T X(t) B_1 \right) \right] \\ &\quad \times \left(\gamma^2 I - B_1^T X(t) B_1 \right)^{-1} \left[\left(B_0^T X(t) + B_1^T X(t) A_1 \right) x(t) - \left(\gamma^2 I - B_1^T X(t) B_1 \right) w(t) \right] \geq 0. \end{aligned} \quad (3.18)$$

On the other hand using the second equations of (2.11) and of (3.11), it follows that

$$\begin{aligned}
E \left[\int_{ih^+}^{(i+1)h} d(x^T X x) \right] &= E \left[x^T((i+1)h) X((i+1)h) x((i+1)h) - x^T(ih^+) X(ih^+) x(ih^+) \right] \\
&= E \left[x^T((i+1)h) X((i+1)h) x((i+1)h) - x^T(ih) X(ih^-) x(ih) + y_d^T(i) y_d(i) \right. \\
&\quad \left. - \gamma^2 \omega_d^T(i) \omega_d(i) + \rho_d(i) \right] - \text{Tr} \left(G_d^T X(ih) G_d \right),
\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}
\rho_d(i) &: \\
&= \left[x^T(ih) \left(A_{0d}^T X(ih) B_{0d} + A_{1d}^T X(ih) B_{1d} \right) - \omega^T(i) \left(\gamma^2 I - B_{1d}^T X(ih) B_{1d} \right) \right] \left(\gamma^2 I - B_{1d}^T X(ih) B_{1d} \right)^{-1} \\
&\quad \times \left[x^T(ih) \left(A_{0d}^T X(ih) B_{0d} + A_{1d}^T X(ih) B_{1d} \right) - \omega^T(i) \left(\gamma^2 I - B_{1d}^T X(ih) B_{1d} \right) \right]^T \geq 0.
\end{aligned} \tag{3.20}$$

Integrating (3.17) from $t = 0$ to ∞ and equalizing it with (3.19) summed up from $i = 0$ to ∞ , based on the fact that $\rho_c(t) \geq 0$ and $\rho_d(i) \geq 0$ and that $X(t)$ is h -periodic, one obtains (3.16). \square

4. State-Feedback Mixed H_2/H_∞ Control Design

Consider the following linear stochastic system with multiplicative noise and finite jumps:

$$\begin{aligned}
dx(t) &= (A_0 x(t) + B_0 w(t) + B_2 u(t)) dt + (A_1 x(t) + B_1 w(t)) dv(t) + G d\eta(t), \quad t \neq ih, \\
x(ih^+) &= A_{0d} x(ih) + B_{0d} w_d(i) + (A_{1d} x(ih) + B_{1d} w_d(i)) v_d(i) + G_d \eta_d(i), \quad i = 0, 1, \dots, \\
y_1(t) &= C x(t) + D u(t), \quad t \neq ih, \\
y_2(t) &= x(t), \quad t \neq ih, \\
y_d(i) &= C_d x(ih), \quad i = 0, 1, \dots,
\end{aligned} \tag{4.1}$$

where $w(t) \in L^2[0, \infty)$ is an exogenous input, $u(t)$ denotes the control variable, $y_1(t)$ stands for the regulated output, and $y_2(t)$ is the measured output. For the simplicity of the computations, the following orthogonality assumption is made:

$$D^T [C \ D] = [0 \ I]. \tag{4.2}$$

As seen from the above system, the state vector $x(t)$ is assumed measurable. It is not the purpose of the present paper to analyze the case when the state variables must be estimated.

For some results concerning the discrete-time filtering methods associated to networked systems, see, for instance, [24–26].

The second equation of the system (4.1) does not include a discrete-time control input since, in the application presented in the previous section, the control law has only a continuous-time component.

The problem analyzed in this section consists in finding a state-feedback gain $F(t), t \neq ih$, such that the resulting system obtained with $u(t) = F(t)x(t), t \neq ih$, satisfies the following conditions.

- (i) It is ESMS.
- (ii) The H_∞ -type norm of the stochastic system with jumps obtained by ignoring the noises $\eta(t)$ and $\eta_d(i)$ is less than a given $\gamma > 0$.
- (iii) The performance index (3.16) is minimized, where $X(t)$ in (3.16) denotes the stabilizing solution of the norm-type Riccati system (3.11) corresponding to the resulting system obtained with $u(t) = F(t)x(t), t \neq ih$, namely, replacing A_0 by $A_0 + B_2F(t)$.

The solution of this problem is given by the following result.

Theorem 4.1. *The solution of the state-feedback mixed H_2/H_∞ control problem considered above is given by*

$$F(t) = -B_2^T X(t), \quad t \neq ih, \quad (4.3)$$

where $X(t)$ denotes the h -periodic stabilizing solution of the game-theoretic Riccati type system with jumps

$$\begin{aligned} -\dot{X}(t) &= A_0^T X(t) + X(t)A_0 + A_1^T X(t)A_1 + C^T C + \left(X(t)B_0 + A_1^T X(t)B_1 \right) \\ &\quad \times \left(\gamma^2 I - B_1^T X(t)B_1 \right)^{-1} \left(B_0^T X(t) + B_1^T X(t)A_1 \right) - X(t)B_2 B_2^T X(t), \quad t \neq ih, \\ X(ih^-) &= A_{0d}^T X(ih)A_{0d} + A_{1d}^T X(ih)A_{1d} + C_d^T C_d + \left(A_{0d}^T X(ih)B_{0d} + A_{1d}^T X(ih)B_{1d} \right) \\ &\quad \times \left(\gamma^2 I - B_{1d}^T X(ih)B_{1d} \right)^{-1} \left(B_{0d}^T X(ih)A_{0d} + B_{1d}^T X(ih)A_{1d} \right), \quad i = 0, 1, \dots \end{aligned} \quad (4.4)$$

Proof. Consider the cost function

$$J(x_0, \tau, w, u) = E \left[\int_0^\tau \left(|y_1(t)|^2 - \gamma^2 |w(t)|^2 \right) dt \right] \quad (4.5)$$

associated with the system

$$\begin{aligned} dx(t) &= (A_0 x(t) + B_0 w(t) + B_2 u(t))dt + (A_1 x(t) + B_1 w(t))dv(t), \quad t \neq ih, \\ x(ih^+) &= A_{0d} x(ih) + B_{0d} w_d(i) + (A_{1d} x(ih) + B_{1d} w_d(i)), \quad i = 0, 1, \dots, \\ y_1(t) &= Cx(t) + Du(t), \quad t \neq ih, \end{aligned} \quad (4.6)$$

with the initial condition $x(0) = x_0$. Applying Proposition 3.4 for $u_1 = w$ and $u_2 = u$, one obtains that

$$\begin{aligned} J(x_0, \tau, w, u) &= x_0^T X(0) x_0 - E \left[x^T(\tau) X(\tau) x(\tau) \right] \\ &\quad + E \int_0^\tau \left[\left(u(t) + B_2^T X(t) x(t) \right)^T \left(u(t) + B_2^T X(t) x(t) \right) - \tilde{\rho}(w(t), x(t), X(t)) \right] dt, \end{aligned} \quad (4.7)$$

where the following notation has been introduced

$$\begin{aligned} \tilde{\rho}(w(t), x(t), X(t)) &:= \left[w(t) - \left(\gamma^2 I - B_1^T X(t) B_1 \right)^{-1} \left(B_0^T X(t) + B_1 X(t) A_1 \right) x(t) \right]^T \\ &\quad \times \left(\gamma^2 I - B_1^T X(t) B_1 \right) \\ &\quad \times \left[w(t) - \left(\gamma^2 I - B_1^T X(t) B_1 \right)^{-1} \left(B_0^T X(t) + B_1 X(t) A_1 \right) x(t) \right], \end{aligned} \quad (4.8)$$

$X(t)$, $t \neq ih$, denoting the stabilizing solution of the Riccati-type system (4.4).

Equation (4.7) shows that the minimum of J with respect to the control input u is obtained for $u(t) = -B_2^T X(t) x(t)$.

Further, consider a stabilizing state-feedback control $\hat{u}(t) = \hat{F}(t) \hat{x}(t)$, $t \neq ih$, for which the H_∞ norm of the resulting system without additive white noise (see the requirement (ii) above)

$$\begin{aligned} d\hat{x}(t) &= \left[\left(A_0 + B_2 \hat{F}(t) \right) \hat{x}(t) + B_0 w(t) \right] dt + \left(A_1 \hat{x}(t) + B_1 w(t) \right) dv(t), \\ y(t) &= (C + DF(t)) \hat{x}(t) \end{aligned} \quad (4.9)$$

is less than γ .

Using again Proposition 3.4 for the system (4.9), direct computations give

$$J(x_0, \tau, w, \hat{u}) = x_0^T \hat{X}(0) x_0 - E \left[\hat{x}^T(\tau) \hat{X}(\tau) \hat{x}(\tau) \right] - E \left[\int_0^\tau \tilde{\rho}(w(t), \hat{x}(t), \hat{X}(t)) dt \right], \quad (4.10)$$

where $\tilde{\rho}(\cdot, \cdot, \cdot)$ is defined by (4.8) and $\hat{X}(t)$ is the stabilizing solution of the Riccati system of form (3.11) corresponding to (4.9). Then, defining $\tilde{u}(t) := F(t) x(t)$ and

$$\tilde{w}(t) := \left(\gamma^2 I - B_1^T X(t) B_1 \right)^{-1} \left(B_0^T X(t) + B_1^T X(t) A_1 \right) x(t), \quad (4.11)$$

one obtains that $J(x_0, \tau, \tilde{w}, \tilde{u}) \leq J(x_0, \tau, \tilde{w}, \hat{u})$; namely,

$$\begin{aligned} x_0^T X(0)x_0 - E \left[x^T(\tau) X(\tau) x(\tau) \right] &\leq x_0^T \hat{X}(0)x_0 - E \left[\hat{x}^T(\tau) \hat{X}(\tau) \hat{x}(\tau) \right], \\ -E \left[\int_0^\tau \tilde{\rho}(\hat{w}(t), \hat{x}(t), \hat{X}(t)) dt \right] &\leq x_0^T \hat{X}(0)x_0 - E \left[\hat{x}^T(\tau) \hat{X}(\tau) \hat{x}(\tau) \right]. \end{aligned} \quad (4.12)$$

Since $X(t)$ and $\hat{X}(t)$ are stabilizing solutions of the Riccati systems, it follows that $\lim_{\tau \rightarrow \infty} x(\tau) = \lim_{\tau \rightarrow \infty} \hat{x}(\tau) = 0$. Therefore, making $\tau \rightarrow \infty$ in (4.12), one obtains that $X(0) \leq \hat{X}(0)$.

Further, using a similar reasoning for the cost function

$$J_t(x_0, \tau, w, u) = E \left[\int_t^\tau \left(|y_1(t)|^2 - \gamma^2 |w(t)|^2 \right) dt \right] \quad (4.13)$$

with $t \in (0, h)$, one obtains that $X(t) \leq \hat{X}(t)$, and; thus, one concludes that the minimum of (3.16) is obtained for the stabilizing solution $X(t)$ of the Riccati-type system (4.4). \square

5. A Numerical Procedure to Compute the Stabilizing Solution of the Riccati System with Jumps

In order to determine J_0 with the expression given in the statement of Theorem 3.5, the stabilizing solution $X(t)$, $t \geq 0$, of the Riccati-type system (3.11) must be determined. Since the two Riccati equations of this system are coupled, an iterative procedure will be used. The proposed iterative method is similar with the iterative numerical methods used to solve Riccati equations of norm in the deterministic continuous-time and discrete-time cases (see, for instance, [27, 28]). These Newton-type iterative procedures are adapted to the particularities of the Riccati systems with jumps derived in the previous sections, and a detailed proof of the convergence towards the stabilizing solution is not the purpose of the present paper. Roughly speaking, the proof based follows showing that the solutions obtained at each iteration determine a monotonic and bounded sequence. An important particular feature of the Riccati systems with jumps, already mentioned above, is that their solution $X(t)$ is h -periodic and right continuous. The proposed iterative procedure is the following:

$$\begin{aligned} \dot{X}_{k+1}(t) + (A_0 + B_0 F_k(t))^T X_{k+1}(t) + X_{k+1}(t)(A_0 + B_0 F_k(t)) + M_k(t) &= 0, \quad t \neq ih, \\ X_{k+1}(ih^-) &= (A_{0d} + B_{0d} F_{d,k}(i))^T X_{k+1}(ih)(A_{0d} + B_{0d} F_{d,k}(i)) + N_k(i), \quad i = 0, 1, \dots, \end{aligned} \quad (5.1)$$

where

$$\begin{aligned} F_k(t) &= \left(\gamma^2 I - B_1^T X_k(t) B_1 \right)^{-1} \left(B_0^T X_k(t) + B_1^T X_k(t) A_1 \right), \\ M_k(t) &= A_1^T X_k(t) B_1 \left(\gamma^2 I - B_1^T X_k(t) B_1 \right)^{-1} B_1^T X_k(t) A_1 \\ &\quad - X_k(t) B_0 \left(\gamma^2 I - B_1^T X_k(t) B_1 \right)^{-1} B_0^T X_k(t) + A_1^T X_k(t) A_1 + C^T C, \end{aligned}$$

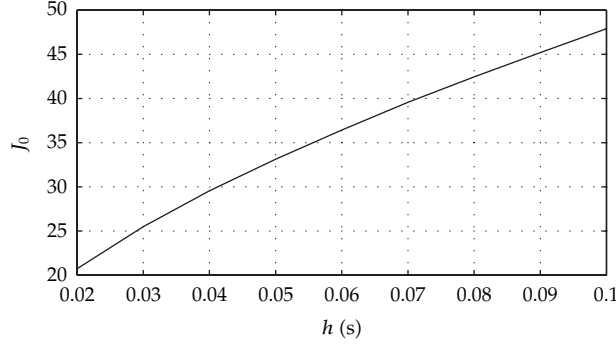


Figure 2: Variation of J_0 with respect to the sampling period h .

$$\begin{aligned}
F_{d,k}(i) &= \left(\gamma^2 I - B_{1d}^T X_k(ih) B_{1d} \right)^{-1} \left(B_{0d}^T X_k(ih) A_{0d} + B_{1d}^T X_k(ih) A_{1d} \right), \\
N_k(i) &= -F_{d,k}^T B_{0d}^T X_k(ih) (A_{0d} + B_{0d} F_{k,d}(i)) - A_{0d}^T X_k(ih) B_{0d} F_{k,d}(i) \\
&\quad + (A_{1d} + B_{1d} F_{k,d}(i))^T X_k(ih) (A_{1d} + B_{1d} F_{k,d}(i)) + C_d^T C_d.
\end{aligned} \tag{5.2}$$

For the initial step of the above iterative procedure, one takes $X(0) = 0$, $t \in (0, h)$, and $F_0(t)$ and $F_{0,d}(i)$ stabilizing (5.1). In order to solve (5.1) at each iteration, one solves the first equation (5.1) obtaining

$$X_{k+1}(ih) = e^{(A_0 + B_0 F_k)^T h} X_{k+1}(ih^-) e^{(A_0 + B_0 F_k)h} + \int_0^h e^{(A_0 + B_0 F_k)^T \tau} M_k(\tau) e^{(A_0 + B_0 F_k)\tau} d\tau, \tag{5.3}$$

which is substituted then in the second equation (5.1) obtaining, thus, a Lyapunov-type equation with the unknown variable $X_{k+1}(ih^-)$. Then, by backward integration on the interval $[(i-1)h, ih^-)$ with the initial condition $X_{k+1}(ih^-)$, one obtains $X_{k+1}(t)$ for $t \in [(i-1)h, ih^-)$.

In the final part of this section, some of the above theoretical results will be used to analyze the mixed performance of the UAVs formation networked with fading communication channel considered in Section 2. The values of the gains considered in this example are $K_d = \text{diag}(0.7, 0.05, 0.05)$ and $K_x = \text{diag}(7, 5, 5)$, for $V_0 = 150 \text{ m/s}$ (see Section 2). One determined the performance index H_2/H_∞ performance computing the value of the index J_0 defined by (3.15). The results are illustrated in Figure 2. One can see the the tracking performances of the flight formation are severely deteriorated when the sampling period of the transmission in the communication channel increases. In Figure 3, the variation of γ_{\min} with respect to the sampling period and the variance of the multiplicative noise, in the absence of the additive white noise, are shown in Figure 3. It can be seen that γ_{\min} is not very much influenced by the multiplicative noise at small sampling periods, but it becomes very sensitive with respect to this noise when the sampling period increases.

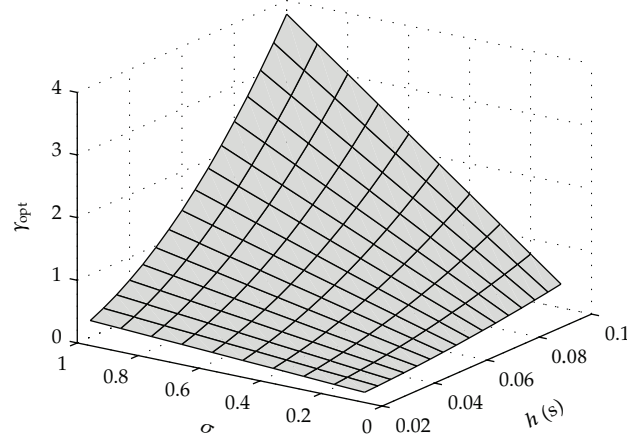


Figure 3: Variation of γ_{\min} with respect to h and σ .

6. Conclusions

The purpose of the paper was to provide an appropriate methodology to evaluate the performance of networked systems interconnected via fading communication channels. The main difficulty arises from the hybrid structure of the resulting system which includes a continuous-time component specific to the network individual members and a discrete-time component given by the communication system. It is shown that such a hybrid configuration can be analyzed from the point of view of stability and disturbance attenuation performances using dynamic models with finite jumps. In the actual stage of the research, a method to compute a mixed H_2/H_∞ -type performance has been developed, and a state-feedback control law to optimize it has been designed. Further research will be focused on the state estimation problems arising in the implementation of such control laws.

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