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Research Article

The Uncertainty Measure of Hierarchical Quotient Space Structure

Qinghua Zhang[1,2] and Guoyin Wang[2]

[1] *College of Mathematics & Physics, Chongqing University of Posts and Telecommunications, Chongqing 400065, China*
[2] *Institute of Computer Science & Technology, Chongqing University of Posts and Telecommunications, Chongqing 400065, China*

Correspondence should be addressed to Qinghua Zhang, zhangqh@cqupt.edu.cn

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In the application of fuzzy reasoning, researchers usually choose the membership function optionally in some degree. Even though the membership functions may be different for the same concept, they can generally get the same (or approximate) results. The robustness of the membership function optionally chosen has brought many researchers' attention. At present, many researchers pay attention to the structural interpretation (definition) of a fuzzy concept, and find that a hierarchical quotient space structure may be a better tool than a fuzzy set for characterizing the essential of fuzzy concept in some degree. In this paper, first the uncertainty of a hierarchical quotient space structure is defined, the information entropy sequence of a hierarchical quotient space structure is proposed, the concept of isomorphism between two hierarchical quotient space structures is defined, and the sufficient condition of isomorphism between two hierarchical quotient space structures is discovered and proved also. Then, the relationships among information entropy sequence, hierarchical quotient space structure, fuzzy equivalence relation, and fuzzy similarity relation are analyzed. Finally, a fast method for constructing a hierarchical quotient space structure is presented.

1. Introduction

Since the fuzzy set theory was proposed by Zadeh in 1965 [1], it has been successfully applied to many application areas, such as fuzzy control, fuzzy reasoning, fuzzy clustering analysis, and fuzzy decision. A fuzzy set interprets fuzzy concept with some membership function, and, as a fundamental tool for revealing and analyzing uncertain problem, it has been frequently used in many real-world applications. When people are faced with the interpretation problem of the membership function generally in the applications, though each person has his/her own option about the meaning of the same subjective concept such as "tall" and "big" and he/she always has his/her membership function for the same fuzzy concept such as triangular, trapezoidal, or Gaussian, they can get the same (or approximate) results. The robustness of the optionally chosen membership functions attracts many researches' attention [2–6]. Lin [2] interpreted memberships as probabilities. Liang and Song [3] regarded the membership function value as an independent and identically distributed random variable and proved that the mean of membership functions exists for all the elements of the universe of discourse. Unfortunately, these results were obtained based on a strong assumption; that is, the membership function value is assumed to be an independent and identically distributed random variable. Lin [4] presented a topological definition of the fuzzy set by using neighborhood systems, discussed the properties of fuzzy set from its structure, and proposed the concept of granular fuzzy set. Lin's works provide a structural interpretation of member function. Afterward, B. Zhang and L. Zhang [5, 6] developed a structural definition of membership function and found that, for a fuzzy set (concept), it may probably be described by different types of membership functions, as long as the structures of these membership functions are the same, they characterize the fuzzy set (i.e., concept) with the same property. That is to say, although these membership functions are different in appearance, they are the same in essence. The structural description is more essential to a fuzzy concept than the membership function. This structure is called hierarchical quotient space structure in quotient space theory developed by B. Zhang and L. Zhang [6]. As a well-known fact, it is one of the basic characteristics in human problem solving that a person has a kind of ability to conceptualize the world at different granularities and translate from one abstraction level to others easily. Such is a powerful ability of human being to deal with complex problem [5]. According to Zadeh's viewpoints [7, 8], both rough set theory [9] and quotient space theory can be used to describe a "crisp" granule world, while fuzzy set can be used to describe a "fuzzy" granule world. However, "fuzzy" granule and "crisp" granule are relative; that is, the "fuzzy" granule and "crisp" granule are two kinds of different manifestations of a concept in different granularity levels, and they can be transformed into each other with the changing of granularity [10]. Li et al. [11] presented that uncertainty (fuzzy) and certainty (crisp) are not opposite absolutely, and they can be transformed into each other in some degree. The fuzzy quotient space theory [10] combining fuzzy set theory and quotient space theory was proposed by B. Zhang and L. Zhang. It was a bridge from "fuzzy" granule world to "crisp" granule world and could better uncover the characteristics of human beings dealing with uncertain problems and better interpret the relationship between "fuzziness" and "crispness." Many important conclusions about the fuzzy quotient space theory could be referred to [10], and the fuzzy quotient space theory for the cut relation of fuzzy equivalence relation with any threshold was discussed by Zhang et al. [12]. In fuzzy quotient space theory, a fuzzy equivalence relation and a hierarchical quotient space structure are one to one, and the hierarchical quotient space structure is a structural description of fuzzy equivalence relation.

The isomorphic fuzzy equivalence relations have the same hierarchical quotient space structure [10]; that is, they have the same classification ability to the objects in domain X. Therefore, the different fuzzy similarity relations may induce the same fuzzy equivalent relation, and the different fuzzy equivalent relations may have the same hierarchical quotient space structure and classification result as long as they are isomorphic. Recently, as a kind of structural description of a fuzzy concept, the hierarchical quotient space structure attracts many researchers' attention. The further study about hierarchical quotient space structure of fuzzy equivalence relation with ε-similarity could be referred to [13]. Tsekouras et al. [14] proposed a hierarchical fuzzy clustering approach. Pedrycz and Reformat [15] presented a hierarchical fuzzy C-means (FCM) method in a stepwise discovery of structure in data. Tang et al. [16] discussed the sufficient condition of isomorphism between fuzzy similarity relations and uncovered the relationships between fuzzy similarity relation and fuzzy equivalence relation. Although each person has his/her own membership function for the same concept, and he/she may get the different fuzzy similarity relations, he/she may finally obtain the same (or isomorphic) fuzzy equivalent relation which can produce the same hierarchical quotient space structure and classification of objects in the domain X. What is the reason that the different fuzzy similarity relations can produce the same hierarchical quotient space structure and the same classification? How can we measure the classification quality of a hierarchical quotient space structure? How much information (knowledge) does a hierarchical quotient space structure contain? How can we measure the uncertainty of a hierarchical quotient space structure? These fundamental issues in hierarchical quotient space theory remain open. In this paper, the fuzzy similarity relation, the fuzzy equivalence relation, the hierarchical quotient space structure, and the entropy sequence of a hierarchical quotient space structure are studied. A fast-constructing hierarchical quotient space structure method is presented. These works uncover the nature of the hierarchical quotient space structure further.

The paper is organized as follows. Some relevant preliminary concepts are reviewed briefly in Section 2. In Section 3, the information entropy sequence of a hierarchical quotient space structure is discussed. In Section 4, a fast-constructing hierarchical quotient space structure method is presented. The paper is concluded in Section 5.

2. Preliminary Concepts

For convenience, some preliminary concepts are reviewed or defined at first. Let X be a Cantor set.

Definition 2.1 (see [6]). Let $\widetilde{R}$ be a fuzzy relation on X. If it satisfies,

(1) for all $x \in X, \widetilde{R}(x,x) = 1$,

(2) for all $x, y \in X, \widetilde{R}(x,y) = \widetilde{R}(y,x)$,

then $\widetilde{R}$ is called a fuzzy similarity relation on X.

Definition 2.2 (see [6]). Let $\widetilde{R}$ be a fuzzy relation on X. If it satisfies,

(1) for all $x \in X, \widetilde{R}(x,x) = 1$,

(2) for all $x, y \in X, \widetilde{R}(x,y) = \widetilde{R}(y,x)$,

(3) for all $x, y, z \in X, \widetilde{R}(x,z) \geq \sup_{y\in X} \min(\widetilde{R}(x,y), \widetilde{R}(y,z))$,

the $\widetilde{R}$ is called a fuzzy equivalence relation on X and is denoted by R.

Proposition 2.3 (see [17]). *Let $\widetilde{R}$ be a fuzzy similarity relation on X, and let $\widehat{R}$ denote the transitive closure of $\widetilde{R}$. Then, for all $m \geq n$, $\widehat{R} = \widetilde{R}^m$.*

According to Proposition 2.3, a fuzzy equivalence relation can be induced from a fuzzy similarity relation by $\widetilde{R} \to \widetilde{R}^2 \to (\widetilde{R}^2)^2 \to \cdots \to \widetilde{R}^{2^k} = \widehat{R}$. Where $k \geq \log_2 n$.

Proposition 2.4 (see [17]). *Let R be a fuzzy equivalence relation on X, and $R_\lambda = \{(x,y) \mid R(x,y) \geq \lambda\}$ $(0 \leq \lambda \leq 1)$, then R_λ is a crisp equivalence relation on X, R_λ is called cut-relation of R.*

Proposition 2.4 shows that R_λ is a crisp equivalence relation on X, and its corresponding quotient space is denoted by $X(\lambda)$. Let R be a fuzzy equivalence relation on X, $S = \{R(x, y) \mid x, y \in X\}$, S is called the value domain of R.

Definition 2.5 (see [6]). Let R be a fuzzy equivalence relation on X, and S is the value domain of X. The set $\pi_X(R) = \{X(\lambda) \mid \lambda \in S\}$ is called the hierarchical quotient space structure of R.

Example 2.6. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, and R_1 is a fuzzy equivalence relation on X, the corresponding relation matrix M_{R_1} is defined as follows,

$$M_{R_1} = \begin{bmatrix} 1 & 0.4 & 0.8 & 0.5 & 0.5 \\ 0.4 & 1 & 0.4 & 0.4 & 0.4 \\ 0.8 & 0.4 & 1 & 0.5 & 0.5 \\ 0.5 & 0.4 & 0.5 & 1 & 0.6 \\ 0.5 & 0.4 & 0.5 & 0.6 & 1 \end{bmatrix}. \tag{2.1}$$

Its corresponding quotient space sequence is defined as follows,

$$\begin{aligned} X(\lambda_1) &= \{\{x_1, x_2, x_3, x_4, x_5\}\}, \quad \text{where } 0 \le \lambda_1 \le 0.4; \\ X(\lambda_2) &= \{\{x_1, x_3, x_4, x_5\}, \{x_2\}\}, \quad \text{where } 0.4 < \lambda_2 \le 0.5; \\ X(\lambda_3) &= \{\{x_1, x_3\}, \{x_2\}, \{x_4, x_5\}\}, \quad \text{where } 0.5 < \lambda_3 \le 0.6; \\ X(\lambda_4) &= \{\{x_1, x_3\}, \{x_2\}, \{x_4\}, \{x_5\}\}, \quad \text{where } 0.6 < \lambda_4 \le 0.8; \\ X(\lambda_5) &= \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\}, \quad \text{where } 0.8 < \lambda_5 \le 1. \end{aligned} \tag{2.2}$$

So, a hierarchical quotient space structure induced by the fuzzy equivalence relation R_1 is $\pi_X(R_1) = \{X(\lambda_1), X(\lambda_2), X(\lambda_3), X(\lambda_4), X(\lambda_5)\}$, which is shown in Figure 1.

In addition, a pyramid model can be established based on the number of block in each layer of the hierarchical quotient space structure, which is shown in Figure 2.

However, the different fuzzy equivalence relations may become into the same hierarchical quotient space structure. In Example 2.6, if the relation matrix M_{R_2} of the fuzzy equivalence relation R_2 is defined as follows:

$$M_{R_2} = \begin{bmatrix} 1 & 0.2 & 0.9 & 0.6 & 0.6 \\ 0.2 & 1 & 0.2 & 0.2 & 0.2 \\ 0.9 & 0.2 & 1 & 0.6 & 0.6 \\ 0.6 & 0.2 & 0.6 & 1 & 0.7 \\ 0.6 & 0.2 & 0.6 & 0.7 & 1 \end{bmatrix} \tag{2.3}$$

then R_2 can become into the same quotient space sequence and the same hierarchical quotient space structure as R_1; that is, $\pi_X(R_1) = \pi_X(R_2)$.

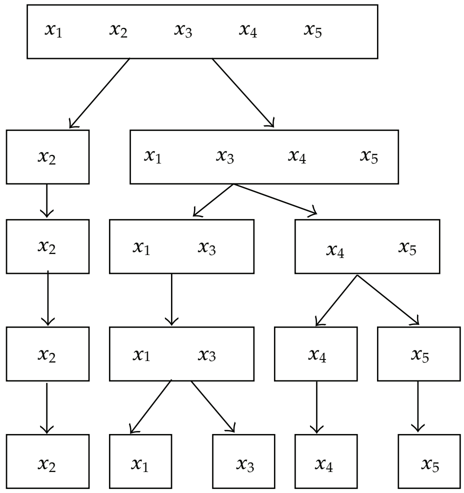

Figure 1: The hierarchical quotient space structure $\pi_X(R_1)$ of R_1.

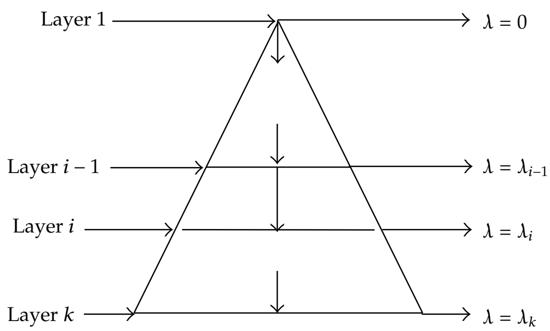

Figure 2: The pyramid model of a hierarchical quotient space structure.

Definition 2.7 (see [16]). Let R_1 and R_2 be two fuzzy equivalence relations on X; if they have the same hierarchical quotient space structure, that is, $\pi_X(R_1) = \pi_X(R_2)$, then R_1 and R_2 are called isomorphic, denoted by $R_1 \cong R_2$.

Proposition 2.8 (see [13]). *Let R_1 and R_2 be two fuzzy equivalence relations on X, for any $x, y, u, v \in X$; if $R_1(x,y) < R_1(u,v) \Leftrightarrow R_2(x,y) < R_2(u,v)$ and $R_1(x,y) = R_1(u,v) \Leftrightarrow R_2(x,y) = R_2(u,v)$, then $R_1 \cong R_2$.*

Definition 2.9 (see [18]). Let R_1 and R_2 be two fuzzy equivalence relations on X; for any $x, y, u, v \in X$, if $R_1(x,y) \le R_1(u,v) \Rightarrow R_2(x,y) \le R_2(u,v)$ and $R_2(x,y) < R_2(u,v) \Rightarrow R_1(x,y) < R_1(u,v)$, then R_1 and R_2 are called homomorphism, denoted by $R_1 \triangleq R_2$.

Proposition 2.10 (see [18]). *Let R_1 and R_2 be two fuzzy equivalence relations on X; if $R_1 \triangleq R_2$, then $\pi_X(R_2) \subseteq \pi_X(R_1)$.*

A fuzzy equivalence relation R can be induced by the fuzzy similarity relation $\widetilde{R}$ by computing the transitive closure of $\widetilde{R}$, then R is called a fuzzy equivalence relation derived from $\widetilde{R}$.

Definition 2.11. Let $\widetilde{R}_1$ and $\widetilde{R}_2$ be two fuzzy similarity relations on X, and R_1 and R_2 are fuzzy equivalence relations derived from $\widetilde{R}_1$ and $\widetilde{R}_2$, respectively. If $R_1 = R_2$, then $\widetilde{R}_1$ and $\widetilde{R}_2$ are called isomorphism, denoted by $\widetilde{R}_1 \eqsim \widetilde{R}_2$. If $R_1 \cong R_2$, then $\widetilde{R}_1$ and $\widetilde{R}_2$ are called similarity, denoted by $\widetilde{R}_1 \approx \widetilde{R}_2$.

From the viewpoints of both clustering and classification, the isomorphic fuzzy similarity relations can induce the same hierarchical quotient space structure, and have the same ability of clustering (or classification) for objects in X. However, the different hierarchical quotient space structures maybe have the same clustering (or classification) ability. There is few correlative research works. For discussing the uncertainty of a hierarchical quotient space structure, in this paper, the information entropy sequence of hierarchical quotient space structure is defined, and the relationship between hierarchical quotient space structure and information entropy sequence is analyzed in detail.

3. The Information Entropy Sequence of Hierarchical Quotient Space Structure

A hierarchical quotient space structure can uncover the essential characteristics of a fuzzy concept better than a fuzzy set. However, how to measure uncertainty of the hierarchical quotient space structure is still an open question. Information entropy is a very useful tool for measuring the uncertainty of vague information, and it has been studied based on the rough set and fuzzy set in the literature. For instance, Liang et al. [19, 20] analyzed the uncertainty of the rough set from the perspective of information entropy, conditional entropy, mutual information, and knowledge granule and presented a new rough entropy. Zhang et al. [21] presented a cognition model based on granular computing and analyzed the uncertainty of human cognition with different granularity levels. Fan [22] Miao et al. [23, 24] discussed the relationships among the knowledge granule, knowledge roughness, and information entropy from the viewpoint of both granular computing and information representation. Wang et al. [25] studied fuzzy entropy of rough sets in different granularity levels. However, in the previous studies, the questions of how to measure the information entropy of a hierarchical quotient space structure and what changing regularities the information entropy of a hierarchical quotient space structure have with different granularity are not discussed. In this section, the information entropy sequence of a hierarchical quotient space structure is proposed and discussed in detail.

Definition 3.1 (see [19]). Let $X = \{x_1, x_2, \ldots, x_n\}$ be a nonempty finite set, $P' = \{P'_1, P'_2, \ldots, P'_l\}$ and $P'' = \{P''_1, P''_2, \ldots, P''_m\}$ are two partition spaces on X. If, for all $P'_i \in P'$, $(\exists_{P''_j \in P''}(P'_i \subseteq P''_j))$, then P' is finer than P'', denoted by $P' \preceq P''$.

Definition 3.2 (see [19]). Let $X = \{x_1, x_2, \ldots, x_n\}$ be a nonempty finite set, $P' = \{P'_1, P'_2, \ldots, P'_l\}$ and $P'' = \{P'_1, P''_2, \ldots, P''_m\}$ are two partition spaces on X. If $P' \preceq P''$ and there exist $P'_i \in P'(\exists_{P_j'' \in P''}(P'_i \subset P''_j))$, then P' is strictly finer than P'', denoted by $P' \prec P''$.

Each layer of the hierarchical quotient space structure $\pi_X(R)$ is a partition on X that is, it is denoted by $X(\lambda)$. If $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_t \le 1$, then $X(\lambda_t) \preceq X(\lambda_{t-1}) \preceq \cdots \preceq X(\lambda_1)$.

Definition 3.3. Let R be a fuzzy equivalence relation on X, let $\pi_X(R) = \{X(\lambda_1), X(\lambda_2), \ldots, X(\lambda_t)\}$ be its corresponding hierarchical quotient space structure, and $X(\lambda_t) \prec X(\lambda_{t-1}) \prec \cdots \prec X(\lambda_1)$. Let $L_i = |X(\lambda_i)|$ $(i = 1, 2, \ldots, t)$, the sequence $L(\pi_X(R)) = \{L_1, L_2, \ldots, L_t\}$ is called the partition sequence of $\pi_X(R)$ (where $|\cdot|$ denotes the cardinality of a set).

Obviously, any one partition sequence of the hierarchical quotient space structure $\pi_X(R)$ is a positive integer sequence, and $L_1 < L_2 < \cdots < L_t$. The information entropy [19, 20, 22–25] of a partition space has been studied by many researchers. Here, we discuss the information entropy sequence of a hierarchical quotient space structure based on classical Shannon information entropy.

Definition 3.4. Let R be a fuzzy equivalence relation on X, let $\pi_X(R) = \{X(\lambda_1), X(\lambda_2), \ldots, X(\lambda_t)\}$ be its corresponding hierarchical quotient space structure, and let $L(\pi_X(R)) = \{L_1, L_2, \ldots, L_t\}$ be a partition sequence of the hierarchical quotient space structure $\pi_X(R)$. Assume that $X(\lambda_i) = \{X_{i1}, X_{i2}, \ldots, X_{iL_i}\}$ (it is a partition on X). The information entropy $H_X(\lambda_i)$ of $X(\lambda_i)$ is defined as follows:

$$H_X(\lambda_i) = -\sum_{k=1}^{L_i} \frac{|X_{ik}|}{|X|} \ln\left(\frac{|X_{ik}|}{|X|}\right). \tag{3.1}$$

Theorem 3.5. *Let R be a fuzzy equivalence relation on X, and let $\pi_X(R) = \{X(\lambda_1), X(\lambda_2), \ldots, X(\lambda_n)\}$ be its hierarchical quotient space structure, then the entropy sequence $H(\pi_X(R)) = \{H_X(\lambda_1), H_X(\lambda_2), \ldots, H_X(\lambda_t)\}$ under Definition 3.4 is a strictly monotonic increasing sequence.*

Proof. It follows from the definition of the hierarchical quotient space structure that $X(\lambda_t) \prec X(\lambda_{t-1}) \prec \cdots \prec X(\lambda_1)$; that is, $X(\lambda_{i-1})$ is a quotient space of $X(\lambda_i)$. Let $X(\lambda_i) = \{X_{i1}, X_{i2}, \ldots, X_{iL_i}\}$ and $X(\lambda_{i-1}) = \{X_{i-1,1}, X_{i-1,2}, \ldots, X_{i-1,L_{i-1}}\}$. Each subblock $X_{i-1,j}$ $(1 \le j \le L_{i-1})$ in $X(\lambda_{i-1})$ is a combination of one or more subblocks in $X(\lambda_i)$. For simplicity, and without any loss of generality, we assume that only a subblock $X_{i-1,j}$ in $X(\lambda_{i-1})$ is a combination of two subblocks in $X(\lambda_i)$; that is, $X_{i-1,j} = X_{i,p} \cup X_{i,q}$, other subblocks in $X(\lambda_{i-1})$ are equal to the corresponding rest subblocks in $X(\lambda_i)$, respectively. Thus

$$\begin{aligned} H_X(\lambda_{i-1}) &= -\sum_{k=1}^{L_{i-1}} \frac{|X_{i-1,k}|}{|X|} \ln\left(\frac{|X_{i-1,k}|}{|X|}\right) \\ &= -\sum_{k=1}^{j-1} \frac{|X_{i-1,k}|}{|X|} \ln\left(\frac{|X_{i-1,k}|}{|X|}\right) - \sum_{k=j+1}^{L_{i-1}} \frac{|X_{i-1,k}|}{|X|} \ln\left(\frac{|X_{i-1,k}|}{|X|}\right) - \frac{\left|X_{i-1,j}\right|}{|X|} \ln\left(\frac{\left|X_{i-1,j}\right|}{|X|}\right) \\ &= -\sum_{k=1}^{j-1} \frac{|X_{i,k}|}{|X|} \ln\left(\frac{|X_{i,k}|}{|X|}\right) - \sum_{k=j+1}^{L_i} \frac{|X_{i,k}|}{|X|} \ln\left(\frac{|X_{i,k}|}{|X|}\right) - \frac{\left|X_{i,p}\right| + \left|X_{i,q}\right|}{|X|} \ln\left(\frac{\left|X_{i,p}\right| + \left|X_{i,q}\right|}{|X|}\right). \end{aligned} \tag{3.2}$$

Because

$$\frac{|X_{i,p}|+|X_{i,q}|}{|X|}\ln\left(\frac{|X_{i,p}|+|X_{i,q}|}{|X|}\right)=\frac{|X_{i,p}|}{|X|}\ln\left(\frac{|X_{i,p}|+|X_{i,q}|}{|X|}\right)+\frac{|X_{i,q}|}{|X|}\ln\left(\frac{|X_{i,p}|+|X_{i,q}|}{|X|}\right)$$
$$>\frac{|X_{i,p}|}{|X|}\ln\left(\frac{|X_{i,p}|}{|X|}\right)+\frac{|X_{i,q}|}{|X|}\ln\left(\frac{|X_{i,q}|}{|X|}\right), \tag{3.3}$$

so

$$\begin{aligned}H_X(\lambda_{i-1})&<-\sum_{k=1}^{j-1}\frac{|X_{i,k}|}{|X|}\ln\left(\frac{|X_{i,k}|}{|X|}\right)-\sum_{k=j+1}^{L_i}\frac{|X_{i,k}|}{|X|}\ln\left(\frac{|X_{i,k}|}{|X|}\right)\\&\quad-\frac{|X_{i,p}|}{|X|}\ln\left(\frac{|X_{i,p}|}{|X|}\right)-\frac{|X_{i,q}|}{|X|}\ln\left(\frac{|X_{i,q}|}{|X|}\right)\\&=H_X(\lambda_i)\quad(2\le i\le n),\end{aligned} \tag{3.4}$$

the sequence $\{H_X(\lambda_1),H_X(\lambda_2),\ldots,H_X(\lambda_t)\}$ is a strictly monotonic increasing sequence; that is, $H_X(\lambda_1)<H_X(\lambda_2)<\cdots<H_X(\lambda_t)$. □

Theorem 3.5 shows that the information entropy sequence of a hierarchical quotient space structure is a monotonic increasing sequence with the partition (or quotient space) becoming finer; that is, the finer the partition in a hierarchical quotient space structure, the bigger its information entropy.

Definition 3.6. Let $X=\{x_1,x_2,\ldots,x_n\}$ be a nonempty finite set, and let $P_k(X)=\{X_1,X_2,\ldots,X_k\}$ be a partition space on X, $P_k(X)$ is called **k**-order partition on X (where $|P_k(X)|=k$).

Definition 3.7. Assume that $P_k(X)=\{X_1,X_2,\ldots,X_k\}$ is a **k**-order partition on X, and let $|X_1|=a'_1,|X_2|=a'_2,\ldots,$ and $|X_k|=a'_k$. Arranging the sequence $\{a'_1,a'_2,\ldots,a'_k\}$ in increasing order, we obtain a new increasing sequence $\{a_1,a_2,\ldots,a_k\}$ denoted by $I(k)$. $I(k)$ is called a subblock sequence of the partition $P_k(X)$.

Obviously, two different **k**-order partitions on X may have the same subblock sequence $I(k)$.

Definition 3.8. Let $X=\{x_1,x_2,\ldots,x_n\}$ be a nonempty finite set, let $P_k(X)=\{X_1,X_2,\ldots,X_k\}$ be a partition space on X, and let $I_1(k)=\{a_1,a_2,\ldots,a_k\}$ be a subblock sequence of the partition $P_k(X)$. $I_2(k)=\{a_1,a_2,\ldots,a_p+1,\ldots,a_q-1,\ldots,a_k\}$ is called a σ-displacement of $I_1(k)$, where $p<q$, $a_p+1<a_q-1$, and $I_2(k)$ keeps its increasing property.

A σ-displacement is equal to minus one from some "bigger" element and add one to some "smaller" element, and the sequence still keeps monotonically increasing in a subblock sequence.

Example 3.9. Let $I_1(4)=\{1,2,4,5\}$ be a subblock sequence, then $I_2(4)=\{1,3,4,4\}$ is derived from $I_1(4)$ through the σ-displacement once.

Theorem 3.10. *If a subblock sequence $I_2(k)$ is derived from $I_1(k)$ through the σ-displacement once, then $H(I_1(k)) < H(I_2(k))$.*

Proof. Let $I_1(k) = \{a_1, a_2, \dots, a_k\}$, $I_2(k) = \{a_1, a_2, \dots, a_i+1, \dots, a_j-1, \dots, a_k\}$, $a_1+a_2+\cdots+a_k = n$, then

$$H(I_2(k)) = -\sum_{i=1}^{k} \frac{a_l}{n} \ln \frac{a_l}{n} + \frac{a_p}{n} \ln \frac{a_p}{n} + \frac{a_q}{n} \ln \frac{a_q}{n} - \frac{a_p+1}{n} \ln \frac{a_p+1}{n} - \frac{a_q-1}{n} \ln \frac{a_q-1}{n}. \tag{3.5}$$

Let $f(x) = -(x/n)\ln(x/n) - ((m-x)/n)\ln((m-x)/n)$ (where $m = a_p + a_q$), then $f'(x) = (1/n)\ln((m-x)/x)$.

Let $f'(x) = 0$, we have a solution; that is, $x = m/2$. In addition, because $f''(x) = (-m/n(m-x)x) < 0$, when $0 \le x \le (m/2)$, $f(x)$ is monotonic increasing function.

Let $x = a_p$ and $y = a_p + 1$, where $a_p + 1 < a_q - 1$; that is, $x < y \le (m/2) = (a_p + a_q)/2$. Due to the monotonicity of $f(x)$, $f(y) - f(x) > 0$; that is,

$$\frac{a_p}{n} \ln \frac{a_p}{n} + \frac{a_q}{n} \ln \frac{a_q}{n} - \frac{a_p+1}{n} \ln \frac{a_p+1}{n} - \frac{a_q-1}{n} \ln \frac{a_q-1}{n} > 0. \tag{3.6}$$

Therefore,

$$\begin{aligned} H(I_2(k)) &= -\sum_{i=1}^{k} \frac{a_l}{n} \ln \frac{a_l}{n} + \left(\frac{a_p}{n} \ln \frac{a_p}{n} + \frac{a_q}{n} \ln \frac{a_q}{n}\right) - \left(\frac{a_p+1}{n} \ln \frac{a_p+1}{n} + \frac{a_q-1}{n} \ln \frac{a_q-1}{n}\right) \\ &> -\left(\frac{a_p+1}{n} \ln \frac{a_p+1}{n} + \frac{a_q-1}{n} \ln \frac{a_q-1}{n}\right) > -\sum_{i=1}^{k} \frac{a_l}{n} \ln \frac{a_l}{n} = H(I_1(k)). \end{aligned} \tag{3.7}$$

□

Theorem 3.10 shows that through the σ-displacement once, the information entropy of a subblock sequence will increase. In the case of nonconfusion, the information entropy of a partition and the information entropy of its corresponding subblock sequence can be deemed as the same value. If two different partitions $P'_k(X)$ and $P''_k(X)$ on X have the same subblock sequence $I(k)$, then they should have the same information entropy sequence. So, from viewpoint of the entropy, they contain the same information. However, if two subblock sequences $I_1(k) \neq I_2(k)$ on X, then $H(I_1(k)) \neq H(I_2(k))$, which is discussed in Theorem 3.11 as follows.

Theorem 3.11. *Let $X = \{x_1, x_2, \dots, x_n\}$ be a nonempty finite set. For any two k-order partitions $P'_k(X)$ and $P''_k(X)$ based on X, their corresponding subblock sequences are $I_1(k)$ and $I_2(k)$, respectively. If $I_1(k) \neq I_2(k)$, then $H(I_1(k)) \neq H(I_2(k))$, where $H(I_1(k))$ and $H(I_2(k))$ denote the information entropy of partitions $P'_k(X)$ and $P''_k(X)$ based on Definition 3.4, respectively* $(2 \le k \le n)$.

Proof. Firstly we prove that the conclusion in Theorem 3.11 holds in the case of $k = 2$. Assume that $P(X) = \{X_1, X_2\}$, let $|X_1| = x$ and $|X_2| = n - x$, then a 2-order subblock sequence is obtained; that is, $I(2) = \{x, n-x\}$. Supposing that $x < (n/2)$ (if n is even) or $x < ((n-1)/2)$ (if n is odd). We have $H(P(X)) = -(x/n)\ln(x/n) - ((n-x)/n)\ln((n-x)/n)$.

Let $f(x) = -(x/n)\ln(x/n) - ((n-x)/n)\ln((n-x)/n)$, then $f'(x) = -(1/n)\ln(x/n) - (1/n) + (1/n)\ln((n-x)/n) + (1/n) = (1/n)\ln((n-x)/x)$. Letting $f'(x) = 0$, we can obtain the solution $x = n/2$. Since $f''(x) = (-1/(n-x)x) < 0$, when $x = n/2$, $f(x)$ achieves its maximum value. Thus, when $x < n/2$ (n is even) or $x < (n-1)/2$ (n is odd), $f(x)$ is a monotonic increasing function; that is, information entropy $H(P(X))$ is a monotonic increasing function. If $x > n/2$ (n is even) or $x > (n+1)/2$ (n is odd), the 2-order subblock sequence $I(2) = \{x, n-x\}$ becomes $I(2) = \{n-x, x\}$; it is the same as that when $x < n/2$ (n is even) or $x < (n-1)/2$ (n is odd). Therefore, when $k = 2$ and $I_1(k) \neq I_2(k)$, $H(I_1(k)) \neq H(I_2(k))$ holds.

When $k \geq 3$, the idea to prove them is the same as $k = 2$. For the limitation of paper length, we only discuss the case of $k = 3$ (similar to other situations) in this paper. Assume that there are two different 3-order partitions $P_3'(X)$ and $P_3''(X)$ and their corresponding subblock sequences $I_1(3)$ and $I_2(3)$ are different. Let $I_1(3) = \{a_1, a_2, a_3\}$, $I_2(3) = \{b_1, b_2, b_3\}$, and $I_1(3) \neq I_2(3)$. If only one element in $I_1(3)$ is equal to the element of $I_2(3)$ (e.g., let $a_1 = b_1$, then $a_2 \neq b_2$ and $a_3 \neq b_3$), due to the former conclusion in case of $k = 2$, $H(P_3'(X)) \neq H(P_3''(X))$ is held. If all elements in $I_1(3)$ are not equal to that in $I_2(3)$, then we get the conclusion from the following three cases (other cases can be transformed into one of the after-mentioned three conditions):

(1) $a_1 > b_1$, $a_2 > b_2$, and $a_3 < b_3$. $I_2(3)$ can be replaced by $I_1(3)$ through many times of σ-displacements. By Theorem 3.10, $H(I_1(3)) \neq H(I_2(3))$.

(2) $a_1 > b_1$, $a_2 < b_2$, and $a_3 < b_3$. $I_2(3)$ can be also replaced by $I_1(3)$ through many times of σ-displacements. By Theorem 3.10, $H(I_1(3)) \neq H(I_2(3))$.

(3) $a_1 > b_1$, $a_2 < b_2$, and $a_3 > b_3$. $I_2(3)$ cannot be replaced by $I_1(3)$ directly through the σ-displacement once, but $I_1(3)$ can be replaced by $I_3(3) = \{a_1, a_2+(a_3-b_3), b_3\}$ through the σ-displacement once. By Theorem 3.10, $H(I_1(3)) < H(I_3(3)) < H(I_2(3))$ holds. So $H(I_1(3)) \neq H(I_2(3))$ holds.

□

Given a hierarchical quotient space structure $\pi_X(R)$, we can easily obtain its corresponding information entropy sequence $\{H(I(n_1)), H(I(n_2)), \ldots, H(I(n_s)), \ldots\}$ (where $\{n_1, n_2, \ldots, n_s, \ldots\}$ is the subsequence of natural number sequence $\{1, 2, 3, \ldots, n, \ldots\}$). Theorem 3.11 shows that the different subblock sequences can have different information entropy sequences. According to Theorem 3.5 and Theorem 3.11, the different hierarchical quotient space structures which have the same subblock sequence should have the same information entropy sequence $\{H(I(n_1)), H(I(n_2)), \ldots, H(I(n_s)), \ldots\}$, the same information, and the same ability for classification (clustering). Both the information entropy sequence and the subblock sequence of a hierarchical quotient space structure are one to one. The information entropy sequence $\{H(I(n_1)), H(I(n_2)), \ldots, H(I(n_s)), \ldots\}$ uncovers the uncertainty of a hierarchical quotient space structure. Therefore, we can analyze the information (uncertainty) of a hierarchical quotient space structure with its information entropy sequence.

Definition 3.12. Let $\pi_X(R_1)$ and $\pi_X(R_2)$ be two hierarchical quotient space structures derived from the fuzzy equivalence relations R_1 and R_2 on X, and their partition sequences are $L(\pi_X(R_1)) = \{L_{11}, L_{12}, \ldots, L_{1r}\}$ and $L(\pi_X(R_2)) = \{L_{21}, L_{22}, \ldots, L_{2s}\}$, respectively. Let $\{I(L_{11}), I(L_{12}), \ldots, I(L_{1r})\}$ and $\{I(L_{21}), I(L_{22}), \ldots, I(L_{2s})\}$ denote their corresponding subblock sequences of each layer in $\pi_X(R_1)$ and $\pi_X(R_2)$, respectively. If $L(\pi_X(R_1)) =$

$L(\pi_X(R_2))$ (obviously, where $r = s$ and $I(L_{1i}) = I(L_{2i})$, $i = 1, 2, \ldots, r$), then $\pi_X(R_1)$ and $\pi_X(R_2)$ are called isomorphism, denoted by $\pi_X(R_1) \simeq \pi_X(R_2)$.

Example 3.13. Assume that $X = \{1, 2, 3, 4, 5, 6\}$, and there are two hierarchical quotient space structures $\pi_X(R_1)$ and $\pi_X(R_2)$ on X as follows:

$$\begin{aligned}\pi_X(R_1) &= \{\{\{1,2,3\},\{4,5,6\}\};\{\{1,2\},\{3\},\{4,5,6\}\};\{\{1\},\{2\},\{3\},\{4,5,6\}\}\}, \\ \pi_X(R_2) &= \{\{\{1,5,6\},\{2,3,4\}\};\{\{1,5\},\{6\},\{2,3,4\}\};\{\{1\},\{5\},\{6\},\{2,3,4\}\}\},\end{aligned} \tag{3.8}$$

namely,

$$\begin{array}{cc} \{\{1,2,3\},\{4,5,6\}\} & \{\{1,5,6\},\{2,3,4\}\} \\ \downarrow & \downarrow \\ \{\{1,2\},\{3\},\{4,5,6\}\} & \{\{1,5\},\{6\},\{2,3,4\}\} \\ \downarrow & \downarrow \\ \{\{1\},\{2\},\{3\},\{4,5,6\}\}, & \{\{1\},\{5\},\{6\},\{2,3,4\}\}. \\ \pi_X(R_1) & \pi_X(R_2) \end{array} \tag{3.9}$$

Obviously, $L(\pi_X(R_1)) = L(\pi_X(R_2)) = \{2, 3, 4\}$, and $I(L_{11}) = I(L_{21}) = \{3, 3\}$, $I(L_{12}) = I(L_{22}) = \{1, 2, 3\}$, and $I(L_{13}) = I(L_{23}) = \{1, 1, 1, 3\}$. Therefore, $\pi_X(R_1) \simeq \pi_X(R_2)$.

From the viewpoint of classification (clustering) analysis, if two hierarchical quotient space structures are isomorphic, they have the same classification abilities of the objects in the set of X. From the viewpoint of information entropy, if two hierarchical quotient space structures are isomorphic, they have the same information entropies. Based on Theorem 3.11, Theorem 3.14 is given as follows.

Theorem 3.14. *Let $\pi_X(R_1)$ and $\pi_X(R_2)$ be two hierarchical quotient space structures derived from fuzzy equivalence relations R_1 and R_2 on X, respectively. If $L(\pi_X(R_1)) = L(\pi_X(R_2))$ and $H(\pi_X(R_1)) = H(\pi_X(R_2))$, then $\pi_X(R_1) \simeq \pi_X(R_2)$.*

Proof. Since $L(\pi_X(R_1)) = L(\pi_X(R_2))$, $\pi_X(R_1)$ has the same number of layer as $\pi_X(R_2)$, each layers of both $\pi_X(R_1)$ and $\pi_X(R_2)$ has the same number of the subblock. Let $L(\pi_X(R_1)) = L(\pi_X(R_2)) = \{k_1, k_2, \ldots, k_r\}$. For any k_i $(1 \le i \le r)$, according to Theorem 3.11, if $I_1(k_i) \ne I_2(k_i)$, $H(I_1(k_i)) \ne H(I_2(k_i))$ $(1 \le i \le r)$. Therefore, if $H(\pi_X(R_1)) = H(\pi_X(R_2))$, $\pi_X(R_1) \simeq \pi_X(R_2)$ is held. □

In Theorem 3.14, the condition $L(\pi_X(R_1)) = L(\pi_X(R_2))$ is very important. If this condition is not held, then Theorem 3.14 is not held. Theorem 3.14 reveals the importance of information entropy sequence. The uncertainty of a hierarchical quotient space structure may be characterized by the information entropy sequence, the partition sequence, and the subblock sequence. The different fuzzy similarity relations may induce the isomorphic fuzzy equivalence relations on X, and the isomorphic fuzzy equivalence relations may produce the same or isomorphic hierarchical quotient space structures which have the same uncertainty (or information entropy sequence).

Table 1: The relationships between fuzzy relations and hierarchical quotient space structure.

$\widetilde{R}_1$ and $\widetilde{R}_2$	R_1 and R_2	$\pi_X(R_1)$ and $\pi_X(R_2)$	$H(\pi_X(R_1))$ and $H(\pi_X(R_2))$
Isogeny but not Similarity ($\doteq$)	Similarity ($\sim$)	Isomorphism ($\simeq$)	Same ($=$)
Similarity but not Isomorphism ($\eqsim$)	Isomorphism ($\cong$)	Same ($=$)	
Isomorphism but not Same ($\overline{\eqsim}$)	Same ($=$)		
Same ($=$)			

Definition 3.15. Let $\widetilde{R}_1$ and $\widetilde{R}_2$ be two fuzzy similarity relations on X, and let R_1 and R_2 be two fuzzy equivalence relations derived from $\widetilde{R}_1$ and $\widetilde{R}_2$, respectively. If $\pi_X(R_1) \simeq \pi_X(R_2)$, then $\widetilde{R}_1$ and $\widetilde{R}_2$ are called isogenous fuzzy similarity relations, denoted by $\widetilde{R}_1 \doteq \widetilde{R}_2$. If R_1 and R_2 are not isomorphic, but $\pi_X(R_1) \simeq \pi_X(R_2)$, then R_1 and R_2 are called similar fuzzy equivalence relation, denoted by $R_1 \sim R_2$.

According to Definitions 2.7, 2.11, and 3.15, the relationships among the information entropy sequence, the hierarchical quotient space structure, the fuzzy equivalence relation and the fuzzy similarity relation are shown in (3.10) and Table 1.

$$\left.\begin{array}{l} R_1 \overset{\text{induce}}{\longrightarrow} \pi_X(R_1) \\ R_2 \overset{\text{induce}}{\longrightarrow} \pi_X(R_2) \end{array}\right\} \Longrightarrow \begin{cases} if\ \pi_X(R_1) = \pi_X(R_2),\ then\ R_1 \cong R_2, \\ if\ \pi_X(R_1) \simeq \pi_X(R_2),\ then\ R_1 \sim R_2, \end{cases}$$
$$\left.\begin{array}{l} \widetilde{R}_1 \overset{\text{induce}}{\longrightarrow} R_1 \\ \widetilde{R}_2 \overset{\text{induce}}{\longrightarrow} R_2 \end{array}\right\} \Longrightarrow \begin{cases} if\ R_1 = R_2,\ then\ \widetilde{R}_1\ \overline{\eqsim}\ \widetilde{R}_2, \\ if\ R_1 \cong R_2,\ then\ \widetilde{R}_1 \eqsim \widetilde{R}_2, \\ if\ R_1 \sim R_2,\ then\ \widetilde{R}_1 \doteq \widetilde{R}_2, \end{cases} \tag{3.10}$$

Note. R_1 and R_2 are two fuzzy equivalence relations derived from two fuzzy similarity relations $\widetilde{R}_1$ and $\widetilde{R}_2$ on X, respectively.

4. A Fast Constructing Hierarchical Quotient Space Structure Method

In Section 3, in order to obtain a classification (clustering) result (or a hierarchical quotient space structure), first we should establish a fuzzy similarity relation $\widetilde{R}$ on X. Then a fuzzy equivalence relation R is derived from the fuzzy similarity relation $\widetilde{R}$ by the transitive closure operation. Finally, a kind of classification (or clustering) result is obtained. Many literature studies [26] have discussed about how to efficiently transform a fuzzy similarity relation into a fuzzy equivalence relation, and many important results have been obtained. In spite of these previous studies, the efficiency of obtaining classification (or clustering) result can be further improved through directly constructing a hierarchical quotient space structure from a fuzzy similarity relation. In this section, a fast constructing hierarchical quotient space structure method is discussed, and this method can effectively improve the efficiency of the classification(or clustering).

Assume that $X = \{x_1, x_2, \ldots, x_n\}$, $\widetilde{R}$ is a fuzzy similarity relation on X, and its corresponding relation matrix is defined by $M(\widetilde{R}) = (r_{ij})_{n\times n}$. Let $\widetilde{r}_{ij} = \widetilde{R}(x_i, x_j)$ stand for the similarity degree between objects x_i and x_j according to some similarity degree criterion. $M(\widetilde{R})$ is a $\mathbf{n} \times \mathbf{n}$ matrix as follows:

$$M\left(\widetilde{R}\right) = \begin{bmatrix} \widetilde{r}_{11} & \widetilde{r}_{12} & \cdots & \widetilde{r}_{1n} \\ \widetilde{r}_{21} & \widetilde{r}_{22} & \cdots & \widetilde{r}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{r}_{n1} & \widetilde{r}_{n2} & \cdots & \widetilde{r}_{nn} \end{bmatrix}. \tag{4.1}$$

Because $M(\widetilde{R})$ is reflexive and symmetric, that is, $\widetilde{r}_{ij} = \widetilde{r}_{ji}$ and $\widetilde{r}_{ii} = 1$ $(i = 1, 2, \ldots, n)$, the matrix $M(\widetilde{R})$ may be constructed by elements $\widetilde{r}_{12}, \widetilde{r}_{13}, \ldots, \widetilde{r}_{1n}, \widetilde{r}_{23}, \widetilde{r}_{24}, \ldots, \widetilde{r}_{2n}, \widetilde{r}_{34}, \ldots, \widetilde{r}_{3n}, \ldots, \widetilde{r}_{n-1,n}$. We reorder the sequence $\widetilde{r}_{12}, \widetilde{r}_{13}, \ldots, \widetilde{r}_{1n}, \widetilde{r}_{23}, \widetilde{r}_{24}, \ldots, \widetilde{r}_{2n}, \widetilde{r}_{34}, \ldots, \widetilde{r}_{3n}, \ldots, \widetilde{r}_{n-1,n}$ in degressive order. If the same element occurs, just leaves one of them, and discards others, then we can obtain a new degressive sequence denoted by $= \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$ $(t \le n(n-1)/2)$, where $1 \ge \lambda_1 > \lambda_2 > \cdots > \lambda_t > 0$. Because the maximal number of layers in hierarchical quotient space structure is n, only anterior n numbers in the sequence $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$ is useful for constructing a hierarchical quotient space structure, and other values are ignored. In some layers (it is a quotient space on X) of the hierarchical quotient space structure, if x_i and x_j are indiscernible, then they must be located in the same subblock.

Theorem 4.1. *Let the relation matrix of the fuzzy similarity relation $\widetilde{R}$ be $M(\widetilde{R}) = (\widetilde{r}_{ij})_{n\times n}$, let R be fuzzy equivalence relation derived from $\widetilde{R}$, and its matrix is denoted by $M(R) = (r_{ij})_{n\times n}$. If $\widetilde{r}_{ij} \ge \lambda$, then x_i and x_j are located in the same subblock in quotient space $X(\lambda)$ of the hierarchical quotient space structure $\pi_X(R)$.*

Proof. If $\widetilde{r}_{ij} \ge \lambda$, $\widetilde{r}_{ij}^{(2)}$ denotes the element of matrix $M(\widetilde{R}^2)$; according to Proposition 2.3, we have $r_{ij}^{(2)} = \bigvee_{1\le k\le n}(r_{ik} \wedge r_{kj})$. Since $r_{ii} = 1$, when $k = i$, we have $r_{ij}^{(2)} \ge \widetilde{r}_{ij}$. $r_{ij} \ge \widetilde{r}_{ij}^{(2)} \ge \widetilde{r}_{ij} \ge \lambda$, that is, $R(x_i, x_j) \ge \lambda$ and $(x_i, x_j) \in R_\lambda$. So, if $\widetilde{r}_{ij} \ge \lambda$ in the quotient space $X(\lambda)$ (the partition $X(\lambda)$ is induced by the equivalence relation R_λ) of the hierarchical quotient space structure $\pi_X(R)$, x_i and x_j are located in the same subblock. □

Corollary 4.2. *Let the matrix of fuzzy similarity relation $\widetilde{R}$ be $M(\widetilde{R}) = (\widetilde{r}_{ij})_{n\times n}$, let R be a fuzzy equivalence relation derived from $\widetilde{R}$, and let its matrix be $M(R) = (r_{ij})_{n\times n}$. If $\widetilde{r}_{ij} \ge \lambda$ and $\widetilde{r}_{jk} \ge \lambda$, then x_i, x_j, and x_k are located in the same subblock in the quotient space $X(\lambda)$ of the hierarchical quotient space structure $\pi_X(R)$.*

Based on Theorem 4.1 and Corollary 4.2, a fast constructing hierarchical quotient space structure method is presented in Algorithm 1

There is an example to illuminate Algorithm 1.

Let $X = \{x_1, x_2, \ldots, x_n\}$.
Input: A fuzzy similarity relation matrix $M(\widetilde{R}) = (\widetilde{r}_{ij})_{n\times n}$
Output: A hierarchical quotient space structure $\pi_X(R)$ (Where R is a fuzzy equivalence relation derived from $\widetilde{R}$)
Step 1 Using the matrix $M(\widetilde{R}) = (\widetilde{r}_{ij})_{n\times n}$, a similarity degree sequence $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$ can be obtained.
Step 2 For $k = 1$ to n
{If $\widetilde{r}_{ij} \geq \lambda_k$, x_i and x_j are located in the same subblock, else x_i and x_j are located in the different subblock respectively. A quotient space $X(\lambda_k)$ is constructed}
{if $X(\lambda_k) = \{X\}$, go to **Step3**}
Step 3 End
Step 4 A hierarchical quotient space structure $\pi_X(R)$ is obtained.

Algorithm 1

Example 4.3. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$; according to some similarity degree criterion, a similarity relation matrix can be established; that is,

$$M\left(\widetilde{R}\right) = \left(\widetilde{r}_{ij}\right)_{n\times n} = \begin{bmatrix} 1 & 0.928 & 0.888 & 0.838 & 0.788 & 0.758 \\ 0.928 & 1 & 0.955 & 0.902 & 0.848 & 0.815 \\ 0.888 & 0.955 & 1 & 0.941 & 0.886 & 0.852 \\ 0.838 & 0.902 & 0.941 & 1 & 0.940 & 0.940 \\ 0.788 & 0.848 & 0.886 & 0.940 & 1 & 0.965 \\ 0.758 & 0.815 & 0.852 & 0.940 & 0.965 & 1 \end{bmatrix}. \tag{4.2}$$

Step 1. Because the similarity relation matrix $M(\widetilde{R}) = (\widetilde{r}_{ij})_{n\times n}$ is reflexive and symmetrical, we only need to consider its upper triangular matrix. A similarity degree sequence $\lambda = \{0.965, 0.955, 0.941, 0.940, 0.928, 0.902\}$ is obtained from $(\widetilde{R})$.

Step 2. When $\lambda_1 = 0.965$, $\widetilde{r}_{56} = 0.965$, the first quotient space

$$X(0.965) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5, x_6\}\} \tag{4.3}$$

is constructed, which is the first layer in $\pi_X(R)$.

When $\lambda_2 = 0.955$, $\widetilde{r}_{56} > 0.955$ and $\widetilde{r}_{23} = 0.955$, the second quotient space

$$X(0.955) = \{\{x_1\}, \{x_2, x_3\}, \{x_4\}, \{x_5, x_6\}\} \tag{4.4}$$

is constructed, which is the second layer in $\pi_X(R)$.

When $\lambda_3 = 0.941$, $\widetilde{r}_{56} > 0.941$, $\widetilde{r}_{23} > 0.941$ and $\widetilde{r}_{34} = 0.941$, the third quotient space

$$X(0.941) = \{\{x_1\}, \{x_2, x_3, x_4\}, \{x_5, x_6\}\} \tag{4.5}$$

is constructed, which is the third layer in $\pi_X(R)$.

When $\lambda_4 = 0.940$, $\tilde{r}_{56} > 0.940$, $\tilde{r}_{23} > 0.940$, $\tilde{r}_{34} > 0.940$, $\tilde{r}_{45} = 0.940$ and $\tilde{r}_{46} = 0.940$, the fourth quotient space

$$X(0.940) = \{\{x_1\}, \{x_2, x_3, x_4, x_5, x_6\}\} \tag{4.6}$$

is constructed, which is the fourth layer in $\pi_X(R)$.

When $\lambda_5 = 0.928$, $\tilde{r}_{56} > 0.928$, $\tilde{r}_{23} > 0.928$, $\tilde{r}_{34} > 0.928$, $\tilde{r}_{45} > 0.928$, $\tilde{r}_{46} > 0.928$ and $\tilde{r}_{12} = 0.928$, the fifth quotient space

$$X(0.928) = \{\{x_1, x_2, x_3, x_4, x_5, x_6\}\} \tag{4.7}$$

is constructed, which is the fifth layer in $\pi_X(R)$.

Since $X(0.928) = \{X\}$, the whole hierarchical quotient space structure is successfully constructed, and it is

$$\pi_X(R) = \{X(0.965), X(0.955), X(0.941), X(0.940), X(0.928)\}. \tag{4.8}$$

5. Conclusions

Due to human various subjective ideas and evaluation criteria, the same fuzzy concept might have different memberships which leads to different fuzzy similarity relations. However, if these fuzzy similarity relations are isomorphic, then they will induce to the same hierarchical quotient space structure. If different hierarchical quotient space structures have the same information entropy sequence and the same subblock sequence, then they have the same classification (clustering) ability and contain the same information. Therefore, the information entropy sequence is a very useful attribute for measuring the uncertainty of a hierarchical quotient space structure. In this paper, the information entropy sequence of a hierarchical quotient space structure is discussed, through the analysis of the relationships among information entropy sequence, hierarchical quotient space structure, fuzzy equivalence relation, and fuzzy similarity relation. A fast-constructing hierarchical quotient space structure method is presented. This further reveals the essential of a hierarchical quotient space structure. In real world, "fuzziness" and "crispness" are relative and can be transformed into each other with different granularity. The hierarchical quotient space structure is just a bridge between the fuzzy granule world and the clear granule world, and the research on uncertainty of the hierarchical quotient space structure will contribute to develop both granular computing theory and information entropy theory.

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References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Computation*, vol. 8, pp. 338–353, 1965.

[2] T. Y. Lin, "Granular fuzzy sets: a view from rough set and probability theories," *International Journal of Fuzzy Systems*, vol. 3, no. 2, pp. 373–381, 2001.

[3] P. Liang and F. Song, "What does a probabilistic interpretation of fuzzy sets mean?" *IEEE Transactions on Fuzzy Systems*, vol. 4, no. 2, pp. 200–205, 1996.

[4] T. Y. Lin, "Neighborhood systems application to qualitative fuzzy and rough sets," in *Advances in Machine Intelligence and Soft-Computing*, P. P. Wang, Ed., pp. 132–155, Department of Electrical Engineering, Duke University, Durham, NC, USA, 1997.

[5] B. Zhang and L. Zhang, *The Theory and Applications of Problem Solving*, Tsinghua University Press, Beijing, China, 1990.

[6] B. Zhang and L. Zhang, *The Theory and Applications of Problem Solving-Quotient Space Based Granular Computing (The Second Version)*, Tsinghua University Press, Beijing, China, 2007.

[7] L. A. Zadeh, "Fuzzy logic = computing with words," *IEEE Transactions on Fuzzy Systems*, vol. 4, no. 2, pp. 103–111, 1996.

[8] L. A. Zadeh, "Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic," *Fuzzy Sets and Systems*, vol. 90, no. 2, pp. 111–127, 1997.

[9] Z. Pawlak, "Rough sets," *International Journal of Computer and Information Sciences*, vol. 11, no. 5, pp. 341–356, 1982.

[10] L. Zhang and B. Zhang, "Theory of fuzzy quotient space (methods of fuzzy granular computing)," *Journal of Software*, vol. 14, no. 4, pp. 770–776, 2003.

[11] D.-Y. Li, C.-Y. Liu, Y. Du, and X. Han, "Artificial intelligence with uncertainty," *Journal of Software*, vol. 15, no. 11, pp. 1583–1594, 2004.

[12] Q. H. Zhang, G. Y. Wang, and X. Q. Liu, "Analysis of the Hierarchical quotient space structure of Fuzzy Quotient Space," *Pattern Recognition and Artificial Intelligence*, vol. 21, no. 5, pp. 627–634, 2008.

[13] L. Zhang and B. Zhang, "The structure analysis of fuzzy sets," *International Journal of Approximate Reasoning*, vol. 40, no. 1-2, pp. 92–108, 2005.

[14] G. Tsekouras, H. Sarimveis, E. Kavakli, and G. Bafas, "A hierarchical fuzzy-clustering approach to fuzzy modeling," *Fuzzy Sets and Systems*, vol. 150, no. 2, pp. 245–266, 2005.

[15] A. Pedrycz and M. Reformat, "Hierarchical FCM in a stepwise discovery of structure in data," *Soft Computing*, vol. 10, no. 3, pp. 244–256, 2006.

[16] X. Q. Tang, P. Zhu, and J. X. Cheng, "Cluster analysis based on fuzzy quotient space," *Journal of Software*, vol. 19, no. 4, pp. 861–868, 2008.

[17] L. B. Yang and Y. Y. Gao, *Fuzzy Mathematics: Theory and Application*, South China University of Technology Press, Guang Zhou, China, 2004.

[18] C.-Y. Zhang, "Fuzzy sets and quotient spaces," in *Proceedings of the IEEE International Conference on Granular Computing*, vol. 2005, pp. 350–353, 2005.

[19] J. Liang, K. S. Chin, C. Dang, and R. C. M. Yam, "A new method for measuring uncertainty and fuzziness in rough set theory," *International Journal of General Systems*, vol. 31, no. 4, pp. 331–342, 2002.

[20] J. Y. Liang and Z. Z. Shi, "The information entropy, rough entropy and knowledge granulation in rough set theory," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 12, no. 1, pp. 37–46, 2004.

[21] Q. H. Zhang, Y. L. Zhou, and H. T. Teng, "Cognition model based on granular computing," *Journal of Chongqing University of Posts and Telecommunications*, vol. 21, no. 4, pp. 494–501, 2009.

[22] S. D. Fan, "Calculation of knowledge granulation and its application," *System Engineering Theory and Practice*, vol. 22, no. 1, pp. 48–56, 2002.

[23] D. Q. Miao and J. Wang, "An information representation of the concepts and operations in rough set theory," *Journal of Software*, vol. 10, no. 2, pp. 113–116, 1999.

[24] D. Q. Miao, G. Y. Wang, Q. Liu et al., *Granular Computing: Past, Present and Future Prospects*, Science Press, Beijing, China, 2007.

[25] G. Y. Wang, D. Q. Miao, W. Z. Wu, and J. Y. Liang, "Uncertain knowledge representation and processing based on rough set," *Journal of Chongqing University of Posts and Telecommunications*, vol. 22, no. 5, pp. 541–544, 2010.

[26] X. Liu, G. Wang, Q. Zhang, J. Hu, and J. Zhou, "Quick building of fuzzy equivalence matrix in granular computing," in *Proceedings of the IEEE International Conference on Granular Computing (GRC '08)*, pp. 466–469, 2008.