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Research Article

Finding Minimum Norm Fixed Point of Nonexpansive Mappings and Applications

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We construct two new methods for finding the minimum norm fixed point of nonexpansive mappings in Hilbert spaces. Some applications are also included.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that a mapping $T : C \to C$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

Iterative algorithms for finding fixed point of nonexpansive mappings are very interesting topic due to the fact that many nonlinear problems can be reformulated as fixed point equations of nonexpansive mappings. Related works can be found in [1–32].

On the other hand, we notice that it is quite often to seek a particular solution of a given nonlinear problem, in particular, the minimum-norm solution. In an abstract way, we may formulate such problems as finding a point $x^\dagger$ with the property

$$x^\dagger \in C, \quad \left\|x^\dagger\right\| = \min_{x \in C}\|x\|, \tag{1.2}$$

where C is a nonempty closed convex subset of a real Hilbert space H. In other words, $x^\dagger$ is the (nearest point or metric) projection of the origin onto C,

$$x^\dagger = P_C(0), \tag{1.3}$$

where P_C is the metric (or nearest point) projection from H onto C.

A typical example is the least-squares solution to the constrained linear inverse problem

$$\begin{aligned} Ax &= b, \\ x &\in C, \end{aligned} \tag{1.4}$$

where A is a bounded linear operator from H to another real Hilbert space H_1 and b is a given point in H_1. The least-squares solution to (1.4) is the least-norm minimizer of the minimization problem

$$\min_{x\in C} \|Ax - b\|^2. \tag{1.5}$$

Let S_b denote the (closed convex) solution set of (1.4) (or equivalently (1.5)). It is known that S_b is nonempty if and only if $P_{\overline{A(C)}}(b) \in A(C)$. In this case, S_b has a unique element with minimum norm (equivalently, (1.4) has a unique least-squares solution); that is, there exists a unique point $x^\dagger \in S_b$ satisfying

$$\left\|x^\dagger\right\| = \min\{\|x\| : x \in S_b\}. \tag{1.6}$$

The so-called C-constrained pseudoinverse of A is then defined as the operator $A_C^\dagger$ with domain and values given by

$$D\left(A_C^\dagger\right) = \left\{b \in H : P_{\overline{A(C)}}(b) \in A(C)\right\}; \quad A_C^\dagger(b) = x^\dagger, \quad b \in D\left(A_C^\dagger\right), \tag{1.7}$$

where $x^\dagger \in S_b$ is the unique solution to (1.6).

Note that the optimality condition for the minimization (1.5) is the variational inequality (VI)

$$\hat{x} \in C, \quad \langle A^*(A\hat{x} - b), x - \hat{x}\rangle \ge 0, \quad x \in C, \tag{1.8}$$

where A^* is the adjoint of A.

If $b \in D(A_C^\dagger)$, then (1.5) is consistent and its solution set S_b coincides with the solution set of VI (1.8). On the other hand, VI (1.8) can be rewritten as

$$\hat{x} \in C, \quad \langle (\hat{x} - \lambda A^*(A\hat{x}) - b) - \hat{x}, x - \hat{x}\rangle \le 0, \quad x \in C, \tag{1.9}$$

where $\lambda > 0$ is any positive scalar. In the terminology of projections, (1.10) is equivalent to the fixed point equation

$$\widehat{x} = P_C(\widehat{x} - \lambda A^*(A\widehat{x} - b)). \tag{1.10}$$

It is not hard to find that for $0 < \lambda < 2/\|A\|^2$, the mapping $x \mapsto P_C(x - \lambda A^*(Ax - b))$ is nonexpansive. Therefore, finding the least-squares solution of the constrained linear inverse problem (1.6) is equivalent to finding the minimum-norm fixed point of the nonexpansive mapping $x \mapsto P_C(x - \lambda A^*(Ax - b))$.

Motivated by the above least-squares solution to constrained linear inverse problems, we will study the general case of finding the minimum-norm fixed point of a nonexpansive mapping $T : C \to C$:

$$x^\dagger \in \mathrm{Fix}(T), \quad \|x^\dagger\| = \min\{\|x\| : x \in \mathrm{Fix}(T)\}, \tag{1.11}$$

where $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$ denotes the set of fixed points of T (throughout we always assume that $\mathrm{Fix}(T) \neq \emptyset$).

We next briefly review two historic approaches which relate to the minimum-norm fixed point problem (1.11).

Browder [1] introduced an implicit scheme as follows. Fix a $u \in C$, and for each $t \in (0,1)$, let x_t be the unique fixed point in C of the contraction T_t which maps C into C:

$$T_t x = tu + (1-t)Tx, \qquad x \in C. \tag{1.12}$$

Browder proved that

$$s - \lim_{t \downarrow 0} x_t = P_{\mathrm{Fix}(T)} u. \tag{1.13}$$

That is, the strong limit of $\{x_t\}$ as $t \to 0^+$ is the fixed point of T which is nearest from $\mathrm{Fix}(T)$ to u.

Halpern [4], on the other hand, introduced an explicit scheme. Again fix a $u \in C$. Then with a sequence $\{t_n\}$ in $(0,1)$ and an arbitrary initial guess $x_0 \in C$, we can define a sequence $\{x_n\}$ through the recursive formula

$$x_{n+1} = t_n u + (1-t_n)Tx_n, \quad n \geq 0. \tag{1.14}$$

It is now known that this sequence $\{x_n\}$ converges in norm to the same limit $P_{\mathrm{Fix}(T)}u$ as Browder's implicit scheme (1.12) if the sequence $\{t_n\}$ satisfies, assumptions (A_1), (A_2), and (A_3) as follows:

(A_1) $\lim_{n\to\infty} t_n = 0$,

(A_2) $\sum_{n=1}^{\infty} t_n = \infty$,

(A_3) either $\sum_{n=1}^{\infty} |t_{n+1} - t_n| = \infty$ or $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$.

Some more progress on the investigation of the implicit and explicit schemes (1.12) and (1.14) can be found in [33–42]. We notice that the above two methods do find the minimum-norm fixed point $x^\dagger$ of T if $0 \in C$. However, if $0 \notin C$, then neither Browder's nor Halpern's method works to find the minimum-norm element $x^\dagger$. The reason is simple: if $0 \notin C$, we cannot take $u = 0$ either in (1.12) or (1.14) since the contraction $x \mapsto (1-t)Tx$ is no longer a self-mapping of C (hence may fail to have a fixed point), or $(1-t_n)Tx_n$ may not belong to C, and consequently, x_{n+1} may be undefined. In order to overcome the difficulties caused by possible exclusion of the origin from C, we introduce the following two remedies.

For Browder's method, we consider the contraction $x \mapsto (1-\beta)P_C[(1-t)x] + \beta Tx$ for some $\beta \in (0,1)$. Since this contraction clearly maps C into C, it has a unique fixed point which is still denoted by x_t, that is, $x_t = (1-\beta)P_C[(1-t)x_t] + \beta Tx_t$. For Halpern's method, we consider the following iterative algorithm $x_{n+1} = (1-\beta)P_C[(1-t_n)x_n] + \beta Tx_n,\ n \geq 0$. It is easily seen that the net $\{x_t\}$ and the sequence $\{x_n\}$ are well defined (i.e., $x_t \in C$ and $x_n \in C$).

The purpose of this paper is to prove that the above both implicit and explicit methods converge strongly to the minimum-norm fixed point $x^\dagger$ of the nonexpansive mapping T. Some applications are also included.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall that the nearest point (or metric) projection from H onto C is defined as follows: for each point $x \in H$, $P_C x$ is the unique point in C with the property

$$\|x - P_C x\| \leq \|x - y\|, \quad y \in C. \tag{2.1}$$

Note that P_C is characterized by the inequality

$$P_C x \in C, \quad \langle x - P_C x, y - P_C x\rangle \leq 0, \quad y \in C. \tag{2.2}$$

Consequently, P_C is nonexpansive.

Below is the so-called demiclosedness principle for nonexpansive mappings.

Lemma 2.1 (cf. [7]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping with fixed points. If (x_n) is a sequence in C such that $x_n \to x^*$ weakly and $x_n - Tx_n \to y$ strongly, then $(I-T)x^* = y$.*

Finally we state the following elementary result on convergence of real sequences.

Lemma 2.2 (see [19]). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\sigma_n, \quad n \geq 0, \tag{2.3}$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ are satisfied that

(i) $\sum_{n=0}^{\infty}\gamma_n = \infty$;

(ii) *either* $\limsup_{n\to\infty}\sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty}|\gamma_n\sigma_n| < \infty$.

Then $\{a_n\}_{n=0}^{\infty}$ converges to 0.

We use the following notation:

(i) $\mathrm{Fix}(T)$ stands for the set of fixed points of T;

(ii) $x_n \rightharpoonup x$ stands for the weak convergence of (x_n) to x;

(iii) $x_n \to x$ stands for the strong convergence of (x_n) to x.

3. Main Results

The aim of this section is to introduce some methods for finding the minimum-norm fixed point of a nonexpansive mapping T. First, we prove the following theorem by using an implicit method.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. For $\beta \in (0,1)$ and each $t \in (0,1)$, let x_t be defined as the unique solution of fixed point equation*

$$x_t = \beta T x_t + (1-\beta) P_C[(1-t)x_t], \quad t \in (0,1). \tag{3.1}$$

Then the net $\{x_t\}$ converges in norm, as $t \to 0^+$, to the minimum-norm fixed point of T.

Proof. First observe that, for each $t \in (0,1)$, x_t is well defined. Indeed, we define a mapping $S_t : C \to C$ by

$$S_t x = \beta T x + (1-\beta) P_C[(1-t)x], \quad x \in C. \tag{3.2}$$

For $x, y \in C$, we have

$$\begin{aligned}
\|S_t x - S_t y\| &= \|\beta(Tx - Ty) + (1-\beta)(P_C[(1-t)x] - P_C[(1-t)y])\| \\
&\leq \beta\|Tx - Ty\| + (1-\beta)\|P_C[(1-t)x] - P_C[(1-t)y]\| \\
&\leq [1 - (1-\beta)t]\|x - y\|,
\end{aligned} \tag{3.3}$$

which implies that S_t is a self-contraction of C. Hence S_t has a unique fixed point $x_t \in C$ which is the unique solution of the fixed point equation (3.1).

Next we prove that $\{x_t\}$ is bounded. Take $u \in \mathrm{Fix}(T)$. From (3.1), we have

$$\begin{aligned}
\|x_t - u\| &= \|\beta T x_t + (1-\beta) P_C[(1-t)x_t] - u\| \\
&\leq \beta\|T x_t - u\| + (1-\beta)\|P_C[(1-t)x_t] - u\| \\
&\leq \beta\|x_t - u\| + (1-\beta)\|(1-t)x_t - u\| \\
&\leq \beta\|x_t - u\| + (1-\beta)[(1-t)\|x_t - u\| + t\|u\|],
\end{aligned} \tag{3.4}$$

that is,

$$\|x_t - u\| \leq \|u\|. \tag{3.5}$$

Hence, $\{x_t\}$ is bounded and so is $\{T x_t\}$.

From (3.1), we have

$$\begin{aligned}\|x_t - Tx_t\| &\le (1-\beta)\|P_C[(1-t)x_t] - P_C[Tx_t]\| \\ &\le (1-\beta)\|x_t - Tx_t - tx_t\| \\ &\le (1-\beta)\|x_t - Tx_t\| + (1-\beta)t\|x_t\|,\end{aligned} \tag{3.6}$$

that is,

$$\|x_t - Tx_t\| \le \frac{1-\beta}{\beta} t\|x_t\| \longrightarrow 0 \quad \text{as } t \longrightarrow 0^+. \tag{3.7}$$

Next we show that $\{x_t\}$ is relatively norm-compact as $t \to 0^+$. Let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0^+$ as $n \to \infty$. Put $x_n := x_{t_n}$. From (3.7), we have

$$\|x_n - Tx_n\| \longrightarrow 0. \tag{3.8}$$

Again from (3.1), we get

$$\begin{aligned}\|x_t - u\|^2 &\le \beta\|Tx_t - u\|^2 + (1-\beta)\|P_C[(1-t)x_t] - u\|^2 \\ &\le \beta\|x_t - u\|^2 + (1-\beta)\|x_t - u - tx_t\|^2 \\ &= \beta\|x_t - u\|^2 + (1-\beta)\left[\|x_t - u\|^2 - 2t\langle x_t - u, x_t - u\rangle - 2t\langle u, x_t - u\rangle + t^2\|x_t\|^2\right].\end{aligned} \tag{3.9}$$

It turns out that

$$\|x_t - u\|^2 \le \langle u, u - x_t\rangle + tM, \tag{3.10}$$

where $M > 0$ is some constant such that $\sup\{(1/2)\|x_t\|^2 : t \in (0,1)\} \le M$. In particular, we get from (3.10)

$$\|x_n - u\|^2 \le \langle u, u - x_n\rangle + t_n M, \quad u \in \operatorname{Fix}(T). \tag{3.11}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $x^* \in C$. Noticing (3.8) we can use Lemma 2.1 to get $x^* \in \operatorname{Fix}(T)$. Therefore we can substitute x^* for u in (3.11) to get

$$\|x_n - x^*\|^2 \le \langle x^*, x^* - x_n\rangle + t_n M. \tag{3.12}$$

However, $x_n \rightharpoonup x^*$. This together with (3.12) guarantees that $x_n \to x^*$. The net $\{x_t\}$ is therefore relatively compact, as $t \to 0^+$, in the norm topology.

Now we return to (3.11) and take the limit as $n \to \infty$ to get

$$\|x^* - u\|^2 \le \langle u, u - x^* \rangle, \quad u \in \mathrm{Fix}(T). \tag{3.13}$$

This is equivalent to

$$0 \le \langle x^*, u - x^* \rangle, \quad u \in \mathrm{Fix}(T). \tag{3.14}$$

Therefore, $x^* = P_{\mathrm{Fix}(T)}0$. This is sufficient to conclude that the entire net $\{x_t\}$ converges in norm to x^* and x^* is the minimum-norm fixed point of T. This completes the proof. □

Next, we introduce an explicit algorithm for finding the minimum norm fixed point of nonexpansive mappings.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. For given $x_0 \in C$, define a sequence $\{x_n\}$ iteratively by*

$$x_{n+1} = \beta T x_n + (1-\beta) P_C[(1-\alpha_n)x_n], \quad n \ge 0, \tag{3.15}$$

where $\beta \in (0,1)$ and $\alpha_n \in (0,1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(C2) $\lim_{n\to\infty}(\alpha_n/\alpha_{n-1}) = 1$.

Then the sequence $\{x_n\}$ converges strongly to the minimum-norm fixed point of T.

Proof. First we prove that the sequence $\{x_n\}$ is bounded. Pick $p \in \mathrm{Fix}(T)$. Then, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta(Tx_n - p) + (1-\beta)(P_C[(1-\alpha_n)x_n] - p)\| \\
&\le \beta\|Tx_n - p\| + (1-\beta)\|P_C[(1-\alpha_n)x_n] - p\| \\
&\le \beta\|x_n - p\| + (1-\beta)\|(1-\alpha_n)(x_n - p) - \alpha_n p\| \\
&\le [1 - (1-\beta)\alpha_n]\|x_n - p\| + (1-\beta)\alpha_n\|p\| \\
&\le \max\{\|x_n - p\|, \|p\|\}.
\end{aligned} \tag{3.16}$$

By induction,

$$\|x_{n+1} - p\| \le \max\{\|x_0 - p\|, \|p\|\}. \tag{3.17}$$

Next, we estimate $\|x_{n+1} - x_n\|$. From (3.15), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \left\|\beta(Tx_n - Tx_{n-1}) + (1-\beta)\left(P_C[(1-\alpha_n)x_n] - P_C[(1-\alpha_{n-1})x_{n-1}]\right)\right\| \\
&\leq \beta\|Tx_n - Tx_{n-1}\| + (1-\beta)\|P_C[(1-\alpha_n)x_n] - P_C[(1-\alpha_{n-1})x_{n-1}]\| \\
&\leq \beta\|x_n - x_{n-1}\| + (1-\beta)\|(1-\alpha_n)(x_n - x_{n-1}) + (\alpha_{n-1} - \alpha_n)x_{n-1}\| \\
&\leq [1-(1-\beta)\alpha_n]\|x_n - x_{n-1}\| + (1-\beta)\alpha_n\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n}\|x_{n-1}\|.
\end{aligned} \tag{3.18}$$

This together with Lemma 2.2 implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.19}$$

Note that

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&\leq \|x_n - x_{n+1}\| + (1-\beta)\|P_C[(1-\alpha_n)x_n] - P_C[Tx_n]\| \\
&\leq \|x_n - x_{n+1}\| + (1-\beta)\|x_n - Tx_n\| + (1-\beta)\alpha_n\|x_n\|.
\end{aligned} \tag{3.20}$$

Thus,

$$\|x_n - Tx_n\| \leq \frac{1}{\beta}\left(\|x_n - x_{n+1}\| + (1-\beta)\alpha_n\|x_n\|\right) \longrightarrow 0. \tag{3.21}$$

We next show that

$$\limsup_{n\to\infty}\langle\tilde{x}, \tilde{x} - x_n\rangle \leq 0, \tag{3.22}$$

where $\tilde{x} = P_{\mathrm{Fix}(T)}0$, the minimum norm fixed point of T. To see this, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying the properties

$$\limsup_{n\to\infty}\langle\tilde{x}, \tilde{x} - x_n\rangle = \lim_{k\to\infty}\langle\tilde{x}, \tilde{x} - x_{n_k}\rangle, \tag{3.23}$$

$$x_{n_k} \rightharpoonup x^* \quad \text{as } k \longrightarrow \infty. \tag{3.24}$$

Now since $x^* \in \mathrm{Fix}(T)$ (this is a consequence of Lemma 2.2 and (3.21)), we get by combining (3.22) and (3.23)

$$\limsup_{n\to\infty}\langle\tilde{x}, \tilde{x} - x_n\rangle = \langle\tilde{x}, \tilde{x} - x^*\rangle \leq 0. \tag{3.25}$$

Finally, we show that $x_n \to \tilde{x}$. As a matter of fact, we have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq \beta\|Tx_n - \tilde{x}\|^2 + (1-\beta)\|P_C[(1-\alpha_n)x_n] - \tilde{x}\|^2 \\
&\leq \beta\|x_n - \tilde{x}\|^2 + (1-\beta)\|x_n - \tilde{x} - \alpha_n x_n\|^2 \\
&= \beta\|x_n - \tilde{x}\|^2 + (1-\beta)\left[(1-2\alpha_n)\|x_n - \tilde{x}\|^2 - 2\alpha_n\langle \tilde{x}, x_n - \tilde{x}\rangle + \alpha_n^2\|\tilde{x}\|^2\right] \\
&= \left[1 - 2(1-\beta)\alpha_n\right]\|x_n - \tilde{x}\|^2 + 2(1-\beta)\alpha_n\left(\langle \tilde{x}, \tilde{x} - x_n\rangle + \frac{\alpha_n\|\tilde{x}\|^2}{2}\right) \\
&= (1-\delta_n)\|x_n - \tilde{x}\|^2 + \delta_n\theta_n.
\end{aligned} \tag{3.26}$$

By (C1) and (3.22), it is easily found that $\lim_{n\to\infty}\delta_n = 0$ and $\limsup_{n\to\infty}\theta_n \leq 0$. We can therefore apply Lemma 2.2 to (3.26) and conclude that $x_{n+1} \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

4. Applications

We consider the following minimization problem

$$\min_{x\in C} \varphi(x), \tag{4.1}$$

where C is a closed convex subset of a real Hilbert space H and $\varphi : C \to \mathbb{R}$ is a continuously Fréchet differentiable convex function. Denote by S the solution set of (4.1); that is,

$$S = \left\{ z \in C : \varphi(z) = \min_{x\in C} \varphi(x) \right\}. \tag{4.2}$$

Assume $S = \emptyset$. It is known that a point $z \in C$ is a solution of (4.1) if and only if the following optimality condition holds:

$$z \in C, \quad \langle \nabla\varphi(z), x - z \rangle \geq 0, \quad x \in C. \tag{4.3}$$

(Here $\nabla\varphi(x)$ denotes the gradient of φ at $x \in C$.) It is also known that the optimality condition (4.3) is equivalent to the following fixed point problem,

$$z = T_\gamma z, \quad T_\gamma = P_C(I - \gamma\nabla\varphi), \tag{4.4}$$

where $\gamma > 0$ is any positive number. Note that the solution set S of (4.1) coincides with the set of fixed points of T_γ (for any $\gamma > 0$).

If the gradient $\nabla\varphi$ is L-Lipschitzian continuous on C, then it is not hard to see that the mapping T_γ is nonexpansive if $0 < \gamma < 2/L$.

Using Theorems 3.1 and 3.2, we immediately obtain the following result.

Theorem 4.1. *Assume φ is continuously (Fréchet) differentiable and convex and its gradient $\nabla\varphi$ is L-Lipschitzian. Assume the solution set S of the minimization* (4.1) *is nonempty. Fix γ such that $0 < \gamma < 2/L$.*

(i) *For each $t \in (0,1)$, let x_t be the unique solution of the fixed point equation*

$$x_t = \beta P_C\left[(I - \gamma\nabla\varphi)x_t\right] + (1-\beta)P_C[(1-t)x_t]. \tag{4.5}$$

Then $\{x_t\}$ converges in norm as $t \to 0^+$ to the minimum-norm solution of the minimization (4.1).

(ii) *Define a sequence $\{x_n\}$ via the recursive algorithm*

$$x_{n+1} = \beta P_C\left[(I - \gamma\nabla\varphi)x_n\right] + (1-\beta)P_C[(1-\alpha_n)x_n], \tag{4.6}$$

where the sequence $\{\alpha_n\}$ satisfies conditions (C1)-(C2) *in Theorem 3.2. Then $\{x_n\}$ converges in norm to the minimum-norm solution of the minimization* (4.1).

We next turn to consider a convexly constrained linear inverse problem

$$\begin{gathered} Ax = b, \\ x \in K, \end{gathered} \tag{4.7}$$

where A is a bounded linear operator with nonclosed range from a real Hilbert space H_1 to another real Hilbert space H_2 and $b \in H_2$ is given.

Problem (4.7) models many applied problems arising from image reconstructions, learning theory, and so on.

Due to some reasons (errors, noises, etc.), (4.7) is often illposed and inconsistent; thus regularization and least-squares are taken into consideration; that is, we look for a solution to the minimization problem

$$\min_{x \in K} \frac{1}{2}\|Ax - b\|^2. \tag{4.8}$$

Let S_b denote the solution set of (4.8). It is always closed convex (but possibly empty). It is known that S_b is nonempty if and only if $P_{\overline{A(K)}}(b) \in A(K)$. In this case, S_b has a unique element with minimum norm; that is, there exists a unique point $x^\dagger \in S_b$ satisfying

$$\left\|x^\dagger\right\| = \min\{\|x\| : x \in S_b\}. \tag{4.9}$$

The K-constrained pseudoinverse of A, $A_K^\dagger$, is defined as

$$D\left(A_K^\dagger\right) = \left\{b \in H_2 : P_{\overline{A(K)}}(b) \in A(K)\right\}, \quad A_K^\dagger(b) = x^\dagger, \quad b \in D\left(A_K^\dagger\right), \tag{4.10}$$

where $x^\dagger \in S_b$ is the unique solution to (4.9).

Set

$$\varphi(x) = \frac{1}{2}\|Ax - b\|^2. \tag{4.11}$$

Then $\varphi(x)$ is quadratic with gradient

$$\nabla\varphi(x) = A^*(Ax - b), \quad x \in H_1, \tag{4.12}$$

where A^* is the adjoint of A. Clearly $\nabla\varphi$ is Lipschitzian with constant $L = \|A^*A\| = \|A\|^2$. Therefore, applying Theorem 4.1, we obtain the following result.

Theorem 4.2. *Let $b \in D(A_K^\dagger)$. Fix γ such that $0 < \gamma < 2/\|A\|^2$.*

(i) *For each $t \in (0,1)$, let x_t be the unique solution of the fixed point equation*

$$x_t = \beta P_K\left[x_t - \gamma A^*(Ax_t - b)\right] + (1-\beta)P_K[(1-t)x_t]. \tag{4.13}$$

Then $\{x_t\}$ converges in norm as $t \to 0^+$ to $A_K^\dagger(b)$.

(ii) *Define a sequence $\{x_n\}$ via the recursive algorithm*

$$x_{n+1} = \beta P_K\left[x_n - \gamma A^*(Ax_n - b)\right] + (1-\beta)P_K[(1-\alpha_n)x_n], \tag{4.14}$$

where the sequence $\{\alpha_n\}$ satisfies conditions $(C1)$-$(C2)$ in Theorem 3.2. Then $\{x_n\}$ converges in norm to $A_K^\dagger(b)$.

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References

[1] F. E. Browder, “Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces,” *Archive for Rational Mechanics and Analysis*, vol. 24, pp. 82–90, 1967.
[2] F. E. Browder, “Convergence theorems for sequences of nonlinear operators in Banach spaces,” *Mathematische Zeitschrift*, vol. 100, pp. 201–225, 1967.
[3] F. E. Browder and W. V. Petryshyn, “Construction of fixed points of nonlinear mappings in Hilbert space,” *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.
[4] B. Halpern, “Fixed points of nonexpanding maps,” *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.
[5] Z. Opial, “Weak convergence of the sequence of successive approximations for nonexpansive mappings,” *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.
[6] P.-L. Lions, “Approximation de points fixes de contractions,” *Comptes Rendus de l’Académie des Sciences. Série I. Mathématique*, vol. 284, no. 21, pp. A1357–A1359, 1977.

[7] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.
[8] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, vol. 83 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1984.
[9] S. Reich, "Strong convergence theorems for resolvents of accretive operators in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 75, no. 1, pp. 287–292, 1980.
[10] S. Reich and H.-K. Xu, "An iterative approach to a constrained least squares problem," *Abstract and Applied Analysis*, vol. 2003, no. 8, pp. 503–512, 2003.
[11] S. Reich and A. J. Zaslavski, "Convergence of Krasnoselskii-Mann iterations of nonexpansive operators," *Mathematical and Computer Modelling*, vol. 32, no. 11–13, pp. 1423–1431, 2000.
[12] A. T.-M. Lau and W. Takahashi, "Fixed point properties for semigroup of nonexpansive mappings on Fréchet spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 11, pp. 3837–3841, 2009.
[13] E. M. Mazcuñán-Navarro, "Three-dimensional convexity and the fixed point property for nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 1-2, pp. 587–592, 2009.
[14] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.
[15] A. Sabharwal and L. C. Potter, "Convexly constrained linear inverse problems: iterative least-squares and regularization," *IEEE Transactions on Signal Processing*, vol. 46, no. 9, pp. 2345–2352, 1998.
[16] Y. J. Cho and X. Qin, "Convergence of a general iterative method for nonexpansive mappings in Hilbert spaces," *Journal of Computational and Applied Mathematics*, vol. 228, no. 1, pp. 458–465, 2009.
[17] Y.-L. Cui and X. Liu, "Notes on Browder's and Halpern's methods for nonexpansive mappings," *Fixed Point Theory*, vol. 10, no. 1, pp. 89–98, 2009.
[18] Y. Yao and H. K. Xu, "Iterative methods for finding minimum-norm fixed points of nonexpansive mappings with applications," *Optimization*. In press.
[19] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.
[20] H.-K. Xu, "Another control condition in an iterative method for nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 65, no. 1, pp. 109–113, 2002.
[21] H.-K. Xu, "Iterative methods for constrained Tikhonov regularization," *Communications on Applied Nonlinear Analysis*, vol. 10, no. 4, pp. 49–58, 2003.
[22] H. K. Xu and T. H. Kim, "Convergence of hybrid steepest-descent methods for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 119, no. 1, pp. 185–201, 2003.
[23] H.-K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.
[24] T. Suzuki, "A sufficient and necessary condition for Halpern-type strong convergence to fixed points of nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 135, no. 1, pp. 99–106, 2007.
[25] Y. Yao, Y. C. Liou, and G. Marino, "Strong convergence of two iterative algorithms for nonexpansive mappings in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 279058, 7 pages, 2009.
[26] X. Liu and Y. Cui, "The common minimal-norm fixed point of a finite family of nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 1, pp. 76–83, 2010.
[27] N. Shahzad, "Approximating fixed points of non-self nonexpansive mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 61, no. 6, pp. 1031–1039, 2005.
[28] H. Zegeye and N. Shahzad, "Viscosity methods of approximation for a common fixed point of a family of quasi-nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 7, pp. 2005–2012, 2008.
[29] H. Zegeye and N. Shahzad, "Strong convergence theorems for a finite family of nonexpansive mappings and semigroups via the hybrid method," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 325–329, 2010.
[30] H. Zegeye and N. Shahzad, "Viscosity approximation methods for a common fixed point of finite family of nonexpansive mappings," *Applied Mathematics and Computation*, vol. 191, no. 1, pp. 155–163, 2007.
[31] L. Wang, "An iteration method for nonexpansive mappings in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2007, Article ID 28619, 8 pages, 2007.
[32] L. Wang, Y.-J. Chen, and R.-C. Du, "Hybrid iteration method for common fixed points of a finite family of nonexpansive mappings in Banach spaces," *Mathematical Problems in Engineering*, vol. 2009, Article ID 678519, 9 pages, 2009.

[33] K. Shimoji and W. Takahashi, "Strong convergence to common fixed points of infinite nonexpansive mappings and applications," *Taiwanese Journal of Mathematics*, vol. 5, no. 2, pp. 387–404, 2001.

[34] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, "Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 8, pp. 2350–2360, 2007.

[35] N. Shioji and W. Takahashi, "Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 125, no. 12, pp. 3641–3645, 1997.

[36] W. Takahashi and K. Shimoji, "Convergence theorems for nonexpansive mappings and feasibility problems," *Mathematical and Computer Modelling*, vol. 32, no. 11–13, pp. 1463–1471, 2000.

[37] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.

[38] M. Kikkawa and W. Takahashi, "Approximating fixed points of infinite nonexpansive mappings by the hybrid method," *Journal of Optimization Theory and Applications*, vol. 117, no. 1, pp. 93–101, 2003.

[39] P. L. Combettes, "The convex feasibility problem in image recovery," in *Advances in Imaging and Electron Physics*, P. Hawkes, Ed., vol. 95, pp. 155–270, Academic Press, New York, NY, USA, 1996.

[40] P. L. Combettes and T. Pennanen, "Generalized Mann iterates for constructing fixed points in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 275, no. 2, pp. 521–536, 2002.

[41] S. A. Hirstoaga, "Iterative selection methods for common fixed point problems," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 1020–1035, 2006.

[42] H. H. Bauschke, "The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 202, no. 1, pp. 150–159, 1996.