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Research Article

Legendre Polynomials Spectral Approximation for the Infinite-Dimensional Hamiltonian Systems

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This paper considers a Legendre polynomials spectral approximation for the infinite-dimensional Hamiltonian systems. As a consequence, the Legendre polynomials spectral semidiscrete system is a Hamiltonian system for the Hamiltonian system whose Hamiltonian operator is a constant differential operator.

1. Introduction

The numerical method for infinite-dimensional Hamiltonian Systems has been widely developed. One of the great challenges in the numerical analysis of PDEs is the development of robust stable numerical algorithms for Hamiltonian PDEs. For the numerical analysis, we always look for those discretizations which can preserve as much as possible some intrinsic properties of Hamiltonian equations. In fact, for Hamiltonian systems, the most important is its Hamiltonian structure. From this point of view, some semidiscrete numerical methods which are based on spectral methods have been developed. Spectral methods have proved to be particularly useful in infinite-dimensional Hamiltonian. Wang [1] discussed the semidiscrete Fourier spectral approximation of infinite-dimensional Hamiltonian systems, Hamiltonian of infinite-dimensional Hamiltonian systems, and Hamiltonian structure. Shen [2] studied the dual-Petrov-Galerbin method for third and higher odd-order equations. Ma and Sun [3] deliberated the third-order equations by using an interesting Legendre-Petrov-Galerbin method. So we consider that the Legendre polynomials basis is very important to analysis of the discretization of Hamiltonian systems.

In this paper, we consider a Legendre polynomials spectral approximation for the KdV equation and the wave equation. As a consequence, we show that the Legendre polynomials spectral semidiscrete system is also a Hamiltonian system for the Hamiltonian system whose Hamiltonian operator is a constant differential operator.

The paper is organized as follows. In Section 2, we give a brief description of infinite-dimensional Hamiltonian equations. In Section 3, we introduce semidiscrete Legendre polynomials spectral approximation. In the last two sections, we consider the Legendre polynomials spectral approximation for the boundary value problem of the KdV equation and the wave equation. Moreover, we give the conclusion about the Hamiltonian structure.

2. A Brief Description of Infinite-Dimensional Hamiltonian Equations

First, we get familiar with some basic knowledge about the infinite-dimensional Hamilton system.

Let the set $A = \{H[u] \equiv H(x, u^{(n)}) \mid \text{H is a infinite differentiable smooth function}\}$; here $u^{(n)} = (u^{1^T}, u^{2^T}, \ldots, u^{n^T})^T$, and u^i denotes the ith derivative of u. To each $H[u] \in A$, there exists a functional $\mathscr{H} = \int H[u]dx$, and the corresponding set of all functional is $\mathcal{F} = \{\mathscr{H} = \int H[u]dx \mid H[u] \in A\}$. $\delta\mathscr{H}/\delta u$ is the variational derivative of the functional $\mathscr{H} \in \mathcal{F}$. With the aid of the differential operator $\mathfrak{D}$, we can define a binary operator on $\mathcal{F}$:

$$\{\mathscr{H}, \mathcal{G}\} = \int \frac{\delta\mathscr{H}}{\delta u}\mathfrak{D}\frac{\delta\mathcal{G}}{\delta u}, \quad \forall\mathscr{H}, \mathcal{G} \in \mathcal{F}. \tag{2.1}$$

If this binary operator satisfies the following conditions:

(i) $\{\ ,\ \}$ is antisymmetric,

$$\{\mathscr{H}, \mathcal{G}\} = -\{\mathcal{G}, \mathscr{H}\}, \tag{2.2}$$

(ii) $\{\ ,\ \}$ is bilinear,

$$\{\alpha\mathscr{H} + \beta\mathcal{G}, \mathscr{K}\} = \alpha\{\mathscr{H}, \mathscr{K}\} + \beta\{\mathcal{G}, \mathscr{K}\}, \qquad \forall\alpha,\ \beta \in R, \tag{2.3}$$

(iii) $\{\ ,\ \}$ satisfies the Jacobi identity,

$$\{\{\mathscr{H}, \mathcal{G}\}, \mathscr{K}\} + \{\{\mathcal{G}, \mathscr{K}\}, \mathscr{H}\} + \{\{\mathscr{K}, \mathscr{H}\}, \mathcal{G}\} = 0, \tag{2.4}$$

for all functionals $\mathscr{H}, \mathcal{G}, \mathscr{K} \in \mathcal{F}$, then, it is called a Poisson bracket. In this case, $\mathfrak{D}$ is called Hamiltonian operator.

For given a Hamiltonian functional $\mathscr{H} \in \mathcal{F}$ and a Hamiltonian operator $\mathfrak{D}$, Hamiltonian equation takes the following form:

$$u_t = \mathfrak{D}\frac{\delta\mathscr{H}}{\delta u}. \tag{2.5}$$

This evolution equation is called an infinite-dimensional Hamiltonian system.

Consider the infinite-dimensional Hamiltonian system of the KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2.6}$$

It has the Hamiltonian structure

$$u_t = \mathfrak{D}\delta\mathscr{H}[u], \tag{2.7}$$

where $\mathfrak{D} = \partial_x$ is the Hamiltonian operator and

$$\mathscr{H}[u] = \int_{-1}^{1}\left(\frac{1}{2}u_x^2 - u^3\right)dx \tag{2.8}$$

is the Hamiltonian functional.

3. Semidiscrete Legendre Polynomials Spectral Approximation

Let $L_n(x)$ be the nth degree Legendre polynomial. The Legendre polynomials satisfy the three-term recurrence relation:

$$\begin{gathered} L_0(x) = 1, \qquad L_1(x) = x, \\ (n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x), \quad n \geq 1 \end{gathered} \tag{3.1}$$

and the orthogonality relation:

$$\begin{gathered} \int_{-1}^{1} L_k(x)L_j(x)dx = \frac{1}{k+(1/2)}\delta_{kj}, \\ L_n(\pm 1) = (\pm 1)^n. \end{gathered} \tag{3.2}$$

As suggested in [4], the choice of compact combinations of orthogonal polynomials as basis functions to minimize the bandwidth and the conditions number of the coefficient matrix is very important. Let $\{L_n\}$ be a sequence of orthogonal polynomials. As a general rule, for differential equations with m boundary conditions, our task is to look for basis functions in the form

$$\phi_k(x) = L_k(x) + \sum_{j=1}^{m} a_j^{(k)} L_{k+j}(x), \tag{3.3}$$

where $a_j^{(k)}$ $(j = 1, 2, \ldots, m)$ are chosen so that $\phi_k(x)$ satisfy the m homogeneous boundary conditions.

Suppose that $U = \{H(x) | H(x)$ is a smooth function, $x \in [-1, 1]\}$, for the fixed homogeneous boundary conditions

$$H(-1) = H(1) = 0. \tag{3.4}$$

As $m = 2$, (3.3) has the form

$$\phi_k(x) = L_k(x) + a_1 L_{k+1}(x) + a_2 L_{k+2}(x). \tag{3.5}$$

Using the basic properties of Legendre polynomials and the boundary value conditions, obviously

$$\phi_k(-1) = 0, \qquad \phi_k(1) = 0. \tag{3.6}$$

We can verify readily that

$$\phi_k(x) = L_k(x) - L_{k+2}(x). \tag{3.7}$$

Easily, we obtain $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x), \ldots$. The L^2-inner product on U is defined by

$$(p, q) = \int_{-1}^{1} p \cdot q dx, \quad \forall p,\ q \in U. \tag{3.8}$$

The basis functions $\phi_k(x)$ $(k = 1, 2, \ldots)$ can be orthogonalized standard on the L^2-inner product. Thus, we can get the sequence of standard orthogonal basis functions $\psi_k(x)$.

After carefully calculation, the orthogonal basis is

$$\begin{aligned} \psi_0 &= \frac{1}{4}\left(\sqrt{15} - 3x^2\right), \\ \psi_1 &= \frac{1}{4}\left(\sqrt{105}x - \sqrt{105}x^3\right), \\ \psi_2 &= \frac{1}{8}\left(-3\sqrt{5} + 24\sqrt{5}x^2 - 21\sqrt{5}x^4\right), \\ &\vdots \end{aligned} \tag{3.9}$$

Set

$$B = \operatorname{span}\{\psi_0, \psi_1, \psi_2, \ldots, \psi_N\} \subset U, \tag{3.10}$$

and set P as an orthogonal projection. $P : U \rightarrow B$,

$$u \longrightarrow \overline{u} = Pu = a_0\psi_0 + a_1\psi_1 + \cdots + a_N\psi_N, \tag{3.11}$$

where

$$a_n = \int_{-1}^{1} u(x)\varphi_n(x)dx, \quad n = 1, 2, 3, \ldots. \tag{3.12}$$

Denote $\widehat{B} = \{\widehat{u} = (a_0, a_1, \ldots, a_N)^T \in R^{N+1}\}$. The inner product of $\widehat{B}$ is usually denoted by Euclidean inner $\langle \cdot, \cdot \rangle$, that is,

$$\langle \widehat{p}, \widehat{q} \rangle = a_0 \cdot \widetilde{a}_0 + a_1 \cdot \widetilde{a}_1 + a_2 \cdot \widetilde{a}_2 + \cdots + a_N \cdot \widetilde{a}_N, \quad \forall \widehat{p}, \widehat{q} \in \widehat{B}, \tag{3.13}$$

where $\widehat{q} = (\widetilde{a}_0, \widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_N)^T$. Set $I : \widehat{B} \to B$,

$$\widehat{u} \longrightarrow \overline{u} = I\widehat{u} = a_0\varphi_0 + a_1\varphi_1 + \cdots + a_N. \tag{3.14}$$

Denote $\widehat{P} = I^{-1} \circ P : U \to \widehat{B}$,

$$u \longrightarrow \widehat{P}u = (a_0, a_1, a_2, \ldots, a_N)^T. \tag{3.15}$$

Hamiltonian equation

$$u_t = \mathfrak{D}\frac{\delta \mathscr{H}}{\delta u} \tag{3.16}$$

has the special Poisson structure; so we can exploit it to design numerical approximations. We can discretize Hamiltonian operator $\mathfrak{D}$ and Hamiltonian functionals; then a numerical bracket can be defined.

The discretization of the Hamiltonian operator $\mathfrak{D}$ is

$$\begin{gathered} \widehat{\mathfrak{D}} = \widehat{P} \circ \mathfrak{D} \circ I : \widehat{B} \longrightarrow \widehat{B}, \\ \widehat{\mathfrak{D}}(\widehat{p})\widehat{q} = \widehat{p} \circ \widehat{\mathfrak{D}}(I\widehat{p}) \cdot I\widehat{q}, \quad \forall \widehat{p}, \widehat{q} \in \widehat{B}. \end{gathered} \tag{3.17}$$

The discretization of a functional $H(x)$ in U is

$$\widehat{H}(\widehat{u}) = \int_{-1}^{1} H(I\widehat{u})dx, \quad \forall \widehat{u} \in \widehat{B}. \tag{3.18}$$

Let $\overline{U}$ be the set of discrete functionals; then we can define a bracket on $\overline{U}$,

$$\begin{gathered} \left\{\widehat{H}, \widehat{G}\right\} = \nabla\widehat{H}\widehat{\mathfrak{D}}\left(\nabla\widehat{G}\right)^T, \\ \nabla\widehat{H} = \left(\frac{\partial\widehat{H}}{\partial\widehat{u}_1}, \frac{\partial\widehat{H}}{\partial\widehat{u}_2}, \ldots, \frac{\partial\widehat{H}}{\partial\widehat{u}_N}\right), \end{gathered} \tag{3.19}$$

which is an approximation of bracket $\{H, G\}$.

Now we define the semidiscrete approximative equation in $\widehat{B}$ of the infinite-dimensional Hamiltonian system $u_t = \mathfrak{D}\dfrac{\delta\mathscr{H}}{\delta u}$ as

$$\frac{d\widehat{u}}{dt} = \widehat{\mathfrak{D}}(\nabla H(\widehat{u}))^T. \tag{3.20}$$

If $\widehat{\mathfrak{D}}$ is still a Hamiltonian operator, then (3.20) is exactly a finite-dimensional Hamiltonian system. The function $\widehat{H}(Pu)$ is a conservation law if and only if (3.20) always preserves conservation law.

4. Legendre Polynomials Spectral Approximation for the Boundary Value Problem of the KdV Equation

We consider the KdV equation with the fixed boundary conditions discussed above:

$$\begin{gathered} u_t + 6uu_x + u_{xxx} = 0, \quad x \in (-1,\ 1),\ t \in [0, T], \\ u(-1,\ t) = u(1,\ t) = 0, \quad t \in [0,\ T]. \end{gathered} \tag{4.1}$$

The KdV equation can be written as Hamiltonian form:

$$u_t = \partial_x \frac{\delta\mathscr{H}_1}{\delta u}, \tag{4.2}$$

where the Hamiltonian operator is $\mathfrak{D}_1 = \partial_x$ and the the Hamiltonian functional is $\mathscr{H}_1 = \int_{-1}^{1}((1/2)u_x^2 - u^3)dx$.

By above analysis and the chosen orthogonal basis, for $N = 2$,

$$\begin{aligned} \widehat{\mathfrak{D}}_1 &= \begin{pmatrix} 0 & \dfrac{\sqrt[2]{7}}{2} & 0 \\ -\dfrac{\sqrt{7}}{2} & 0 & \dfrac{\sqrt{21}}{2} \\ 0 & -\dfrac{\sqrt{21}}{2} & 0 \end{pmatrix}, \\ \widehat{H}_1(\widehat{u}) &= \int_{-1}^{1} \left[\frac{1}{2}(a_0\psi_0 + a_1\psi_1 + a_2\psi_2)_x^2 - (a_0\psi_0 + a_1\psi_1 + a_2\psi_2)^3\right] dx \\ &= \frac{5}{4}a_0^2 + \frac{21}{4}a_1^2 + \frac{\sqrt{3}}{2}a_0a_2 + \frac{51}{4}a_2^2 + \frac{3\sqrt{15}}{14}a_0^3 - \frac{3\sqrt{15}}{14}a_0^2a_2 \\ &\quad + \frac{15\sqrt{5}}{22}a_1^2a_2 + \frac{423\sqrt{5}}{2002}a_2^3 + \frac{\sqrt{15}}{2}a_0a_1^2 + \frac{69\sqrt{5}}{154}a_0a_2^2. \end{aligned} \tag{4.3}$$

Then

$$\nabla \widehat{H}_1^T = \begin{pmatrix} \frac{5}{2}a_0 + \frac{\sqrt{3}}{2}a_2 + \frac{9\sqrt{15}}{14}a_0^2 - \frac{3\sqrt{5}}{7}a_0a_2 + \frac{\sqrt{15}}{2}a_1^2 + \frac{\sqrt{5}}{154}a_2^2 \\ \frac{21}{2}a_1 + \frac{15\sqrt{5}}{11}a_1a_2 + \frac{\sqrt{15}}{7}a_0a_1 \\ \frac{\sqrt{3}}{2}a_0 + \frac{51}{2}a_2 - \frac{3\sqrt{5}}{14}a_0^2 + \frac{15\sqrt{5}}{22}a_1^2 + \frac{1269\sqrt{5}}{2002}a_2^2 + \frac{69\sqrt{5}}{77}a_0a_2 \end{pmatrix}. \tag{4.4}$$

This equations can be written as Hamiltonian form in another way, that is,

$$u_t = (\partial_{xxx} + 4u\partial_x + 2u_x I)\frac{\delta \mathscr{H}_2}{\delta u}, \tag{4.5}$$

where the Hamiltonian operator is $\mathfrak{D}_2 = \partial_{xxx} + 4u\partial_x + 2u_x I$ abd the Hamiltonian functional is $\mathscr{H}_2 = \int_{-1}^{1}(-(1/2)u^2)dx$.

In the same theory, for $N = 2$, we can get

$$\widehat{\mathfrak{D}}_2 = \begin{pmatrix} -2\sqrt{\frac{15}{7}}a_1 & -\frac{13\sqrt{7}}{2} + 4\sqrt{\frac{15}{7}}a_0 - 6\sqrt{\frac{5}{7}}a_2 & 10\sqrt{\frac{5}{7}}a_1 \\ -\sqrt{7} - 2\sqrt{\frac{15}{7}}a_0 - 4\sqrt{\frac{5}{7}}a_2 & 0 & \mathcal{R} \\ -4\sqrt{\frac{5}{7}}a_1 & -\frac{5\sqrt{21}}{2} - 6\sqrt{\frac{5}{7}}a_0 - \frac{16}{11}\sqrt{\frac{15}{7}}a_2 & \frac{8}{11}\sqrt{\frac{15}{7}}a_1 \end{pmatrix}$$
$$\widehat{H}_2(\widehat{u}) = \int_{-1}^{1}\left[-\frac{1}{2}(a_0\psi_0 + a_1\psi_1 + a_2\psi_2)^2\right]dx = -\frac{1}{2}\left(a_0^2 + a_1^2 + a_2^2\right) \tag{4.6}$$

where $\mathcal{R}$ denotes $-20\sqrt{21} + 10\sqrt{\frac{5}{7}}a_0 + \frac{8}{10}\sqrt{\frac{15}{7}}a_2$.

Then

$$\nabla \widehat{H}_2^T = (-a_0, \ -a_1, \ -a_2). \tag{4.7}$$

The corresponding semidiscrete approximation is

$$\frac{d\widehat{u}}{dt} = \widehat{\mathfrak{D}}(\nabla H(\widehat{u}))^T, \quad \widehat{u} = (a_0, a_1, a_2)^T. \tag{4.8}$$

It is easy to verify that $\widehat{\mathfrak{D}}_1$ is Hamiltonian operator; so the approximating system can be written as

$$\frac{d\widehat{u}}{dt} = \widehat{\mathfrak{D}}_1(\nabla H_1(\widehat{u}))^T, \tag{4.9}$$

adifferent Hamiltonian form, and it can be verified that $\widehat{\mathfrak{D}}_2$ is not a Hamiltonian operator. As $\widehat{\mathfrak{D}}_1$ is a constant antisymmetric matrix,

$$\frac{d\hat{u}}{dt} = \widehat{P} \circ \partial_x \circ I \cdot \frac{\delta H_1}{\delta u}(I\hat{u}) \tag{4.10}$$

is a finite-dimensional Hamiltonian system. So the approximating system can preserve the Poisson structure given by Hamiltonian operator $\widehat{\mathfrak{D}}_1$.

Theorem 4.1. *The equation $d\hat{u}/dt = \widehat{\mathfrak{D}}_1(\nabla H_1(\hat{u}))^T$ is the discretization of the KdV equation $u_t + 6uu_x + u_{xxx} = 0$; then $d\hat{u}/dt = \widehat{\mathfrak{D}}_1(\nabla H_1(\hat{u}))^T$ has the property of energy conservation law.*

Proof.

$$\begin{gathered}
\widehat{\mathfrak{D}} = \widehat{P} \circ \mathfrak{D} \circ I : \widehat{B} \longrightarrow \widehat{B}, \\
I : \widehat{B} \longrightarrow B, \\
\hat{u} \longrightarrow \overline{u} = I\hat{u} = a_0\psi_0 + a_1\psi_1 + \cdots + a_N\psi_N, \\
P : U \longrightarrow B, \\
u \longrightarrow \overline{u} = Pu = a_0\psi_0 + a_1\psi_1 + \cdots + a_N\psi_N, \\
\widehat{P} = I^{-1} \circ P : U \longrightarrow \widehat{B}, \\
u \longrightarrow \widehat{P}u = (a_0, a_1, a_2, \ldots, a_N)^T, \\
\widehat{\mathfrak{D}} = \begin{pmatrix}
\frac{(\mathfrak{D}\psi_1, \psi_1)}{(\psi_1, \psi_1)} & \frac{(\mathfrak{D}\psi_2, \psi_1)}{(\psi_1, \psi_1)} & \cdots & \frac{(\mathfrak{D}\psi_N, \psi_1)}{(\psi_1, \psi_1)} \\
\frac{(\mathfrak{D}\psi_1, \psi_2)}{(\psi_2, \psi_2)} & \frac{(\mathfrak{D}\psi_2, \psi_2)}{(\psi_2, \psi_2)} & \cdots & \frac{(\mathfrak{D}\psi_N, \psi_2)}{(\psi_2, \psi_2)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{(\mathfrak{D}\psi_1, \psi_N)}{(\psi_N, \psi_N)} & \frac{(\mathfrak{D}\psi_2, \psi_N)}{(\psi_N, \psi_N)} & \cdots & \frac{(\mathfrak{D}\psi_N, \psi_N)}{(\psi_N, \psi_N)}
\end{pmatrix}.
\end{gathered} \tag{4.11}$$

$\psi_0, \psi_1, \psi_2, \ldots, \psi_N$ are a sequence of standard orthogonal basis,

$$\begin{gathered}
(\psi_i, \psi_i) = 1, \quad i = 1, 2, \ldots, N, \\
(\mathfrak{D}\psi_i, \psi_j) = -(\mathfrak{D}\psi_j, \psi_i), \quad (i \neq j), \\
(\mathfrak{D}\psi_i, \psi_i) = (\partial_x\psi_i, \psi_i) = 0, \quad i = 1, 2, \ldots, N.
\end{gathered} \tag{4.12}$$

$\widehat{\mathfrak{D}}_1$ is a constant antisymmetric matrix.

According to $\{\widehat{H}, \widehat{G}\} = \nabla\widehat{H}\widehat{\mathfrak{D}}(\nabla\widehat{G})^T$, then $\{\widehat{H}_1, \widehat{H}_1\} = \nabla\widehat{H}_1\widehat{\mathfrak{D}}(\nabla\widehat{H}_1)^T = 0$.

The function $\widehat{H}_1(\hat{u})$ is a conservation law of energy, that is, $\dfrac{d\hat{u}}{dt} = \widehat{\mathfrak{D}}_1(\nabla H_1(\hat{u}))^T$ has the property of energy conservation law. □

5. Legendre Polynomials Spectral Approximation for the Boundary Value Problem of the Wave Equation

Now we consider the wave equation with the fixed boundary conditions discussed above:

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad x \in (-1, 1),\ t \in [0, T], \\ u(-1, t) = u(1, t) = 0, \quad t \in [0, T].\end{gathered} \tag{5.1}$$

It can be rewritten as two forms of the first-order equations:

$$\begin{gathered}\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x}, \\ \frac{\partial v}{\partial t} = \frac{\partial u}{\partial x}.\end{gathered} \tag{5.2}$$

This equation can be written as Hamiltonian form:

$$\frac{\partial \overline{u}}{\partial t} = \mathfrak{D}_1 \frac{\delta H_1}{\delta u}, \quad \overline{u} = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{5.3}$$

The Hamiltonian operator is $\mathfrak{D}_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}$, and the corresponding Hamiltonian functional is $H_1 = 1/2 \int_{-1}^{1} (u^2 + v^2) dx$.

There is another way to write the equation into Hamiltonian form, that is,

$$\frac{\partial \overline{u}}{\partial t} = \mathfrak{D}_2 \frac{\delta H_1}{\delta u}, \quad \overline{u} = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{5.4}$$

The corresponding Hamiltonian operator is $\mathfrak{D}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and the Hamiltonian functional is $H_2 = 1/2 \int_{-1}^{1} (u_x^2 + v^2) dx$.

In this case, the element in U is denoted by $u = (u_1, u_2)^T$. The inner product is denoted by $(u, v) = \sum_{i=1}^{2} (u_i, v_i)$, $u, v \in U$, where $(u_i, v_i) = \int_{-1}^{1} u_i v_i dx$.

Take the orthogonal basis:

$$\begin{pmatrix} \psi_0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} \psi_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}, \ldots. \tag{5.5}$$

Set

$$B = \operatorname{span}\left\{ \begin{pmatrix} \psi_0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_1 \end{pmatrix}, \ldots, \begin{pmatrix} \psi_N \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_N \end{pmatrix} \right\} \subset U. \tag{5.6}$$

That is, B is $2N + 1$-dimensional subspace of U.

The orthogonal projection is $P : U \rightarrow B$,

$$u \longrightarrow \overline{u} = Pu = a_0 \begin{pmatrix} \psi_0 \\ 0 \end{pmatrix} + \tilde{a}_0 \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix} + \cdots + a_N \begin{pmatrix} \psi_N \\ 0 \end{pmatrix} + \tilde{a}_N \begin{pmatrix} 0 \\ \psi_N \end{pmatrix}. \tag{5.7}$$

Denote $\widehat{B} = \{\widehat{u} = (a_0, \tilde{a}_0, a_1, \tilde{a}_1, \ldots, a_N, \tilde{a}_N)^T \in R^{2N+2}\}$. The inner product of $\widehat{B}$ is usually denoted by Euclidean inner $\langle \cdot, \cdot \rangle$, that is,

$$\langle \widehat{p}, \widehat{q} \rangle = a_0 b_0 + \tilde{a}_0 \tilde{b}_0 + \cdots + a_N b_N + \tilde{a}_N \tilde{b}_N, \tag{5.8}$$

where $\widehat{p} = (a_0, \tilde{a}_0, \ldots, a_N, \tilde{a}_N)^T$ and $\widehat{q} = (b_0, \tilde{b}_0, \ldots, b_N, \tilde{b}_N)^T$.

Set $I : \widehat{B} \rightarrow B$,

$$\widehat{u} \longrightarrow \overline{u} = I\widehat{u} = a_0 \begin{pmatrix} \psi_0 \\ 0 \end{pmatrix} + \tilde{a}_0 \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix} + \cdots + a_N \begin{pmatrix} \psi_N \\ 0 \end{pmatrix} + \tilde{a}_N \begin{pmatrix} 0 \\ \psi_N \end{pmatrix}. \tag{5.9}$$

Denote $\widehat{P} = I^{-1} \circ P : U \rightarrow \widehat{B}$,

$$u \longrightarrow \widehat{P}u = (a_0, \tilde{a}_0, a_1, \tilde{a}_1, a_2, \tilde{a}_2, \ldots, a_N, \tilde{a}_N)^T. \tag{5.10}$$

The discretization of the Hamiltonian operator $\mathfrak{D}$ is

$$\begin{gathered} \widehat{\mathfrak{D}} = \widehat{P} \circ \mathfrak{D} \circ I : \widehat{B} \longrightarrow \widehat{B}, \\ \widehat{\mathfrak{D}}(\widehat{p})\widehat{q} = \widehat{p} \circ \widehat{\mathfrak{D}}(I\widehat{p}) \cdot I\widehat{q}, \quad \forall \widehat{p}, \widehat{q} \in \widehat{B}. \end{gathered} \tag{5.11}$$

The discretization of a functionals $H(x)$ in U is

$$\widehat{H}(\widehat{u}) = \int_{-1}^{1} H(I\widehat{u})dx, \quad \forall \widehat{u} \in \widehat{B}. \tag{5.12}$$

By the above analysis and the chosen orthogonal basis, for $N = 2$,

$$\widehat{\mathfrak{D}}_1 = \begin{pmatrix} 0 & 0 & \frac{\sqrt{7}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{7}}{2} & 0 & 0 \\ -\frac{\sqrt{7}}{2} & 0 & 0 & 0 & \frac{\sqrt{21}}{2} & 0 \\ 0 & -\frac{\sqrt{7}}{2} & 0 & 0 & 0 & \frac{\sqrt{21}}{2} \\ 0 & 0 & -\frac{\sqrt{21}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{21}}{2} & 0 & 0 \end{pmatrix},$$
$$\begin{aligned}\widehat{H}_1(\widehat{u}) &= \frac{1}{2}\int_{-1}^{1} \left[\left(a_0\psi_0 + a_1\psi_1 + a_2\psi_2\right)^2 + \left(\widetilde{a}_0\psi_0 + \widetilde{a}_1\psi_1 + \widetilde{a}_2\psi_2\right)^2\right]dx \\ &= \frac{1}{2}\left(a_0^2 + \widetilde{a}_0^2 + a_1^2 + \widetilde{a}_1^2 + a_2^2 + \widetilde{a}_2^2\right).\end{aligned} \tag{5.13}$$

Then

$$\nabla \widehat{H}_1^T = (a_0, \widetilde{a}_0, a_1, \widetilde{a}_1, a_2, \widetilde{a}_2). \tag{5.14}$$

The corresponding semidiscrete approximation is

$$\begin{aligned}\frac{da_0}{dt} &= \frac{\sqrt{7}}{2}a_1, \\ \frac{d\widetilde{a}_0}{dt} &= \frac{\sqrt{7}}{2}\widetilde{a}_1, \\ \frac{da_1}{dt} &= -\frac{\sqrt{7}}{2}a_0 + \frac{\sqrt{21}}{2}a_2, \\ \frac{d\widetilde{a}_1}{dt} &= -\frac{\sqrt{7}}{2}\widetilde{a}_0 + \frac{\sqrt{21}}{2}\widetilde{a}_2, \\ \frac{da_2}{dt} &= -\frac{\sqrt{21}}{2}a_1, \\ \frac{d\widetilde{a}_2}{dt} &= -\frac{\sqrt{21}}{2}\widetilde{a}_1.\end{aligned} \tag{5.15}$$

For the other form, we can also obtain

$$\widehat{\mathfrak{D}}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{5.16}$$

$$\begin{aligned} \widehat{H}_2(\widehat{u}) &= \frac{1}{2}\int_{-1}^{1}\left[\left(a_0\psi_0 + a_1\psi_1 + a_2\psi_2\right)_x^2 + \left(\tilde{a}_0\psi_0 + \tilde{a}_1\psi_1 + \tilde{a}_2\psi_2\right)^2\right]dx \\ &= \frac{1}{2}\left(\frac{5}{2}a_0^2 + \tilde{a}_0^2 + \frac{21}{2}a_1^2 + \tilde{a}_1^2 + \sqrt{3}a_0a_2 + \tilde{a}_2^2\right). \end{aligned} \tag{5.17}$$

Then

$$\nabla\widehat{H}_2 = \left(\frac{5}{2}a_0 + \frac{\sqrt{3}}{2}a_2,\ \tilde{a}_0,\ \frac{21}{2}a_1,\ \tilde{a}_1,\ \frac{51}{2}a_2 + \frac{\sqrt{3}}{2}a_0,\ \tilde{a}_2\right). \tag{5.18}$$

The corresponding semidiscrete approximation is

$$\begin{aligned} \frac{da_0}{dt} &= \tilde{a}_0, \\ \frac{d\tilde{a}_0}{dt} &= -\frac{5}{2}a_0 - \frac{\sqrt{3}}{2}a_2, \\ \frac{da_1}{dt} &= \tilde{a}_1, \\ \frac{d\tilde{a}_1}{dt} &= -\frac{21}{2}a_1, \\ \frac{da_2}{dt} &= \tilde{a}_2, \\ \frac{d\tilde{a}_2}{dt} &= -\frac{\sqrt{3}}{2}a_0 - \frac{51}{2}a_2. \end{aligned} \tag{5.19}$$

Similar to the analysis of the KdV equation, for the situation of $N = 2$, we can verify that $\widehat{\mathfrak{D}}_1$ and $\widehat{\mathfrak{D}}_2$ are all Hamiltonian operators; so the approximating system can be written as

$d\hat{u}/dt = \widehat{\mathfrak{D}}_1(\nabla H_1(\hat{u}))^T$ and $d\hat{u}/dt = \widehat{\mathfrak{D}}_2(\nabla H_2(\hat{u}))^T$, two different Hamiltonian forms. As D_1 and $\widehat{\mathfrak{D}}_2$ both are constant antisymmetric matrix for $N > 2$,

$$\frac{d\hat{u}}{dt} = \widehat{P} \circ \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \circ I \cdot \frac{\delta H_1}{\delta u}(I\hat{u}),$$
$$\frac{d\hat{u}}{dt} = \widehat{P} \circ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \circ I \cdot \frac{\delta H_2}{\delta u}(I\hat{u}) . \tag{5.20}$$

are finite-dimensional Hamiltonian systems. The approximating systems can preserve the Poisson structure given by Hamiltonian operators $\widehat{\mathfrak{D}}_1$ and $\widehat{\mathfrak{D}}_2$.

Theorem 5.1. *The equations $d\hat{u}/dt = \widehat{\mathfrak{D}}_1(\nabla H_1(\hat{u}))^T$ and $d\hat{u}/dt = \widehat{\mathfrak{D}}_2(\nabla H_2(\hat{u}))^T$ are the discretizations of the 1-dim wave equation*

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial t} = \frac{\partial u}{\partial x}. \tag{5.21}$$

Then $d\hat{u}/dt = \widehat{\mathfrak{D}}_1(\nabla H_1(\hat{u}))^T$ and $d\hat{u}/dt = \widehat{\mathfrak{D}}_2(\nabla H_2(\hat{u}))^T$ both have the property of energy conservation law.

The proof of Theorem 5.1 is similar to that of Theorem 4.1.

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