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Research Article

Coupled Fixed-Point Theorems for Contractions in Partially Ordered Metric Spaces and Applications

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Bhaskar and Lakshmikantham (2006) showed the existence of coupled coincidence points of a mapping F from $X \times X$ into X and a mapping g from X into X with some applications. The aim of this paper is to extend the results of Bhaskar and Lakshmikantham and improve the recent fixed-point theorems due to Bessem Samet (2010). Indeed, we introduce the definition of generalized g-Meir-Keeler type contractions and prove some coupled fixed point theorems under a generalized g-Meir-Keeler-contractive condition. Also, some applications of the main results in this paper are given.

1. Introduction

The Banach contraction principle [1] is a classical and powerful tool in nonlinear analysis and has been generalized by many authors (see [2–15] and others).

Recently, Bhaskar and Lakshmikantham [16] introduced the notion of a coupled fixed-point of the given two variables mapping. More precisely, let X be a nonempty set and $F : X \times X \to X$ be a given mapping. An element $(x, y) \in X \times X$ is called a *coupled fixed-point* of the mapping F if

$$F(x, y) = x, \qquad F(y, x) = y. \tag{1.1}$$

They also showed the uniqueness of a coupled fixed-point of the mapping F and applied their theorems to the problems of the existence and uniqueness of a solution for a periodic boundary value problem.

Theorem 1.1 (see Zeidler [15]). *Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)] \tag{1.2}$$

for all $x \geq u$ and $y \leq v$. Moreover, if there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0), \qquad y_0 \geq F(y_0, x_0), \tag{1.3}$$

then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Later, in [17], Lakshmikantham and Ćirić investigated some more coupled fixed-point theorems in partially ordered sets, and some others obtained many results on coupled fixed-point theorems in cone metric spaces, intuitionistic fuzzy normed spaces, ordered cone metric spaces and topological spaces (see, e.g., [18–25]).

In [9], Meir and Keeler generalized the well-known Banach fixed-point theorem [1] as follows.

Theorem 1.2 (Meir and Keeler [9]). *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that*

$$\epsilon \leq d(x, y) < \epsilon + \delta(\epsilon) \Longrightarrow d(T(x), T(y)) < \epsilon \tag{1.4}$$

for all $x, y \in X$. Then T admits a unique fixed-point $x_0 \in X$ and, for all $x \in X$, the sequence $\{T^n(x)\}$ converges to x_0.

Proposition 1.3 (see [17]). *Let (X, d) be a partially ordered metric space and $F : X \times X \to X$ be a given mapping. If the contraction* (1.2) *is satisfied, then F is a generalized Meir-Keeler type contraction.*

Motivated by the results of Bhaskar and Lakshmikantham [16], Lakshmikantham and Ćirić [17], and Samet [26], in this paper, we introduce the definition of g-Meir-Keeler-contractive mappings and prove some coupled fixed-point theorems under a generalized g-Meir-Keeler contractive condition.

2. Main Results

Let X be a nonempty set. We note that an element $(x, y) \in X \times X$ is called a *coupled coincidence point* of a mapping $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$ for all $x, y \in X$. Also, we say that F and g are *commutative* (or *commuting*) if $g(F(x, y)) = F(g(x), g(y))$ for all $x, y \in X$.

We introduce the following two definitions.

Definition 2.1. Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say that F has the *mixed strict g-monotone property* if, for any $x, y \in X$,

$$\begin{aligned} x_1, x_2 \in X, \quad g(x_1) < g(x_2) \Longrightarrow F(x_1, y) < F(x_2, y), \\ y_1, y_2 \in X, \quad g(y_1) < g(y_2) \Longrightarrow F(x, y_1) > F(x, y_2). \end{aligned} \tag{2.1}$$

Definition 2.2. Let $(X, \leq)$ be a partially ordered set and d be a metric on X. Let $F : X \times X \to X$ and $g : X \to X$ be two given mappings. We say that F is a *generalized g-Meir-Keeler type contraction* if, for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that, for all $x, y, u, v \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$,

$$\epsilon \leq \frac{1}{2}\left[d(g(x), g(u)) + d(g(y), g(v))\right] < \epsilon + \delta(\epsilon) \Longrightarrow d(F(x, y), F(u, v)) < \epsilon. \tag{2.2}$$

Lemma 2.3. *Let $(X, \leq)$ be a partially ordered set and d be a metric on X. Let $F : X \times X \to X$ and $g : X \to X$ be two given mappings. If F is a generalized g-Meir-Keeler type contraction, then we have*

$$d(F(x, y), F(u, v)) < \frac{1}{2}\left[d(g(x), g(u)) + d(g(y), g(v))\right] \tag{2.3}$$

for all x, y, u, v with $g(x) < g(u)$, $g(y) \geq g(v)$ or $g(x) \leq g(u)$, $g(y) > g(v)$.

Proof. Let $x, y, u, v \in X$ such that $g(x) < g(u)$ and $g(y) \geq g(v)$ or $g(x) \leq g(u)$ and $g(y) > g(v)$. Then $d(g(x), g(u)) + d(g(y), g(v)) > 0$. Since F is a generalized g-Meir-Keeler type contraction, for $\epsilon = (1/2)[d(g(x), g(u)) + d(g(y), g(v))]$, there exists $\delta(\epsilon) > 0$ such that, for all $x_0, y_0, u_0, v_0 \in X$ with $g(x_0) \leq g(u_0)$ and $g(y_0) \geq g(v_0)$,

$$\epsilon \leq \frac{1}{2}\left[d(g(x_0), g(u_0)) + d(g(y_0), g(v_0))\right] < \epsilon + \delta(\epsilon) \Longrightarrow d(F(x_0, y_0), F(u_0, v_0)) < \epsilon. \tag{2.4}$$

Therefore, putting $x_0 = x$, $y_0 = y$, $u_0 = u$ and $v_0 = v$, we have

$$d(F(x, y), F(u, v)) < \frac{1}{2}\left[d(g(x), g(u)) + d(g(y), g(v))\right]. \tag{2.5}$$

This completes the proof. □

From now on, we suppose that $(X, \leq)$ is a partially ordered set, and there exists a metric d on X such that (X, d) is a complete metric space.

Theorem 2.4. *Let $F : X \times X \to X$ and $g : X \to X$ be such that $F(X \times X) \subseteq g(X)$, g is continuous and commutative with F. Also, suppose that*

(a) *F has the mixed strict g-monotone property;*

(b) *F is a generalized g-Meir-keeler type contraction;*

(c) *there exist $x_0, y_0 \in X$ such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) > F(y_0, x_0)$.*

Then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$; that is, F and g have a coupled coincidence in $X \times X$.

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) > F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Again, from $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$.

Continuing this process, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n), \qquad g(y_{n+1}) = F(y_n, x_n) \tag{2.6}$$

for all $n \geq 0$.

Now, we show that

$$g(x_n) < g(x_{n+1}), \qquad g(y_n) > g(y_{n+1}) \tag{2.7}$$

for all $n \geq 0$. For $n = 0$, we have

$$g(x_0) < F(x_0, y_0) = g(x_1), \qquad g(y_0) > F(y_0, x_0) = g(y_1). \tag{2.8}$$

Since F has the mixed strict g-monotone property, then we have

$$\begin{aligned} g(x_0) < g(x_1) &\Longrightarrow F(x_0, y_1) < F(x_1, y_1), \\ g(y_0) > g(y_1) &\Longrightarrow F(x_0, y_0) < F(x_0, y_1). \end{aligned} \tag{2.9}$$

It follows that $F(x_0, y_0) < F(x_1, y_1)$, that is, $g(x_1) < g(x_2)$.

Similarly, we have

$$\begin{aligned} g(y_1) < g(y_0) &\Longrightarrow F(y_1, x_0) < F(y_0, x_0), \\ g(x_1) > g(x_0) &\Longrightarrow F(y_1, x_1) < F(y_1, x_0). \end{aligned} \tag{2.10}$$

Thus it follows that $F(y_1, x_1) < F(y_0, x_0)$, that is, $g(y_2) < g(y_1)$.

Again, we have

$$\begin{aligned} g(x_1) < g(x_2) &\Longrightarrow F(x_1, y_2) < F(x_2, y_2), \\ g(y_1) > g(y_2) &\Longrightarrow F(x_1, y_1) < F(x_1, y_2). \end{aligned} \tag{2.11}$$

Thus it follows that $F(x_1, y_1) < F(x_2, y_2)$, that is, $g(x_2) < g(x_3)$.

Similarly, we have

$$g(y_2) < g(y_1) \Longrightarrow F(y_2, x_1) < F(y_1, x_1),$$
$$g(x_2) > g(x_1) \Longrightarrow F(y_2, x_2) < F(y_2, x_1). \tag{2.12}$$

Thus it follows that $F(y_2, x_2) < F(y_1, x_1)$, that is, $g(y_3) < g(y_2)$.

Continuing this process for each $n \geq 1$, we get the following:

$$g(x_0) < g(x_1) < g(x_2) < \cdots < g(x_n) < g(x_{n+1}) < \cdots,$$
$$g(y_0) > g(y_1) > g(y_2) > \cdots g(y_n) > g(y_{n+1}) > \cdots. \tag{2.13}$$

Denote that

$$\delta_n := d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})). \tag{2.14}$$

Since $g(x_{n-1}) < g(x_n)$ and $g(y_{n-1}) > g(y_n)$, it follows from (2.6) and Lemma 2.3 that

$$\begin{aligned} d(g(x_n), g(x_{n+1})) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &< \frac{1}{2}[d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n))]. \end{aligned} \tag{2.15}$$

Since $g(y_n) < g(y_{n-1})$ and $g(x_n) > g(x_{n-1})$, it follows from (2.6) and Lemma 2.3 that

$$\begin{aligned} d(g(y_{n+1}), g(y_n)) &= d(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\ &< \frac{1}{2}[d(g(y_n), g(y_{n-1})) + d(g(x_n), g(x_{n-1}))]. \end{aligned} \tag{2.16}$$

Thus it follows from (2.14)–(2.16) that $\delta_n < \delta_{n-1}$. This means that the sequence $\{\delta_n/2\}$ is monotone decreasing. Therefore, there exists $\delta^* \geq 0$ such that $\lim_{n\to\infty} \delta_n/2 = \delta^*$, that is,

$$\lim_{n\to\infty} \frac{1}{2}[d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))] = \delta^*. \tag{2.17}$$

Now, we show that $\delta^* = 0$. Suppose that $\delta^* > 0$ hold. Let $\delta^* = \epsilon$. Then there exists a positive integer m such that

$$\epsilon \leq \frac{1}{2}[d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1}))] < \epsilon + \delta(\epsilon). \tag{2.18}$$

Then, by using (2.7) and the condition (b), we have

$$d(F(x_m, y_m),\ F(x_{m+1}, y_{m+1})) < \epsilon, \tag{2.19}$$

and so, by (2.6), we have

$$d(g(x_{m+1}), g(x_{m+2})) < \epsilon. \tag{2.20}$$

On the other hand, by (2.15), we have

$$\frac{1}{2}\left[d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1}))\right] < \epsilon, \tag{2.21}$$

which is a contradiction with (2.18). Thus we have $\epsilon = \delta^* = 0$, that is,

$$\lim_{n\to\infty}\frac{1}{2}\left[d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))\right] = 0, \tag{2.22}$$

that is,

$$\lim_{n\to\infty}\delta_n = 0. \tag{2.23}$$

Now, we prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in X. Suppose that at least one of $\{g(x_n)\}$ or $\{g(y_n)\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and two subsequences $\{l_k\}$, $\{m_k\}$ of integers such that $m_k > l_k \geq k$ and

$$d(g(x_{l_k}), g(x_{m_k})) \geq \frac{\epsilon}{2}, \qquad d(g(y_{l_k}), g(y_{m_k})) \geq \frac{\epsilon}{2} \tag{2.24}$$

for all $k \geq 1$. Thus we have

$$r_k = d(g(x_{l_k}), g(x_{m_k})) + d(g(y_{l_k}), g(y_{m_k})) \geq \epsilon \tag{2.25}$$

for all $k \geq 1$. Let m_k be the smallest number exceeding l_k such that (2.25) holds. Then we have

$$d(g(x_{l_k}), g(x_{m_k-1})) + d(g(y_{l_k}), g(y_{m_k-1})) < \epsilon. \tag{2.26}$$

Thus, from (2.14), (2.25), (2.26) and the triangle inequality, it follows that

$$\begin{aligned} \epsilon &\leq r_k \\ &\leq d(g(x_{l_k}), g(x_{m_k-1})) + d(g(x_{m_k-1}), g(x_{m_k})) \\ &\quad + d(g(y_{l_k}), g(y_{m_k-1})) + d(g(y_{m_k-1}), g(y_{m_k})) \\ &< \epsilon + \delta_{m_k-1} \end{aligned} \tag{2.27}$$

and so

$$\epsilon \leq \lim_{k\to\infty} r_k \leq \lim_{k\to\infty}(\epsilon + \delta_{m_k-1}). \tag{2.28}$$

Hence, by (2.23), we have

$$\lim_{k\to\infty} r_k = \epsilon^+. \tag{2.29}$$

It follows from (2.6), (2.14), and the triangle inequality that

$$\begin{aligned} r_k &= d(g(x_{l_k}), g(x_{m_k})) + d(g(y_{l_k}), g(y_{m_k})) \\ &\le d(g(x_{l_k}), g(x_{l_k+1})) + d(g(x_{l_k+1}), g(x_{m_k+1})) + d(g(x_{m_k+1}), g(x_{m_k})) \\ &\quad + d(g(y_{l_k}), g(y_{l_k+1})) + d(g(y_{l_k+1}), g(y_{m_k+1})) + d(g(y_{m_k+1}), g(y_{m_k})) \\ &= \delta_{l_k} + \delta_{m_k} + d(g(x_{l_k+1}), g(x_{m_k+1})) + d(g(y_{l_k+1}), g(y_{m_k+1})) \\ &= \delta_{l_k} + \delta_{m_k} + d(F(x_{l_k}, y_{l_k}), F(x_{m_k}, y_{m_k})) + d(F(y_{l_k}, x_{l_k}), F(y_{m_k}, x_{m_k})). \end{aligned} \tag{2.30}$$

Form (2.13) we have $g(x_{l_k}) < g(x_{m_k})$ and $g(y_{l_k}) > g(y_{m_k})$. Now, it follows from Lemma 2.3 and (2.30) that

$$r_k < \delta_{l_k} + \delta_{m_k} + d(g(x_{l_k}), g(x_{m_k})) + d(g(y_{l_k}), g(y_{m_k})), \tag{2.31}$$

that is,

$$r_k < \delta_{l_k} + \delta_{m_k} + r_k. \tag{2.32}$$

This is a contradiction. Therefore, $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Since X is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x, \qquad \lim_{n\to\infty} g(y_n) = y. \tag{2.33}$$

Since $\{g(x_n)\}$ is monotone increasing and $\{g(y_n)\}$ is monotone decreasing, we have

$$g(x_n) < x, \qquad g(y_n) > y \tag{2.34}$$

for all $n \ge 1$. Thus it follows from (2.33) and the continuity of g that

$$\lim_{n\to\infty} g(g(x_n)) = g(x), \qquad \lim_{n\to\infty} g(g(y_n)) = g(y). \tag{2.35}$$

Thus, for all $m \ge 1$, there exists a positive integer n_0 such that, for all $n \ge n_0$,

$$d(g(g(x_n)), g(x)) < \frac{1}{4m}, \qquad d(g(g(y_n)), g(y)) < \frac{1}{4m}. \tag{2.36}$$

Hence, from (2.6), the commutativity of F and g and the triangle inequality, we have

$$\begin{aligned} d(F(x,y),g(x)) &\leq d(F(x,y),g(g(x_n))) + d(g(g(x_n)),g(x)) \\ &= d(F(x,y),g(F(x_{n-1},y_{n-1}))) + d(g(g(x_n)),g(x)) \\ &= d(F(x,y),F(g(x_{n-1}),g(y_{n-1}))) + d(g(g(x_n)),g(x)). \end{aligned} \tag{2.37}$$

Thus, it follows from (2.34), (2.36), and Lemma 2.3 that

$$\begin{aligned} &d(F(x,y),g(x)) \\ &\quad < \frac{1}{2}\left[d(g(g(x_{n-1})),g(x)) + d(g(g(y_{n-1})),g(y))\right] + d(g(g(x_n)),g(x)) \\ &\quad < \frac{1}{8m} + \frac{1}{8m} + \frac{1}{4m} \\ &\quad = \frac{1}{2m} \longrightarrow 0 \end{aligned} \tag{2.38}$$

as $m \to \infty$. Therefore, we have $F(x,y) = g(x)$. Similarly, we can show that $F(y,x) = g(y)$. This means that F and g have a coupled coincidence point in $X \times X$. This completes the proof. □

Corollary 2.5. *Let $F : X \times X \to X$ be a mapping satisfying the following conditions:*

(a) *F has the mixed strict monotone property;*

(b) *F is a generalized Meir-Keeler type contraction;*

(c) *there exists $x_0, y_0 \in X$ such that $x_0 < F(x_0, y_0)$ and $y_0 > F(y_0, x_0)$.*

Then there exist $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$.

Proof. The conclusion follows from Theorem 2.4 by putting $g = I$ (: the identity mapping) on X. □

Now, we introduce the product space $X \times X$ with the following partial order: for all $(x,y),(u,v) \in X \times X$,

$$(u,v) \leq (x,y) \Longleftrightarrow u < x, \quad v \geq y. \tag{2.39}$$

Theorem 2.6. *Suppose that all the hypotheses of Theorem 2.4 hold and, further, for all $(x,y),(x^*,y^*) \in X \times X$, there exists $(u,v) \in X \times X$ such that $(F(u,v),F(v,u))$ is comparable with $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$. Then F and g have a unique coupled common fixed-point, that is, there exists a unique $(x,y) \in X \times X$ such that*

$$x = g(x) = F(x,y), \qquad y = g(y) = F(y,x). \tag{2.40}$$

Proof. By Theorem 2.4, the set of coupled coincidences of the mapping F and g is nonempty.

First, we show that, if (x, y) and (x^*, y^*) are coupled coincidence points of F and g, that is, if

$$g(x) = F(x, y), \qquad g(y) = F(y, x), \qquad g(x^*) = F(x^*, y^*), \qquad g(y^*) = F(y^*, x^*), \tag{2.41}$$

then we have

$$g(x) = g(x^*), \qquad g(y) = g(y^*). \tag{2.42}$$

Put $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ such that $g(u_1) = F(u_0, v_0)$ and $g(v_1) = F(v_0, u_0)$. Then, similarly as in the proof of Theorem 2.4, we can inductively define the sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ such that

$$g(u_{n+1}) = F(u_n, v_n), \qquad g(v_{n+1}) = F(v_n, u_n) \tag{2.43}$$

for all $n \geq 0$. Also, if we set $x_0 = x$, $y_0 = y$, $x_0^* = x^*$, and $y_0^* = y^*$, then we can define the sequences $\{g(x_n)\}$, $\{g(y_n)\}$, $\{g(x_n^*)\}$, and $\{g(y_n^*)\}$ as follows:

$$\begin{gathered} g(x_{n+1}) = F(x_n, y_n), \qquad g(y_{n+1}) = F(y_n, x_n), \\ g(x_{n+1}^*) = F(x_n^*, y_n^*), \qquad g(y_{n+1}^*) = F(y_n^*, x_n^*) \end{gathered} \tag{2.44}$$

for all $n \geq 0$. Since

$$\begin{gathered} (F(x, y), F(y, x)) = (g(x_1), g(y_1)) = (g(x), g(y)), \\ (F(u, v), F(v, u)) = (g(u_1), g(v_1)) \end{gathered} \tag{2.45}$$

are comparable each other, then $g(x) < g(u_1)$ and $g(y) \geq g(v_1)$. It is easy to show that $(g(x), g(y))$, and $(g(u_n), g(v_n))$ are comparable each other, that is, $g(x) < g(u_n)$ and $g(y) \geq g(v_n)$ for all $n \geq 1$. Thus it follows from Lemma 2.3 that

$$\begin{aligned} &d(g(x), g(u_{n+1})) + d(g(y), g(v_{n+1})) \\ &\quad = d(F(x, y), F(u_n, v_n)) + d(F(y, x), F(v_n, u_n)) \\ &\quad < \frac{1}{2}[d(g(x), g(u_n)) + d(g(y), g(v_n))] + \frac{1}{2}[d(g(y), g(v_n)) + d(g(x), g(u_n))] \\ &\quad = d(g(x), g(u_n)) + d(g(y), g(v_n)) \end{aligned} \tag{2.46}$$

and so

$$\frac{1}{2}[d(g(x), g(u_{n+1})) + d(g(y), g(v_{n+1}))] < \frac{1}{2^n}[d(g(x), g(u_1)) + d(g(y), g(v_1))] \longrightarrow 0 \tag{2.47}$$

as $n \longrightarrow \infty$. Therefore, we have

$$\lim_{n\to\infty} d(g(x), g(u_{n+1})) = 0, \qquad \lim_{n\to\infty} d(g(y), g(v_{n+1})) = 0. \tag{2.48}$$

Similarly, we can prove that

$$\lim_{n\to\infty} d(g(x^*), g(u_{n+1})) = 0, \qquad \lim_{n\to\infty} d(g(y^*), g(v_{n+1})) = 0. \tag{2.49}$$

Thus, by the triangle inequality, (2.48) and (2.49), we have

$$\begin{gathered} d(g(x), g(x^*)) \leq d(g(x), g(u_{n+1})) + d(g(x^*), g(u_{n+1})) \longrightarrow 0, \\ d(g(y), g(y^*)) \leq d(g(y), g(v_{n+1})) + d(g(y^*), g(v_{n+1})) \longrightarrow 0 \end{gathered} \tag{2.50}$$

as $n \longrightarrow \infty$, which imply that $g(x) = g(x^*)$ and $g(y) = g(y^*)$.

Now, we prove that $g(x) = x$ and $g(y) = y$. Denote that $g(x) = z$ and $g(x) = w$. Since $g(x) = F(x, y)$ and $g(y) = F(y, x)$, by the commutativity of F and g, we have

$$g(z) = g(g(x)) = g(F(x, y)) = F(g(x), g(y)) = F(z, w), \tag{2.51}$$

$$g(w) = g(g(y)) = g(F(y, x)) = F(g(y), g(x)) = F(w, z). \tag{2.52}$$

Thus, (z, w) is a coupled coincidence point of F and g.

Putting $x^* = z$ and $y^* = w$ in (2.52), it follows from (2.42) that

$$z = g(x) = g(x^*) = g(z), \qquad w = g(y) = g(y^*) = g(w) \tag{2.53}$$

and so, from (2.51) and (2.52),

$$z = g(z) = F(z, w), \qquad w = g(w) = F(w, z). \tag{2.54}$$

Therefore, (z, w) is a coupled common fixed-point of F and g.

Finally, to prove the uniqueness of the coupled common fixed-point of F and g, assume that (p, q) is another coupled common fixed-point of F and g. Then, by (2.42), we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$. This completes the proof. □

Corollary 2.7. *Suppose that all the hypotheses of* Corollary 2.5 *hold and, further, for all* (x, y) *and* $(x^*, y^*) \in X \times X$, *there exists* $(u, v) \in X \times X$ *that is comparable with* (x, y) *and* (x^*, y^*). *Then there exists a unique* $x \in X$ *such that* $x = F(x, x)$.

3. Applications

Now, we give some applications of the main results in Section 2.

Theorem 3.1. *Let $F : X \times X \to X$ and $g : X \to X$ be two given mappings. Assume that there exists a function $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying the following conditions:*

(a) $\varphi(0) = 0$ *and* $\varphi(t) > 0$ *for any* $t > 0$;

(b) φ *is nondecreasing and right continuous;*

(c) *for any* $\epsilon > 0$, *there exists* $\delta(\epsilon) > 0$ *such that, for all* $x, y, u, v \in X$ *with* $g(x) \leq g(u)$ *and* $g(y) \geq g(v)$,

$$\epsilon \leq \varphi\left(\frac{1}{2}\left[d\left(g(x), g(u)\right) + d\left(g(y), g(v)\right)\right]\right) < \epsilon + \delta(\epsilon) \Longrightarrow \varphi\left[d\left(F(x,y), F(u,v)\right)\right] < \epsilon. \tag{3.1}$$

Then F is a generalized g-Meir-Keeler type contraction.

Proof. For any $\epsilon > 0$, it follows from (a) that $\varphi(\epsilon) > 0$ and so there exists $\alpha > 0$ such that, for all $u, v, u^*, v^* \in X$ with $g(u) \leq g(u^*)$ and $g(v) \geq g(v^*)$,

$$\begin{aligned} \varphi(\epsilon) &\leq \varphi\left(\frac{1}{2}\left[d\left(g(u), g(u^*)\right) + d\left(g(v), g(v^*)\right)\right]\right) < \varphi(\epsilon) + \alpha \\ &\Longrightarrow \varphi[d(F(u,v), F(u^*,v^*))] < \varphi(\epsilon). \end{aligned} \tag{3.2}$$

From the right continuity of φ, there exists $\delta > 0$ such that $\varphi(\epsilon + \delta) < \varphi(\epsilon) + \alpha$. For any $x, y, u, v \in X$ such that $g(x) \leq g(u)$, $g(y) \geq g(v)$ and

$$\epsilon \leq \frac{1}{2}\left[d\left(g(x), g(u)\right) + d\left(g(y), g(v)\right)\right] < \epsilon + \delta, \tag{3.3}$$

since φ is nondecreasing function, we get the following:

$$\varphi(\epsilon) \leq \varphi\left(\frac{1}{2}\left[d\left(g(x), g(u)\right) + d\left(g(y), g(v)\right)\right]\right) < \varphi(\epsilon + \alpha) < \varphi(\epsilon) + \alpha. \tag{3.4}$$

By (3.2), we have $\varphi[d(F(x,y), F(u,v))] < \varphi(\epsilon)$ and so $d(F(x,y), F(u,v)) < \epsilon$. Therefore, it follows that F is a generalized g-Meir-Keeler type contraction. This completes the proof. □

Corollary 3.2 (see [26, Theorem 3.1]). *Let $F : X \times X \to X$ be a given mapping. Assume that there exists a function $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying the following conditions:*

(a) $\varphi(0) = 0$ *and* $\varphi(t) > 0$ *for any* $t > 0$;

(b) φ *is nondecreasing and right continuous;*

(c) *for any* $\epsilon > 0$, *there exists* $\delta(\epsilon) > 0$ *such that* $x \leq u$, $y \geq v$ *and*

$$\epsilon \leq \varphi\left(\frac{1}{2}[d(x,u) + d(y,v)]\right) < \epsilon + \delta(\epsilon) \Longrightarrow \varphi[d(F(x,y), F(u,v))] < \epsilon. \tag{3.5}$$

Then F *is a generalized Meir-Keeler type contraction.*

The following result is an immediate consequence of Theorems 2.4 and 3.1.

Corollary 3.3. *Let* $F : X \times X \to X$ *and* $g : X \to X$ *be two given mappings such that* $F(X \times X) \subseteq g(X)$, g *is continuous and commutative with* F. *Also, suppose that*

(a) F *has the mixed strict* g*-monotone property;*

(b) *for any* $\epsilon > 0$, *there exists* $\delta(\epsilon) > 0$ *such that, for all* $x, y, u, v \in X$ *with* $g(x) \leq g(u)$ *and* $g(y) \geq g(v)$,

$$\epsilon \leq \int_0^{(1/2)[d(g(x),g(u))+d(g(y),g(v))]} \varphi(t)dt < \epsilon + \delta(\epsilon) \Longrightarrow \int_0^{d(F(x,y),F(u,v))} \varphi(t)dt < \epsilon, \tag{3.6}$$

where φ *is a locally integrable function from* $[0, +\infty)$ *into itself satisfying the following condition:*

$$\int_0^s \varphi(t)dt > 0 \tag{3.7}$$

for all $s > 0$;

(c) *there exist* $x_0, y_0 \in X$ *such that* $g(x_0) < F(x_0, y_0)$ *and* $g(y_0) > F(y_0, x_0)$.

Then there exists $(x, y) \in X \times X$ *such that* $g(x) = F(x, y)$ *and* $g(y) = F(y, x)$. *Moreover, if* $g(x_0)$ *and* $g(y_0)$ *are comparable to each other, then* F *and* g *have a unique coupled common fixed-point in* $X \times X$.

Corollary 3.4. *Let* $F : X \times X \to X$ *be a mapping satisfying the following conditions:*

(a) F *has the mixed strict monotone property;*

(b) *for any* $\epsilon > 0$, *there exists* $\delta(\epsilon) > 0$ *such that* $x \leq u$, $y \geq v$ *and*

$$\epsilon \leq \int_0^{(1/2)[d(x,u)+d(y,v)]} \varphi(t)dt < \epsilon + \delta(\epsilon) \Longrightarrow \int_0^{[d(F(x,y),F(u,v))]} \varphi(t)dt < \epsilon, \tag{3.8}$$

where φ is a locally integrable function from $[0, +\infty)$ into itself satisfying

$$\int_0^s \varphi(t)dt > 0 \tag{3.9}$$

for all $s > 0$;

(c) *there exist $x_0, y_0 \in X$ such that $x_0 < F(x_0, y_0)$ and $y_0 > F(y_0, x_0)$.*

Then there exists $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$. Moreover, if x_0 and y_0 are comparable to each other, then F has a unique coupled common fixed-point in $X \times X$.

Corollary 3.5. *Let $F : X \times X \to X$ and $g : X \to X$ be two given mappings such that $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F. Also, suppose that*

(a) *F has the mixed strict g-monotone property;*

(b) *for any $x, y, u, v \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$,*

$$\int_0^{[d(F(x,y),F(u,v))]} \varphi(t)dt \leq k \int_0^{(1/2)[d(g(x),g(u))+d(g(y),g(v))]} \varphi(t)dt, \tag{3.10}$$

where $k \in (0, 1)$ and φ is a locally integrable function from $[0, +\infty)$ into itself satisfying

$$\int_0^s \varphi(t)dt > 0 \tag{3.11}$$

for all $s > 0$;

(c) *there exist $x_0, y_0 \in X$ such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) > F(y_0, x_0)$.*

Then there exists $(x, y) \in X \times X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$. Moreover, if $g(x_0)$ and $g(y_0)$ are comparable to each other, then F and g have a unique coupled common fixed-point in $X \times X$.

Proof. For any $\epsilon > 0$, if we take $\delta(\epsilon) = (1/k - 1)\epsilon$ and apply Corollary 3.3, then we can get the conclusion. □

Corollary 3.6. *Let $F : X \times X \to X$ be a mapping satisfying the following conditions:*

(a) *F has the mixed strict monotone property,*

(b) *for any $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$,*

$$\int_0^{d(F(x,y),F(u,v))} \varphi(t)dt \leq k \int_0^{(1/2)[d(x,u)+d(y,v)]} \varphi(t)dt, \tag{3.12}$$

where $k \in (0,1)$ and φ is a locally integrable function from $[0,+\infty)$ into itself satisfying

$$\int_0^s \varphi(t)dt > 0 \tag{3.13}$$

for all $s > 0$;

(c) *there exist $x_0, y_0 \in X$ such that $x_0 < F(x_0, y_0)$ and $y_0 > F(y_0, x_0)$.*

Then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$. Moreover, if x_0 and y_0 are comparable to each other, then F has a unique coupled common fixed-point in $X \times X$.

Finally, by using the above results, we show the existence of solutions for the following integral equation:

$$\begin{aligned}(x(t), y(t)) = \Bigg(&\int_0^T G(t,s)\left[(f(s,x(s)) + \lambda x(s)) - (f(s,y(s)) \ + \lambda y(s))\right]ds, \\ &\int_0^T G(t,s)\left[(f(s,y(s)) + \lambda y(s)) - (f(s,x(s)) + \lambda x(s))\right]ds\Bigg),\end{aligned} \tag{3.14}$$

where $x, y \in C(I, \mathbb{R})$ (: the set of continuous functions from I into $\mathbb{R}$), $T > 0$, $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function and

$$G(t,s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}, & \text{if } 0 \le s < t \le T; \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}, & \text{if } 0 \le t < s \le T. \end{cases} \tag{3.15}$$

Definition 3.7. A *lower solution* for the integral equation (3.14) is an element $(\alpha, \beta) \in C^1(I, \mathbb{R}) \times C^1(I, \mathbb{R})$ such that

$$\begin{aligned}&\alpha'(t) + \lambda\beta(t) \le f(t, \alpha(t)) - f(t, \beta(t)), \quad \alpha(0) < \alpha(T), \\ &\beta'(t) + \lambda\alpha(t) \ge f(t, \beta(t)) - f(t, \alpha(t)), \quad \beta(0) \ge \beta(T),\end{aligned} \tag{3.16}$$

where $C^1(I, \mathbb{R})$ denotes the set of differentiable functions from I into $\mathbb{R}$.

Now, we prove the existence of solutions for the integral equation (3.14) by using the existence of a lower solution for the integral equation (3.14).

Theorem 3.8. *Let $\mathcal{A}$ be the class of the functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:*

(a) *φ is increasing;*

(b) *for any $x \ge 0$, there exists $k \in [0,1)$ such that $\varphi(x) < (k/2)x$.*

In the integral equation (3.14)*, suppose that there exists* $\lambda > 0$ *such that, for all* $x, y \in \mathbb{R}$ *with* $y > x$*,*

$$0 < f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \lambda \varphi(y - x), \tag{3.17}$$

where $\varphi \in \mathcal{A}$*. If a lower solution of the integral equation* (3.14) *exists, then a solution of the integral equation* (3.14) *exists.*

Proof. Define a mapping $F : C(I, \mathbb{R}) \times C(I, \mathbb{R}) \to C(I, \mathbb{R})$ by

$$F(x(t), y(t)) = \int_0^T G(t, s)[(f(s, x(s)) + \lambda x(s)) - (f(s, y(s)) + \lambda y(s))]ds. \tag{3.18}$$

Note that, if $(x(t), y(t)) \in C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is a coupled fixed-point of F, then $(x(t), y(t))$ is a solution of the integral equation (3.14).

Now, we check the hypotheses in Corollary 2.5 as follows:

(1) $X \times X = C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is a partially ordered set if we define the order relation in $X \times X$ as follows:

$$(u(t), v(t)) \leq (x(t), y(t)) \quad \text{iff } u(t) < x(t),\ v(t) \geq y(t) \tag{3.19}$$

for all $(x(t), y(t)), (u(t), v(t)) \in X \times X$ and $t \in I$.

(2) (X, d) is a complete metric space if we define a metric d as follows:

$$d(x(t), y(t)) = \sup_{t \in I}\{|x(t) - y(t)| : x(t), y(t) \in X\}. \tag{3.20}$$

(3) The mapping F has the mixed strict monotone property. In fact, by hypothesis, if $x_2 > x_1$, then we have

$$f(t, x_2) + \lambda x_2 > f(t, x_1) + \lambda x_1, \tag{3.21}$$

which implies that, for any $t \in I$,

$$\begin{aligned} &\int_0^T [f(s, x_2(s)) + \lambda x_2(s) - f(s, y(s)) - \lambda y(s)]G(t, s)ds \\ &\quad > \int_0^T [f(s, x_1(s)) + \lambda x_1(s) - f(s, y(s)) - \lambda y(s)]G(t, s)ds, \end{aligned} \tag{3.22}$$

that is,

$$F(x_2(t), y(t)) > F(x_1(t), y(t)). \tag{3.23}$$

Similarly, if $y_1 < y_2$, then we have

$$f(t, y_2) + \lambda y_2 > f(t, y_1) + \lambda y_1, \tag{3.24}$$

which implies that, for any $t \in I$,

$$\begin{aligned}\int_0^T &[f(s, x(s)) + \lambda x(s) - f(s, y_2(s)) - \lambda y_2(s)] G(t,s) ds \\ &< \int_0^T [f(s, x(s)) + \lambda x(s) - f(s, y_1(s)) - \lambda y_1(s)] G(t,s) ds,\end{aligned} \tag{3.25}$$

that is,

$$F(x(t), y_2(t)) < F(x(t), y_1(t)). \tag{3.26}$$

Now, we show that F satisfies (1.2). In fact, let $(\mathrm{x}, y) \leq (u, v)$ and $t \in I$. Then we have

$$\begin{aligned}&d(F(x(t), y(t)), F(u(t), v(t))) \\ &\quad = \sup\{|F(x(t), y(t)) - F(u(t), v(t))| : t \in I\} \\ &\quad = \sup_{t \in I}\left\{\left|\int_0^T G(t,s)[f(s, x(s)) + \lambda x(s) - f(s, y(s)) - \lambda y(s)] ds\right.\right. \\ &\qquad\qquad \left.\left. - \int_0^T G(t,s)[f(s, u(s)) + \lambda u(s) - f(s, v(s)) - \lambda v(s)] ds\right|\right\} \\ &\quad \leq \sup_{t \in I} \int_0^T G(t,s)|f(s, x(s)) + \lambda x(s) - f(s, u(s)) - \lambda u(s) \\ &\qquad\qquad + f(s, v(s)) + \lambda v(s) - f(s, y(s)) - \lambda y(s)| ds.\end{aligned} \tag{3.27}$$

Since the function $\varphi(x)$ is increasing and $(x, y) \leq (u, v)$, we have

$$\varphi(x(s) - u(s)) \leq \varphi(d(x(s), u(s))), \qquad \varphi(v(s) - y(s)) \leq \varphi(d(v(s), y(s))), \tag{3.28}$$

we obtain the following:

$$
\begin{aligned}
&d(F(x(t),y(t)),F(u(t),v(t)))\\
&\quad\le \sup_{t\in I}\int_0^T G(t,s)\left|\lambda\varphi(x(s)-u(s))+\lambda\varphi(v(s)-y(s))\right|ds\\
&\quad\le \lambda\sup_{t\in I}\int_0^T G(t,s)\left|\varphi(d(x(s),u(s)))+\varphi(d(v(s),y(s)))\right|ds\\
&\quad= \lambda(\varphi(d(x(s),u(s)))+\varphi(d(v(s),y(s))))\cdot\sup_{t\in I}\int_0^T G(t,s)ds\\
&\quad= \lambda(\varphi(d(x(s),u(s)))+\varphi(d(v(s),y(s))))\cdot\sup_{t\in I}\frac{1}{e^{\lambda T}-1}\left(\left[\frac{1}{\lambda}e^{\lambda(T+s-t)}\right]_0^t+\left[\frac{1}{\lambda}e^{\lambda(s-t)}\right]_t^T\right)\\
&\quad= \lambda(\varphi(d(x(s),u(s)))+\varphi(d(v(s),y(s))))\cdot\frac{1}{\lambda e^{\lambda T}-1}\left(e^{\lambda T}-1\right)\\
&\quad= \varphi(d(x(s),u(s)))+\varphi(d(v(s),y(s)))\\
&\quad< \frac{k}{2}\left[d(x(s),u(s))+d(v(s),y(s))\right]\\
&\quad\le \frac{k}{2}\sup\{|x(t)-u(t)|:t\in I\}+\frac{k}{2}\sup\{|v(t)-y(t)|:t\in I\}\\
&\quad= \frac{k}{2}\left[d(x(t),u(t))+d(y(t),v(t))\right].
\end{aligned}
\tag{3.29}
$$

Then, by Proposition 1.3, F is a generalized Meir-Keeler type contraction.

Finally, let $(\alpha(t),\beta(t))\in C^1(I,\mathbb{R})\times C^1(I,\mathbb{R})$ be a lower solution for the integral equation (3.14). Then we show that

$$\alpha < F(\alpha,\beta),\quad \beta\ge F(\beta,\alpha). \tag{3.30}$$

Indeed, we have $\alpha'(t)+\lambda\beta(t)\le f(t,\alpha(t))-f(t,\beta(t))$ for any $t\in I$ and so

$$\alpha'(t)+\lambda\alpha(t)\le f(t,\alpha(t))-f(t,\beta(t))+\lambda\alpha(t)-\lambda\beta(t) \tag{3.31}$$

for any $t\in I$. Multiplying by $e^{\lambda t}$ in (3.31), we get the following:

$$\left(\alpha(t)e^{\lambda t}\right)'\le\left[(f(t,\alpha(t))+\lambda\alpha(t))-(f(t,\beta(t))+\lambda\beta(t))\right]e^{\lambda t} \tag{3.32}$$

for any $t\in I$, which implies that

$$\alpha(t)e^{\lambda t}\le\alpha(0)+\int_0^t\left[(f(s,\alpha(s))+\lambda\alpha(s))-f(s,\beta(s))-\lambda\beta(s)\right]e^{\lambda s}ds \tag{3.33}$$

for any $t \in I$. This implies that

$$\alpha(0)e^{\lambda t} < \alpha(T)e^{\lambda T} \leq \alpha(0) + \int_0^T \left[f(s,\alpha(s)) + \lambda\alpha(s) - f\left(s,\beta(s)\right) - \lambda\beta(s)\right]e^{\lambda s}ds \tag{3.34}$$

and so

$$\alpha(0) < \int_0^T \frac{e^{\lambda s}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s) - f\left(s,\beta(s)\right) - \lambda\beta(s)\right]ds. \tag{3.35}$$

Thus it follows from (3.35) and (3.33) that

$$\begin{aligned} \alpha(t)e^{\lambda t} < &\int_t^T \frac{e^{\lambda s}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s) - f\left(s,\beta(s)\right) - \lambda\beta(s)\right]ds \\ &+ \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s) - f\left(s,\beta(s)\right) - \lambda\beta(s)\right]ds, \end{aligned} \tag{3.36}$$

and so

$$\begin{aligned} \alpha(t) < &\int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s) - f\left(s,\beta(s)\right) - \lambda\beta(s)\right]ds \\ &+ \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s) - f\left(s,\beta(s)\right) - \lambda\beta(s)\right]ds. \end{aligned} \tag{3.37}$$

Hence we have

$$\alpha(t) < \int_0^T G(t,s)\left[f(s,\alpha(s)) + \lambda\alpha(s) - f\left(s,\beta(s)\right) - \lambda\beta(s)\right]ds = F\left(\alpha(t),\beta(t)\right) \tag{3.38}$$

for any $t \in I$.

Similarly, we have $\beta(t) \geq F(\beta(t),\alpha(t))$. Therefore, by Corollary 2.5, F has a coupled fixed-point. □

Example 3.9. In the integral equation (3.14), we put $\lambda = 1.5$, $f(u,v) = u-v$ for all $(u,v) \in I\times\mathbb{R}$ and $T = 0.5$. Then f is a continuous function, and we have

$$\left(x(t),y(t)\right) = \left(\int_0^{0.5} G(t,s)\left[0.5x(s) - 0.5y(s)\right]ds,\ \int_0^{0.5} G(t,s)\left[0.5y(s) - 0.5x(s)\right]ds\right), \tag{3.39}$$

where $x, y \in C(I, \mathbb{R})$, and

$$G(t,s) = \begin{cases} \dfrac{e^{1.5(0.5+s-t)}}{e^{0.75}-1}, & \text{if } 0 \le s < t \le 0.5, \\ \dfrac{e^{1.5(s-t)}}{e^{0.75}-1}, & \text{if } 0 \le t < s \le 0.5. \end{cases} \tag{3.40}$$

Also, $(\alpha(t), \beta(t)) = (-2e^{-0.5t}, 3e^{-0.5t})$ is a lower solution of (3.39). Moreover, if we define $\varphi(x) = x/3$ for all $x \in [0, \infty)$, then φ is increasing and, for any $x > 0$, there exists $k = 1/1.1 \in [0, 1)$ such that $\varphi(x) = x/3 < (k/2)x = x/2.2$. For all $x, y \in \mathbb{R}$ with $y > x$, we have

$$0 < f(t,y) + \lambda y - [f(t,x) + \lambda x] = 0.5(y-x) \le \lambda\varphi(y-x) = 1.5\frac{y-x}{3} = 0.5(y-x). \tag{3.41}$$

Therefore, all the conditions of Theorem 3.8 hold, and a solution of (3.39) exists.

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