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Research Article

Thin Film Limits in Magnetoelastic Interactions

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This paper deals with classical dimensional reductions 3D-2D and 3D-1D in magnetoelastic interactions. We adopt a model described by the Landau-Lifshitz-Gilbert equation for the magnetization field coupled to an evolution equation for the displacement. We identify the limit problem both for flat and slender media by using the so-called energy method.

1. Introduction

Magnetostrictive solids are those in which reversible elastic deformations are caused by changes in the magnetization. These materials have a coupling of ferromagnetic energies with elastic energies. Magnetostriction is observed, to differing degrees, in all ferromagnetic materials. To explain the observed magnetic behaviour, there have been a number of theoretical developments for magnetostrictive materials, including the works by Brown [1] and Landau and Lifshitz [2]. A well-established variational model, called Micromagnetics [1, 2], is in principle available to describe the magnetomechanical response of magnetostrictive solids. Treatments on micromagnetic processes are also available in Aharoni [3] and Hubert and Schäfer [4]. The general micromagnetic problem for reasonably large samples is, however, difficult. That is because of the necessity of resolving exceedingly complex three-dimensional domain structures. For sufficiently small thin films, numerical simulations are now routinely used to explore the energy landscape. But simulations are simply experiments. To interpret them, it is natural to do analysis as well. The understanding of thin film behaviour has been helped by the mathematical asymptotic analysis of energies defined on three-dimensional domains of vanishing thickness, through the use of Γ-convergence techniques Gioia and James [5], Desimone [6], Desimone et al. [7], DeSimone and James [8], Alicandro and Leone [9], Kohn and Slastikov [10], Alouges and Labbé [11], Chipot et al. [12, 13]. The focus in all these papers is energy minimization both with and without magnetostriction, but the dynamics and switching of thin film ferromagnets is also an important topic. The papers by Ammari et al. [14], Carbou [15] and Kohn and Slastikov [16] propose a two-dimensional reduction of the Landau-Lifshitz-Gilbert (LLG) equations of micromagnetic switching. For dynamics including magnetostriction we refer, for example, to Visintin [17], Valente [18], Valente and Vergara Caffarelli [19] and Carbou et al. [20] where some results concerning global existence of weak solutions are established. Finally, some computational aspects of magnetostriction are presented in Cerimele et al. [21] and Baňas [22].

The main goal of this paper is to combine the ideas of the references cited above from the elasticity and micromagnetics literature to derive asymptotic models for magnetostrictive films and nanowires. We are concerned with the passage 3D-2D and 3D-1D in the dynamical theory of thin magnetoelastic films. Our investigation has its starting point in the work by Valente and Vergara Caffarelli [19], where the authors establish the existence of weak solutions to a three-dimensional model for the LLG equation with magnetostriction. We shall assume antiplane displacement. We intend to analyze the behaviour of these solutions with one and two diminishing edges. In order to identify the limit problem we make use of the scaling techniques which are well known in elasticity, see, for example, Ciarlet [23], Ciarlet and Destuynder [24]. Here we try to extend in some sense the work [25].

This paper is organized as follows. In Section 2 we present the general three-dimensional model with corresponding energy estimate and a global existence result of weak solutions. Section 3 describes the antiplane model equations which will be the subject of an asymptotic study. In Section 4, we first consider the dimensional reduction from 3D to 2D. We introduce the natural scaling for the problem and prove uniform bounds for the solutions, with respect to vanishing parameter, which allows us to identify the limit problem. We then consider the 3D-1D reduction and state the limit problem. The last section concludes the paper and provides future directions for this work.

2. The 3D Model and Preliminary Results

To describe the model equations we consider Ω a bounded open set of $\mathbb{R}^3$. The generic point of Ω is denoted by $(\widehat{x}, x_3)$ with $\widehat{x} = (x_1, x_2)$. Here and throughout the paper we use bold characters to denote the vector-valued functions. The model combines phenomenological constitutive equations for the magnetization $\mathbf{M}$ and the displacement $\mathbf{u}$. The nonlinear parabolic hyperbolic coupled system describing the dynamics in $Q = (0,T) \times \Omega$ is given by (see [19])

$$\begin{gathered}\gamma^{-1}\partial_t\mathbf{M} - \mathbf{M} \times (a\Delta\mathbf{M} - \partial_t\mathbf{M} - \mathbf{l}(\mathbf{M},\mathbf{u})) = 0,\\ \rho\partial_{tt}\mathbf{u} - \operatorname{div}\left(\mathbf{S}(\mathbf{u}) + \frac{1}{2}\mathbf{L}(\mathbf{M})\right) + \mathbf{f} = 0,\end{gathered} \tag{2.1}$$

where the components of the vector $\mathbf{l}(\mathbf{M},\mathbf{u})$ and the tensors $\mathbf{S}(\mathbf{u}), \mathbf{L}(\mathbf{M})$ are given by

$$\begin{aligned} l_i(\mathbf{M},\mathbf{u}) &= \lambda_{ijkl} M_j \varepsilon_{kl}(\mathbf{u}),\\ \mathbf{S}_{kl}(\mathbf{u}) &= \sigma_{ijkl}\varepsilon_{ij}(\mathbf{u}),\\ \mathbf{L}_{kl}(\mathbf{M}) &= \lambda_{ijkl} M_i M_j. \end{aligned} \tag{2.2}$$

Here ε_{ij} stand for the components of the linearized strain tensor $\boldsymbol{\varepsilon}$, $\lambda_{ijkl} = \lambda_1\delta_{ijkl} + \lambda_2\delta_{ij}\delta_{kl} + \lambda_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ and $\sigma_{ijkl} = \tau_1(\delta_{ijkl} - \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) + \tau_2\delta_{ij}\delta_{kl}$ with $\delta_{ijkl} = 1$ if $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise. The elasticity tensor σ_{ijkl} is assumed to satisfy the following symmetry property,

$$\sigma_{ijkl} = \sigma_{jikl} = \sigma_{klij}, \tag{2.3}$$

and moreover the inequality

$$\sum\left(\sigma_{klij}\varepsilon_{kl}\varepsilon_{ij}\right) \geq \beta\sum|\varepsilon_{kl}|^2 \tag{2.4}$$

holds for some $\beta > 0$.

As initial and boundary conditions we assume

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0, \quad \partial_t\mathbf{u}(0,\cdot) = \mathbf{u}_1, \qquad \mathbf{M}(0,\cdot) = \mathbf{M}_0, \quad |\mathbf{M}_0| = 1 \quad \text{in } \Omega, \tag{2.5}$$

$$\mathbf{u} = 0, \quad \partial_\nu\mathbf{M} = 0 \quad \text{on } \Sigma = (0,T)\times\partial\Omega, \tag{2.6}$$

where ν is the outer unit normal at the boundary $\partial\Omega$.

The first equation in (2.1), well known in the literature, is the modified LLG equation. The modification lies in the presence of the term $\mathbf{l}(\mathbf{M},\mathbf{u})$. The unknown $\mathbf{M}$, the magnetization vector, is a map from Ω to $\mathcal{S}^2$ (the unit sphere of $\mathbb{R}^3$). The symbol $\times$ denotes the vector cross product in $\mathbb{R}^3$. Moreover we denote by $M_i, i = 1,2,3$ the components of $\mathbf{M}$. The constant γ represents the damping parameter while a is the exchange coefficient. The second equation in (2.1) describes the evolution of the displacement $\mathbf{u}$ where ρ is a positive constant and $\mathbf{f}$ is a given external force.

We introduce the functional $\mathcal{E}(t)$ defined as

$$\mathcal{E}(t) = \frac{a}{2}\int_\Omega|\nabla\mathbf{M}|^2 d\Omega + \frac{\beta}{4}\int_\Omega|\boldsymbol{\varepsilon}(\mathbf{u})|^2 d\Omega + \frac{\rho}{2}\int_\Omega|\partial_t\mathbf{u}|^2\, d\Omega \tag{2.7}$$

and put ($\tau > 0$)

$$\mathcal{E}_0 = \frac{a}{2}\int_\Omega|\nabla\mathbf{M}_0|^2 d\Omega + \frac{\tau}{4}\int_\Omega|\nabla\mathbf{u}_0|^2 d\Omega + \frac{\rho}{2}\int_\Omega|\mathbf{u}_1|^2\, d\Omega. \tag{2.8}$$

In the sequel and without loss of generality, we assume that $\mathbf{f} \equiv 0$.

Lemma 2.1 (energy estimate). *Let* $\mathbf{u}_0 \in H_0^1(\Omega,\mathbb{R}^3)$, $\mathbf{u}_1 \in L^2(\Omega,\mathbb{R}^3)$, $\mathbf{M}_0 \in H^1(\Omega,\mathbb{R}^3)$ *with* $|\mathbf{M}_0| = 1$ *a.e in* Ω*, then there exist positive constants* c_1 *and* c_2 *depending on* T *such that the following estimate holds:*

$$\mathcal{E}(t) + \int_0^t\int_\Omega|\partial_t\mathbf{M}|^2 d\Omega\, dt \leq c_1\mathcal{E}_0 + c_2. \tag{2.9}$$

Proof. We refer the reader to Valente and Vergara Caffarelli [19]. We mention that to obtain the above energy estimate it is assumed that $\sigma_{ijkl}\varepsilon_{ij}(\mathbf{u})\varepsilon_{kl}(\mathbf{u}) \le \tau|\nabla \mathbf{u}|^2$. □

We have the following global existence result for problem (2.1)–(2.6).

Theorem 2.2 (global existence). *(Valente and Vergara Caffarelli [19]). Given* $\mathbf{u}_0 \in H^1(\Omega), \mathbf{u}_1 \in L^2(\Omega)$ *and* $\mathbf{M}_0 \in H^1(\Omega)$ *with* $|\mathbf{M}_0| = 1$ *a.e. in* Ω*, there exists a weak solution* $(\mathbf{M}, \mathbf{u})$ *to the problems* (2.1)–(2.6) *in the sense that*

(i) $\mathbf{M} \in H^1(Q)$ *with* $|\mathbf{M}| = 1$ *a.e. in* $Q, \mathbf{u} \in L^2(0,T;H_0^1(\Omega)),\ \partial_t \mathbf{u} \in L^2(0,T;L^2(\Omega))$*,*

(ii) *for each couple* $(\mathbf{p}, \mathbf{g})$ *such that* $\mathbf{p} \in C^\infty(\overline{Q})$ *vanishing at* $t = 0$ *and* $t = T$*, and* $\mathbf{g} \in H^1(Q) \cap C_0(Q)$*, one has*

$$\int_Q \left(\gamma^{-1}\partial_t \mathbf{M}\cdot \mathbf{p} + a\sum_{j=1}^{2} \mathbf{M} \times \partial_{x_j}\mathbf{M}\cdot\partial_{x_j}\mathbf{p} + \mathbf{M}\times(\partial_t\mathbf{M} + \mathbf{l}(\mathbf{M},\mathbf{u}))\cdot\mathbf{p} \right) d\Omega\, dt = 0, \tag{2.10}$$

$$\int_Q \left[-\rho\partial_t\mathbf{u}\partial_t\mathbf{g} + \left(\mathbf{S}(\mathbf{u}) + \frac{1}{2}\mathbf{L}(\mathbf{m})\right) : \varepsilon(\mathbf{g})\right] d\Omega\, dt = 0, \tag{2.11}$$

Moreover the energy estimate (2.9) *holds true.*

In the sequel, the operators $\widehat{\mathrm{div}}, \widehat{\mathrm{grad}}$, and $\widehat{\Delta}$ will represent divergence, gradient, and laplacian operators, respectively, with respect to the variable $\widehat{x}$.

3. The Antiplane Case

Our asymptotic analysis will be performed in the framework of a simplified model in which we neglect the in-plane components of displacement; that is, we assume that $\mathbf{u} = (0, 0, W)$. So the model (2.1) reduces to the following system (we let $\tau_2 = 0, \lambda_1 = 0, \tau_1 = 2\tau$ and $\lambda_2 = \lambda_3 = \lambda$):

$$\gamma^{-1}\partial_t\mathbf{M} - \mathbf{M}\times(a\Delta\mathbf{M} - \partial_t\mathbf{M} - \lambda\mathbf{V}) = 0 \tag{3.1}$$

coupled with

$$\begin{gathered} \rho\partial_{tt}W - \tau\widehat{\Delta}W - 2\tau\partial_{x_3x_3}W - \lambda\left(\partial_{x_1}(M_1M_3) + \partial_{x_2}(M_2M_3) + \partial_{x_3}\left(M_3^2\right)\right) = 0, \\ 2\tau\partial_{x_1x_3}W - \lambda\partial_{x_1}M_1^2 - \lambda\partial_{x_2}(M_1M_2) - \tau\partial_{x_3}\partial_{x_1}W - \lambda\partial_{x_3}(M_1M_3) = 0, \\ -\lambda\partial_{x_1}(M_1M_2) + 2\tau\partial_{x_2x_3}W - \lambda\partial_{x_2}\left(M_2^2\right) - \tau\partial_{x_3x_2}W - \lambda\partial_{x_3}(M_2M_3) = 0 \end{gathered} \tag{3.2}$$

in $Q = (0,T)\times\Omega$, where the vector $\mathbf{V} = \mathbf{V}(\mathbf{M}, \nabla W)$ is given by

$$\mathbf{V} = (M_3\partial_{x_1}W + M_1\partial_{x_3}W,\ M_3\partial_{x_2}W + M_2\partial_{x_3}W,\ M_1\partial_{x_1}W + M_2\partial_{x_2}W + 3M_3\partial_{x_3}W). \tag{3.3}$$

The associated initial and boundary conditions writes

$$W(0,\cdot) = W_0, \quad \partial_t W(0,\cdot) = W_1, \qquad \mathbf{M}(0,\cdot) = \mathbf{M}_0, \quad |\mathbf{M}_0| = 1 \quad \text{in } \Omega, \tag{3.4}$$

$$W = 0, \qquad \partial_\nu \mathbf{M} = 0 \quad \text{on } \Sigma = (0,T) \times \partial\Omega. \tag{3.5}$$

4. Dimensional Reduction

4.1. Flat Domains

Let ε be a real parameter taking values in a sequence of positive numbers converging to zero. We consider flat magnetoelastic domains represented by $\Omega^\varepsilon = \omega \times (0,\varepsilon)$, where ω is a regular and bounded subset of $\mathbb{R}^2$. We shall be interested in getting the asymptotic behaviour of the solutions, when $\varepsilon \to 0$.

4.1.1. Scaling and Uniform Bounds

Let $(\mathbf{M}, W)$ be a solution of the problem (3.1)–(3.5) posed in Ω^ε. We introduce the change of variables $(x_1, x_2, x_3) = (x, y, \varepsilon z)$ with $(x, y, z) = \mathrm{X} \in \Omega = \omega \times (0,1)$. For functions $\mathbf{R}(x_1, x_2, x_3)$ and $S(x_1, x_2, x_3)$ defined in Ω^ε we introduce the functions $\mathbf{r}^\varepsilon(x, y, z)$ and $s^\varepsilon(x, y, z)$ defined on Ω by setting

$$\mathbf{R}(x_1, x_2, x_3) = \mathbf{r}^\varepsilon(x, y, z); \qquad S(x_1, x_2, x_3) = s^\varepsilon(x, y, z). \tag{4.1}$$

Let $(\mathbf{m}^\varepsilon, w^\varepsilon)$ be the fields associated with $(\mathbf{M}, W)$. The scaled equations satisfied by $(\mathbf{m}^\varepsilon)$ are the following:

$$\gamma^{-1}\partial_t \mathbf{m}^\varepsilon - \mathbf{m}^\varepsilon \times \left(a\left(\widehat{\Delta}\mathbf{m}^\varepsilon + \frac{1}{\varepsilon^2}\partial_{zz}\mathbf{m}^\varepsilon \right) - \partial_t \mathbf{m}^\varepsilon - \lambda \widetilde{\mathbf{V}}^\varepsilon \right) = 0. \tag{4.2}$$

The vector $\widetilde{\mathbf{V}}^\varepsilon$ is defined by

$$\widetilde{\mathbf{V}}^\varepsilon = \left(m_3^\varepsilon \partial_x w^\varepsilon + \frac{1}{\varepsilon} m_1^\varepsilon \partial_z w^\varepsilon, m_3^\varepsilon \partial_y w^\varepsilon + \frac{1}{\varepsilon} m_2^\varepsilon \partial_z w^\varepsilon, m_1^\varepsilon \partial_x w^\varepsilon + m_2^\varepsilon \partial_y w^\varepsilon + \frac{3}{\varepsilon} m_3^\varepsilon \partial_z w^\varepsilon \right). \tag{4.3}$$

For the scaled displacement $\mathbf{u}^\varepsilon = (0, 0, w^\varepsilon)$ we have

$$\begin{gathered}
\rho \partial_{tt} w^\varepsilon - \tau\left(\widehat{\Delta} w^\varepsilon + \frac{2}{\varepsilon^2}\partial_{zz} w^\varepsilon \right) - \lambda\left(\partial_x (m_1^\varepsilon m_3^\varepsilon) + \partial_y (m_2^\varepsilon m_3^\varepsilon) - \frac{\lambda}{\varepsilon}\partial_z (m_3^\varepsilon)^2 \right) = 0, \\
2\frac{\tau}{\varepsilon}\partial_{xz} w^\varepsilon - \lambda \partial_x (m_1^\varepsilon)^2 - \lambda \partial_y (m_1^\varepsilon m_2^\varepsilon) - \frac{\tau}{\varepsilon}\partial_{zx} w^\varepsilon - \frac{\lambda}{\varepsilon}\partial_z (m_1^\varepsilon m_3^\varepsilon) = 0, \\
-\lambda \partial_x (m_1^\varepsilon m_2^\varepsilon) + 2\frac{\tau}{\varepsilon}\partial_{yz} w^\varepsilon - \lambda \partial_y \left((m_2^\varepsilon)^2 \right) - \frac{\tau}{\varepsilon}\partial_{zx} w^\varepsilon - \frac{\lambda}{\varepsilon}\partial_z (m_2^\varepsilon m_3^\varepsilon) = 0.
\end{gathered} \tag{4.4}$$

The associated energy $\mathcal{E}^\varepsilon(t)$, defined in (2.7), becomes

$$\mathcal{E}^\varepsilon(t) = \frac{a}{2}\int_\Omega \left|\widehat{\operatorname{grad}}\,\mathbf{m}^\varepsilon\right|^2 d\Omega + \frac{a}{2\varepsilon^2}\int_\Omega |\partial_z\,\mathbf{m}^\varepsilon|^2 d\Omega + \frac{\beta}{4}\int_\Omega \left|\widehat{\operatorname{grad}}\,w^\varepsilon\right|^2 d\Omega + \frac{\beta}{4\varepsilon^2}\int_\Omega |\partial_z w^\varepsilon|^2 d\Omega + \frac{\rho}{2}\int_\Omega |\partial_t w^\varepsilon|^2 d\Omega. \tag{4.5}$$

The energy equation remains unchanged as well as the saturation constraint on magnetization (see (2.5)) which is written as

$$|\mathbf{m}^\varepsilon(t,X)|^2 = \left|\mathbf{m}_0^\varepsilon(X)\right|^2 = 1 \tag{4.6}$$

for almost every (t,X). The following estimates hold true for all $t \geq 0$

$$\mathcal{E}^\varepsilon(t) + \int_0^t\int_\Omega |\partial_t\mathbf{m}^\varepsilon|^2 d\Omega\, dt \leq c_1\mathcal{E}_0^\varepsilon + c_2, \tag{4.7}$$

where $\mathcal{E}_0^\varepsilon$ is given by

$$\mathcal{E}_0^\varepsilon = \frac{a}{2}\int_\Omega \left|\widehat{\operatorname{grad}}\,\mathbf{m}_0^\varepsilon\right|^2 d\Omega + \frac{a}{2\varepsilon^2}\int_\Omega \left|\partial_z m_0^\varepsilon\right|^2 d\Omega + \frac{\tau}{4}\int_\Omega \left|\widehat{\operatorname{grad}}\,w_0^\varepsilon\right|^2 d\Omega + \frac{\tau}{4\varepsilon^2}\int_\Omega \left|\partial_z w_0^\varepsilon\right|^2 d\Omega + \frac{\rho}{2}\int_\Omega \left|w_1^\varepsilon\right|^2 d\Omega. \tag{4.8}$$

To get uniform bounds for the solutions we discuss the admissibility criterion for the initial data. An initial datum $(\mathbf{m}_0^\varepsilon, w_0^\varepsilon)$ is said to be admissible if we have

$$\mathcal{E}_0^\varepsilon < +\infty. \tag{4.9}$$

The admissibility criterion means

$$\frac{a}{2}\int_\Omega \left|\widehat{\operatorname{grad}}\,\mathbf{m}_0^\varepsilon\right|^2 d\Omega + \frac{a}{2\varepsilon^2}\int_\Omega \left|\partial_z m_0^\varepsilon\right|^2 d\Omega + \frac{\tau}{4}\int_\Omega \left|\widehat{\operatorname{grad}}\,w_0^\varepsilon\right|^2 d\Omega + \frac{\tau}{4\varepsilon^2}\int_\Omega \left|\partial_z w_0^\varepsilon\right|^2 d\Omega + \frac{\rho}{2}\int_\Omega \left|w_1^\varepsilon\right|^2 d\Omega < +\infty. \tag{4.10}$$

Thus, since $|\mathbf{m}_0^\varepsilon|^2 = 1$ a.e., to satisfy the criterion, we assume that there exists $C > 0$ independent of ε such that

$$\begin{gathered}\left|\widehat{\operatorname{grad}}\,\mathbf{m}_0^\varepsilon\right|_{L^2(\Omega)} \leq C, \qquad \left|\partial_z\mathbf{m}_0^\varepsilon\right|_{L^2(\Omega)} \leq C\varepsilon, \qquad \left|\mathbf{m}_0^\varepsilon(x,y)\right|^2 = 1 \quad \text{a.e.}, \\ \left|\widehat{\operatorname{grad}}\,w_0^\varepsilon\right|_{L^2(\Omega)} \leq C, \qquad \left|\partial_z w_0^\varepsilon\right|_{L^2(\Omega)} \leq C\varepsilon, \qquad \left|w_1^\varepsilon\right|_{L^2(\Omega)} \leq C.\end{gathered} \tag{4.11}$$

Condition (4.11) means that the couple $(\mathbf{m}_0^\varepsilon, w_0^\varepsilon)$ is essentially independent of the variable z and its strong limit $(\mathbf{m}_0, w_0)$ is independent of z.

Remark 4.1. If the initial data are not admissible, then we expect an initial layer to occur when ε tends to zero.

4.1.2. Passing to the Limit

Let $(\mathbf{m}^\varepsilon, w^\varepsilon)$ be a solution of the problem associated to an admissible initial datum $(\mathbf{m}_0^\varepsilon, w_0^\varepsilon)$. We have

$$\begin{aligned} \mathbf{m}_0^\varepsilon \rightharpoonup \mathbf{m}_0 \text{ weakly-}\star \quad &\text{in } L^\infty(\Omega) \text{ and weakly in } H^1(\Omega), \\ w_0^\varepsilon \rightharpoonup w_0 \text{ weakly} \quad &\text{in } H^1(\Omega). \end{aligned} \tag{4.12}$$

Moreover $\mathbf{m}_0(\hat{x}, z) = \mathbf{m}_0(\hat{x})$ is independent of z. For subsequences, the solutions verify the convergences

$$\begin{aligned} \mathbf{m}^\varepsilon \rightharpoonup \mathbf{m} \text{ weakly-}\star \quad &\text{in } L^\infty(\mathbb{R}^+ \times \Omega) \cap L^\infty\left(\mathbb{R}^+, H^1(\Omega)\right), \\ w^\varepsilon \rightharpoonup w \text{ weakly} \quad &\text{in } L^2\left(0, T, H_0^1(\Omega)\right), \end{aligned} \tag{4.13}$$

$$\begin{aligned} \partial_z \mathbf{m}^\varepsilon \longrightarrow 0 \text{ strongly} \quad &\text{in } L^\infty\left(\mathbb{R}^+, L^2(\Omega)\right), \\ \partial_z w^\varepsilon \longrightarrow 0 \text{ strongly} \quad &\text{in } L^\infty\left(\mathbb{R}^+, L^2(\Omega)\right), \\ \partial_t \mathbf{m}^\varepsilon \rightharpoonup \partial_t \mathbf{m} \text{ weakly} \quad &\text{in } L^2\left(\mathbb{R}^+, L^2(\Omega)\right), \\ \partial_t w^\varepsilon \rightharpoonup \partial_t w \text{ weakly} \quad &\text{in } L^2\left(0, T; L^2(\Omega)\right). \end{aligned} \tag{4.14}$$

Hence, the couple $(\mathbf{m}, w)$ is independent of the variable z. By Aubin's compactness results, we have

$$(\mathbf{m}^\varepsilon, w^\varepsilon) \longrightarrow (\mathbf{m}, w) \text{ strongly} \quad \text{in } L^2_{\text{loc}}\left(\mathbb{R}^+, L^2(\Omega)\right). \tag{4.15}$$

Moreover from the Sobolev embedding theorem $W^{1,2}(Q) \to L^q(Q)$ $(2 \le q \le 6)$, the further compactness result follows

$$m_i^\varepsilon m_j^\varepsilon \longrightarrow m_i m_j \text{ strongly} \quad \text{in } L^2(Q), i, j = 1, 2, 3. \tag{4.16}$$

Recall that $Q = (0, T) \times \Omega$ with $\Omega = \omega \times (0, 1)$.

In order to pass to the limit we look at the variational formulation of the scaled problem (4.2)–(4.4) by using an oscillating test functions. Let $\psi^\varepsilon(t, \hat{x}, z)$ and $g^\varepsilon(t, \hat{x}, z)$ be a regular test functions depending on ε. Multiplying (4.2) by ψ^ε, each Equation (4.4) by g^ε and integrating by parts, we get the weak formulations

$$\begin{aligned}
&\gamma^{-1}\int_Q \partial_t \mathbf{m}^\varepsilon \cdot \psi^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t + \int_Q \mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon \cdot \psi^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \\
&\quad = -\lambda \int_Q \mathbf{m}^\varepsilon \times \tilde{\mathbf{V}}^\varepsilon \cdot \psi^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t - a\int_Q \mathbf{m}^\varepsilon \times \widehat{\operatorname{grad}}\,\mathbf{m}^\varepsilon \cdot \widehat{\operatorname{grad}}\,\psi^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \\
&\quad -\frac{a}{\varepsilon^2}\int_Q \mathbf{m}^\varepsilon \times \partial_z \mathbf{m}^\varepsilon \cdot \partial_z \psi^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t,
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
&-\rho\int_Q \partial_t w^\varepsilon \partial_t g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t + \tau\int_Q \widehat{\operatorname{grad}}\,w^\varepsilon \widehat{\operatorname{grad}}\,g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t + \frac{2\tau}{\varepsilon^2}\int_Q \partial_z w^\varepsilon \partial_z g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \\
&+\lambda\int_Q m_1^\varepsilon m_3^\varepsilon \partial_x g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t + \lambda\int_Q m_2^\varepsilon m_3^\varepsilon \partial_y g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t + \frac{\lambda}{\varepsilon}\int_Q \left(m_3^\varepsilon\right)^2 \partial_z g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t = 0, \\
&-2\frac{\tau}{\varepsilon}\int_Q \partial_z w^\varepsilon \partial_x g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t + \lambda\int_Q \left(m_1^\varepsilon\right)^2 \partial_x g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t + \lambda\int_Q m_1^\varepsilon m_2^\varepsilon \partial_y g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \\
&+\frac{\tau}{\varepsilon}\int_Q \partial_x w^\varepsilon \partial_z g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t + \frac{\lambda}{\varepsilon}\int_Q m_1^\varepsilon m_2^\varepsilon \partial_z g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t = 0, \\
&\lambda\int_Q m_1^\varepsilon m_2^\varepsilon \partial_x g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t - \frac{2\tau}{\varepsilon}\int_Q \partial_z w^\varepsilon \partial_y g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t + \lambda\int_Q \left(m_2^\varepsilon\right)^2 \partial_y g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \\
&+\frac{\tau}{\varepsilon}\int_Q \partial_x w^\varepsilon \partial_z g^\varepsilon \mathrm{d}\Omega\,\mathrm{d}t + \frac{\lambda}{\varepsilon}\int_Q m_2^\varepsilon m_3^\varepsilon \partial_z g^\varepsilon \mathrm{d}\Omega\,\mathrm{d}t = 0.
\end{aligned} \tag{4.18}$$

To pass to the limit in these equations we need the following convergence result.

Lemma 4.2. *Defining* $\Theta^\varepsilon := (1/\varepsilon)\partial_z w^\varepsilon$, *then*

$$\Theta^\varepsilon \rightharpoonup \Theta = -\frac{\lambda}{2\tau}(m_3)^2 + K \quad \text{weakly-}\star \qquad \text{in } L^\infty\left(\mathbb{R}^+, L^2(\Omega)\right), \tag{4.19}$$

where K *is a function of the variable* $\hat{x}$.

Proof. We multiply the first equation of (4.18) by ε and choose $g^\varepsilon = g \in \mathfrak{D}(Q)$ independent of ε. We get

$$\begin{aligned}
&\varepsilon\left(-\rho\int_Q \partial_t w^\varepsilon \partial_t g \,\mathrm{d}\Omega\,\mathrm{d}t + \tau\int_Q \widehat{\operatorname{grad}}\,w^\varepsilon \widehat{\operatorname{grad}}\,g \,\mathrm{d}\Omega\,\mathrm{d}t + \lambda\int_Q m_1^\varepsilon m_3^\varepsilon \partial_x g \,\mathrm{d}\Omega\,\mathrm{d}t\right. \\
&\left.+\lambda\int_Q m_2^\varepsilon m_3^\varepsilon \partial_y g \,\mathrm{d}\Omega\,\mathrm{d}t\right) + \frac{2\tau}{\varepsilon}\int_Q \partial_z w^\varepsilon \partial_z g \,\mathrm{d}\Omega\,\mathrm{d}t + \lambda\int_Q \left(m_3^\varepsilon\right)^2 \partial_z g \,\mathrm{d}\Omega\,\mathrm{d}t = 0.
\end{aligned} \tag{4.20}$$

Hence, passing to the limit, by using convergences (4.14), (4.15), and (4.16), we deduce that the weak-$\star$ limit Θ of the sequence Θ^ε satisfies $\partial_z(2\tau\Theta + \lambda m_3^2) = 0$ which allows to get (4.19). □

Remark 4.3. In the sequel and without loss of generality we will assume that $K \equiv 0$.

Now we are able to pass to the limit. We set $Q_\tau = \mathbb{R}^+ \times \omega$. We choose in the above weak formulations test functions of the form

$$\begin{aligned} \psi^\varepsilon(t,\widehat{x},z) &= \psi_0(t,\widehat{x}) + \varepsilon\psi(t,\widehat{x},\varepsilon z), \\ g^\varepsilon(t,\widehat{x},z) &= g(t,\widehat{x}) + \varepsilon g_0(t,\widehat{x})h(\varepsilon z). \end{aligned} \tag{4.21}$$

We pass to the limit in each term of (4.17) by using the convergence results (4.14), (4.15), (4.16) and the following facts, $\partial_z\psi^\varepsilon = \varepsilon^2(\partial_z\psi)(\varepsilon z)$ and $\partial_{\widehat{x}}\psi^\varepsilon = \partial_{\widehat{x}}\psi_0 + \varepsilon\partial_{\widehat{x}}\psi(\varepsilon z)$. Hence we first get

$$\begin{aligned} \int_Q \partial_t \mathbf{m}^\varepsilon \cdot \psi^\varepsilon \, d\Omega \, dt &\longrightarrow \int_{Q_\tau} \partial_t \mathbf{m} \cdot \psi_0 \, d\widehat{x} \, dt, \\ \int_Q \mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon \cdot \psi^\varepsilon \, d\Omega \, dt &\longrightarrow \int_{Q_\tau} \mathbf{m} \times \partial_t \mathbf{m} \cdot \psi_0 \, d\widehat{x} \, dt. \end{aligned} \tag{4.22}$$

Next, we have

$$\int_Q \mathbf{m}^\varepsilon \times \widehat{\operatorname{grad}}\, \mathbf{m}^\varepsilon \cdot \widehat{\operatorname{grad}}\, \psi^\varepsilon \, d\Omega \, dt \longrightarrow \int_{Q_\tau} \mathbf{m} \times \widehat{\operatorname{grad}}\, \mathbf{m} \cdot \widehat{\operatorname{grad}}\, \psi_0 \, d\widehat{x} \, dt. \tag{4.23}$$

We also get

$$\frac{1}{\varepsilon^2} \int_Q a\mathbf{m}^\varepsilon \times \partial_z \mathbf{m}^\varepsilon \cdot \partial_z \psi^\varepsilon \, d\Omega \, dt \longrightarrow 0. \tag{4.24}$$

Recall that

$$\widetilde{\mathbf{V}}^\varepsilon = \left(m_3^\varepsilon \partial_x w^\varepsilon + \frac{1}{\varepsilon} m_1^\varepsilon \partial_z w^\varepsilon,\ m_3^\varepsilon \partial_y w^\varepsilon + \frac{1}{\varepsilon} m_2^\varepsilon \partial_z w^\varepsilon,\ m_1^\varepsilon \partial_x w^\varepsilon + m_2^\varepsilon \partial_y w^\varepsilon + \frac{3}{\varepsilon} m_3^\varepsilon \partial_z w^\varepsilon \right). \tag{4.25}$$

To pass to the limit in the term with $\widetilde{\mathbf{V}}^\varepsilon$ we make use of the convergence of Lemma 4.2.

Similarly we pass to the limit in the weak formulation (4.18). The convergences (4.14) and (4.15) allow to get

$$\begin{aligned} \int_Q \partial_t w^\varepsilon \partial_t g^\varepsilon \, d\Omega \, dt &\longrightarrow \int_{Q_\tau} \partial_t w \partial_t g \, d\widehat{x} \, dt, \\ \int_Q \widehat{\operatorname{grad}}\, w^\varepsilon \widehat{\operatorname{grad}}\, g^\varepsilon \, d\Omega \, dt &\longrightarrow \int_{Q_\tau} \widehat{\operatorname{grad}}\, w \, \widehat{\operatorname{grad}}\, g \, d\widehat{x} \, dt. \end{aligned} \tag{4.26}$$

Next, we have

$$\begin{aligned}
&\int_Q m_1^\varepsilon m_3^\varepsilon \partial_x g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \longrightarrow \int_{Q\tau} m_1 m_3 \partial_x g \,\mathrm{d}\widehat{x}\,\mathrm{d}t,\\
&\int_Q m_2^\varepsilon m_3^\varepsilon \partial_y g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \longrightarrow \int_{Q\tau} m_2 m_3 \partial_y g \,\mathrm{d}\widehat{x}\,\mathrm{d}t.
\end{aligned} \tag{4.27}$$

We also get

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_Q \partial_z w^\varepsilon \partial_z g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \longrightarrow 0,\\
&\frac{1}{\varepsilon}\int_Q \left(m_3^\varepsilon\right)^2 \partial_z g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \longrightarrow 0.
\end{aligned} \tag{4.28}$$

It remains to identify the limit of the two last equations of (4.18). Let us pass to the limit in the second equation of (4.18). We make use of the Lemma 4.2 to get

$$\frac{1}{\varepsilon}\int_Q \partial_z w^\varepsilon \partial_x g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \longrightarrow -\int_{Q\tau} \frac{\lambda}{2\tau}(m_3)^2 \partial_x g \,\mathrm{d}\widehat{x}\,\mathrm{d}t. \tag{4.29}$$

Similarly, by the same arguments above, we also get the limit both for the other terms and for the last equation of (4.18).

We proved the result.

Theorem 4.4. *Let $(\mathbf{m}^\varepsilon, w^\varepsilon)$ be a solution of the problem associated with the admissible initial datum $(\mathbf{m}_0^\varepsilon, w_0^\varepsilon)$. Then, one has $(\mathbf{m}^\varepsilon, w^\varepsilon) \rightarrow (\mathbf{m}, w)$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^+, L^2(\Omega))$, $\mathbf{m}^\varepsilon \rightharpoonup \mathbf{m}$ weakly-$\star$ in $L^\infty(\mathbb{R}^+, H^1(\Omega))$ and $w^\varepsilon \rightharpoonup w$ weakly in $L^2(\mathbb{R}^+, H_0^1(\Omega))$. The couple $(\mathbf{m}, w)$ is independent of the variable z and satisfies in $\mathbb{R}^+ \times \omega$, $|\mathbf{m}(t,\widehat{x})|^2 = 1$ and the following two-dimensional coupled system*

$$\begin{aligned}
&\gamma^{-1}\partial_t \mathbf{m} - \mathbf{m} \times \left(a\widehat{\Delta}\mathbf{m} - \partial_t \mathbf{m} - \lambda \widetilde{\mathbf{V}}\right) = 0,\\
&\rho \partial_{tt} w - \tau \widehat{\Delta} w - \lambda \partial_x (m_1 m_3) - \lambda \partial_y (m_2 m_3) = 0,\\
&\partial_x\left(m_1^2 + m_3^2\right) + \partial_y (m_1 m_2) = 0,\\
&\partial_y\left(m_2^2 + m_3^2\right) + \partial_x (m_1 m_2) = 0,
\end{aligned} \tag{4.30}$$

where

$$\widetilde{\mathbf{V}} = \left(m_3 \partial_x w - \frac{\lambda}{2\tau} m_1 m_3^2,\ m_3 \partial_y w - \frac{\lambda}{2\tau} m_2 m_3^2,\ m_1 \partial_x w + m_2 \partial_y w - \frac{3\lambda}{2\tau} m_3^3\right). \tag{4.31}$$

The associated initial and boundary conditions are given by

$$
\begin{gathered}
w(0,\hat{x}) = w_0, \quad \partial_t w(0,\hat{x}) = w_1, \qquad \mathbf{m}(0,\hat{x}) = \mathbf{m}_0, \quad |\mathbf{m}_0| = 1 \quad \text{in } \omega \\
w = 0, \quad \partial_\nu \mathbf{m} = 0 \text{ on } \partial\omega,
\end{gathered}
\tag{4.32}
$$

where w_1 is the weak limit of w_1^ε in $L^2(\Omega)$.

Remark 4.5. Note that if the function K introduced in Lemma 4.2 is such that $K \not\equiv 0$, then the vector $\widetilde{\mathbf{V}}$ in (4.30) becomes

$$
\begin{aligned}
\widetilde{\mathbf{V}} = \Bigg(& m_3\partial_x w - \left(\frac{\lambda}{2\tau}m_3^2 + K\right)m_1, m_3\partial_y w - \left(\frac{\lambda}{2\tau}m_3^2 + K\right)m_2, m_1\partial_x w \\
& + m_2\partial_y w - 3\left(\frac{\lambda}{2\tau}m_3^2 + K\right)m_3\Bigg).
\end{aligned}
\tag{4.33}
$$

4.2. Slender Domains

Note that we can proceed as above to get result for the 3D-1D dimensional reduction. In fact, we have the following theorem which we state without proof.

Theorem 4.6. *Let $(\mathbf{m}^\varepsilon, w^\varepsilon)$ be a solution of the problem associated with the admissible initial datum $(\mathbf{m}_0^\varepsilon, w_0^\varepsilon)$. Then, one has $(\mathbf{m}^\varepsilon, w^\varepsilon) \to (\mathbf{m}, w)$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^+, L^2(\Omega))$, $\mathbf{m}^\varepsilon \rightharpoonup \mathbf{m}$ weakly-$\star$ in $L^\infty(\mathbb{R}^+, H^1(\Omega))$ and $w^\varepsilon \rightharpoonup w$ weakly in $L^2(\mathbb{R}^+, H_0^1(\Omega))$. The couple $(\mathbf{m}, w)$ is independent of the variable $\hat{x}$ and satisfies in $\mathbb{R}^+ \times (0,1)$, $|\mathbf{m}(t,z)|^2 = 1$ and the following one-dimensional coupled system*

$$
\begin{gathered}
\gamma^{-1}\partial_t \mathbf{m} - \mathbf{m} \times \left(a\partial_{zz}\mathbf{m} - \partial_t \mathbf{m} - \lambda\widetilde{\mathbf{V}}\right) = 0, \\
\rho\partial_{tt} w - 2\tau\partial_{zz} w - \lambda\partial_z\left(m_3^2\right) = 0,
\end{gathered}
\tag{4.34}
$$

where

$$
\widetilde{\mathbf{V}} = \left(-\frac{\lambda}{\tau}m_1m_3^2 + m_1\partial_z w, -\frac{\lambda}{\tau}m_2m_3^2 - m_2\partial_z w, -\frac{\lambda}{\tau}m_1m_3^2 - \frac{\lambda}{\tau}m_2m_3^2 + 3m_3\partial_z w\right).
\tag{4.35}
$$

The associated initial and boundary conditions are given by

$$
\begin{gathered}
w(0,z) = w_0, \quad \partial_t w(0,z) = w_1, \qquad \mathbf{m}(0,z) = \mathbf{m}_0, \quad |\mathbf{m}_0| = 1 \quad \text{in } (0,1) \\
w(t,j) = 0, \quad \partial_z \mathbf{m}(t,j) = 0 \quad \text{for} \quad j = 0,1,
\end{gathered}
\tag{4.36}
$$

where w_1 is the weak limit of w_1^ε in $L^2(\Omega)$.

5. Concluding Remarks

We remark that the 1D model obtained in Theorem 4.6 slightly differs from the one derived in [25]. This is simply due to the original 2D model considered in [18] which is obtained by making some approximations on the total energy. The limiting behaviours obtained in this work concern magnetoelastic interactions system when we neglect in-plane components of displacement. The present paper is our first attempt at scrutinizing the reduced theories from the complete model. The analysis with arbitrary displacement seems to be more difficult and needs much more investigations. In fact, it would be interesting to consider the general model which consists of the three-dimensional case with total energy (see [18])

$$\begin{aligned}\mathcal{E}(t) = \frac{1}{2}\int_{\Omega} & a|\nabla \mathbf{M}|^2 + \tau_1|\nabla U|^2 + \tau_2(\operatorname{div} U)^2 \\ & + \lambda_1\delta_{klij}\nabla_i U_j M_k M_l + \lambda_2|\mathbf{M}|^2 \operatorname{div} U + 2\lambda_3(\nabla U_i \cdot \mathbf{M})M_i,\end{aligned} \tag{5.1}$$

where $\delta_{ijkl} = 1$ if $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise. The parameters τ_1, τ_2, λ_1, λ_2, and λ_3 are positive constants.

Another direction for future research concerns magnetic domain walls (DWs) which are boundaries in magnetic materials that divide regions with distinct magnetization directions. The manipulation and control of DWs in ferromagnetic nanowires (essentially one dimensional models) has recently become a subject of intense experimental and theoretical research, see, for example, Carbou and Labbé [26]. The rapidly growing interest in the physics of the DW motion can be mainly explained by a promising possibility of using DWs as the basis for next-generation memory and logic devices. However, in order to realize such devices in practice it is essential to be able to position individual DWs precisely along magnetic nanowires. It would be interesting to address within the context of the present paper the stability of the propagation of such processing DWs with respect to perturbations of the initial magnetization profile, some anisotropy properties of the nanowire, and applied magnetic field.

Finally, a valuable direction for future research is the effect of very small domain irregularities on the limiting problems. More precisely, the roughness may be defined by means of a periodical function h^ε with period, for example, of order ε^2 ($\varepsilon > 0$). So that the three-dimensional domain may be represented by $\Omega^\varepsilon = \{(x,z) \in \mathbb{R}^2 \times \mathbb{R}; x \in \omega, 0 < z < h^\varepsilon(x)\}$ where ω is a domain of $\mathbb{R}^2$. Various limit models may be obtained depending on the ratio between the size of rugosities and the mean height of the domain.

References

[1] W. F. Brown, *Magnetoelastic Interactions*, vol. 9 of *Springer Tracts in Naturals Philosophy*, Springer, Berlin, Germany, 1966.

[2] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon, 1986.

[3] A. Aharoni, *Introduction to the Theory of Ferromagnetism*, Oxford University Press, London, UK, 1996.

[4] A. Hubert and R. Schäfer, *Magnetic Domains. The Analysis of Magnetic Microstructures*, Springer, New York, NY, USA, 1998.

[5] G. Gioia and R. D. James, "Micromagnetics of very thin fims," *Proceedings of the Royal Society A*, vol. 453, pp. 213–223, 1997.

[6] A. DeSimone, "Energy minimizers for large ferromagnetic bodies," *Archive for Rational Mechanics and Analysis*, vol. 125, no. 2, pp. 99–143, 1993.

[7] A. Desimone, R. V. Kohn, S. Müller, and F. Otto, "A reduced theory for thin-film micromagnetics," *Communications on Pure and Applied Mathematics*, vol. 55, no. 11, pp. 1408–1460, 2002.
[8] A. DeSimone and R. D. James, "A theory of magnetostriction oriented towards applications," *Journal of Applied Physics*, vol. 81, no. 8, pp. 5706–5708, 1997.
[9] R. Alicandro and C. Leone, "3D-2D asymptotic analysis for micromagnetic thin films," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 6, pp. 489–498, 2001.
[10] R. V. Kohn and V. V. Slastikov, "Another thin-film limit of micromagnetics," *Archive for Rational Mechanics and Analysis*, vol. 178, no. 2, pp. 227–245, 2005.
[11] F. Alouges and S. Labbé, "Convergence of a ferromagnetic film model," *Comptes Rendus Mathématique, Académie des Sciences Paris*, vol. 344, no. 2, pp. 77–82, 2007.
[12] M. Chipot, I. Shafrir, V. Valente, and G. Vergara Caffarelli, "On a hyperbolic-parabolic system arising in magnetoelasticity," *Journal of Mathematical Analysis and Applications*, vol. 352, no. 1, pp. 120–131, 2009.
[13] M. Chipot, I. Shafrir, V. Valente, and G. Vergara Caffarelli, "A nonlocal problem arising in the study of magneto-elastic interactions," *Bollettino della Unione Matematica Italiana*, vol. 9, no. 1, pp. 197–221, 2008.
[14] H. Ammari, L. Halpern, and K. Hamdache, "Asymptotic behaviour of thin ferromagnetic films," *Asymptotic Analysis*, vol. 24, no. 3-4, pp. 277–294, 2000.
[15] G. Carbou, "Thin layers in micromagnetism," *Mathematical Models & Methods in Applied Sciences*, vol. 11, no. 9, pp. 1529–1546, 2001.
[16] R. V. Kohn and V. V. Slastikov, "Effective dynamics for ferromagnetic thin films: a rigorous justification," *Proceedings of The Royal Society of London Series A*, vol. 461, no. 2053, pp. 143–154, 2005.
[17] A. Visintin, "On Landau-Lifshitz' equations for ferromagnetism," *Japan Journal of Applied Mathematics*, vol. 2, no. 1, pp. 69–84, 1985.
[18] V. Valente, ".An evolutive model for magnetorestrictive interactions: existence of weak solutions," in *Smart Structures and Materials, Modeling, Signal Processing and Control*, Proceedings of SPIE, 2006.
[19] V. Valente and G. Vergara Caffarelli, "On the dynamics of magneto-elastic interactions: existence of solutions and limit behaviors," *Asymptotic Analysis*, vol. 51, no. 3-4, pp. 319–333, 2007.
[20] G. Carbou, M. A. Efendiev, and P. Fabrie, "Global weak solutions for the Landau-Lifschitz equation with magnetostriction," *Mathematical Methods in the Applied Sciences*, vol. 34, no. 10, pp. 1274–1288, 2011.
[21] M. M. Cerimele, F. Pistella, and V. Valente, "Numerical study of an evolutive model for magnetostrictive materials," *Mathematics and Computers in Simulation*, vol. 77, no. 1, pp. 22–33, 2008.
[22] L. Baňas, "Adaptive techniques for Landau-Lifshitz-Gilbert equation with magnetostriction," *Journal of Computational and Applied Mathematics*, vol. 215, no. 2, pp. 304–310, 2008.
[23] P. G. Ciarlet, *Introduction to Linear Shell Theory*, vol. 1 of *Series in Applied Mathematics (Paris)*, North-Holland, Amsterdam, The Netherlands, 1998.
[24] P. G. Ciarlet and P. Destuynder, "A justification of the two-dimensional linear plate model," *Journal de Mécanique*, vol. 18, no. 2, pp. 315–344, 1979.
[25] M. Tilioua, "2D-1D dimensional reduction in a toy model for magnetoelastic interactions," *Applications of Mathematics*, vol. 56, no. 3, pp. 287–295, 2011.
[26] G. Carbou and S. Labbé, "Stabilization of walls for nano-wires of finite length," *ESAIM: Control Optimisation and Calculus of Variations*, vol. 18, no. 1, pp. 1–24, 2012.