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*Research Article*

# Moving Heat Source Reconstruction from the Cauchy Boundary Data

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We consider the problem of reconstruction of an unknown characteristic transient thermal source inside a domain. By introducing the definition of an extended dirichlet-to-Neumann map in the time-space cylinder and the adoption of the anisotropic Sobolev-Hilbert spaces, we can treat the problem with methods similar to those used in the analysis of the stationary source reconstruction problem. Further, the finite difference $\theta$ scheme applied to the transient heat conduction equation leads to a model based on a sequence of modified Helmholtz equation solutions. For each modified Helmholtz equation the characteristic star-shape source function may be reconstructed uniquely from the Cauchy boundary data. Using representation formula, we establish reciprocity functional mapping functions that are solutions of the modified Helmholtz equation to their integral in the unknown characteristic support.

## 1. Introduction

Inverse source transient heat problem has been studied by a huge number of authors. In relation to books with specific chapters in the subject, we can give special attention to Anger [1] and Isakov [2]. Those gives specific results for the problem of source reconstruction in models with different operators and overspecification of boundary conditions, and specifically demonstrates an uniqueness theorem related with the moving characteristic source studied in this work. Early works by Cannon and Pérez Esteva [3] studied stationary support reconstruction under hypotheses of a known intensity function, $f(t,x) = f(t)\chi_\omega(x)$. Some years later, Cannon and Pérez Esteva studied the same problem in a three-dimensional case [4]. More recently, Lefevre and Le Niliot [5] used the boundary elements method for identification of static and moving point sources. Also, it is important to mention the authors Hettlich and Rundell [6] who model the unmoving characteristic domain and El Badia and Ha Duong [7], whose stationary source reconstruction and the transient point sources reconstructions have a fundamental influence in the present work. The present work has come from the investigation of the stationary source reconstruction by the fundamental solution method in Alves et al. [8]. The adoption of the reciprocity gap functional method to solve the stationary source in the Laplace Poisson equation, Roberty and Alves [9], and the solution of the full identification of sources with the Helmholtz Poisson model, Alves et al. [10], have developed to the modeling adopted to the transient heat transient characteristic source reconstruction in this work. The model is based on the modified Helmholtz Poisson equation that is obtained from the transient equation through the $\theta$-scheme related to time finite differences discretization. Analysis of the related mathematical and computational work involved has been presented by the author in national conferences, Roberty and Alves [11, 12], Roberty and Sousa [13], and Roberty and Rainha [14, 15].

This paper will be structured as follows. Some definitions and mathematical results extracted from Lions and Magenes [16] that are important to the understanding of the inverse problems are presented in Section 2. There we introduce the concept of consistent Cauchy data, extended Dirichlet-to-Neumann map and the Green function formalism. The inverse transient heat source problem is introduced in Section 3. Basic lemma and theorems about the relative extended Dirichlet-to-Neumann map, existence of solution, reciprocity gap functional in the transient model, uniqueness of solution are demonstrated and discussed. These concepts present original aspects that we are introducing in the present work. We show that the transient problem can be studied with aid of results demonstrated for the Modified Helmholtz Dirichlet Source problem, in Section 4. There the iterative source reconstruction scheme, the uniqueness theorem for these sequences of stationary problems, and the reciprocity gap methodology is presented. We conclude by pointing out the advances introduced by the present work.

## 2. Direct Transient Heat Equation Problem

By $\Omega \subset \Re^d$, $d = 1, 2, 3$ we denote a bounded space domain with Lipschitz boundary $\Gamma = \partial\Omega$, which means that $\Omega$ is locally on one side of its connected boundary. The boundary may be locally parametrized by a Lipschitz continuous function. The more regular boundary will be named as smooth and will be locally parametrized with $C^\infty$ functions. In the spatial surface $\Gamma$ the normal $\nu$ is defined almost everywhere and the induced measure on the surface is denoted by $d\sigma$. In the time-space $\Re^{d+1}$, we consider the time interval $I := (0, T)$, $T > 0$ to form the bounded cylinder $Q := I \times \Omega$, whose lateral time-space surface is $\Sigma := I \times \Gamma$. A section in this cylinder is $\Omega_t := \{t\} \times \Omega$ and the complete cylinder boundary is

$$\partial Q = \overline{\Sigma} \cup \Omega_0 \cup \Omega_T, \tag{2.1}$$

where $\Omega_0$ and $\Omega_T$ are, respectively, the cylinders' bottom and top sections. At cylinder top and bottom there exist the corners $\Gamma_0 = \overline{\Omega_0} \cap \overline{\Sigma} \subset \Re^{d-1}$ and $\Gamma_T = \overline{\Omega_T} \cap \overline{\Sigma} \subset \Re^{d-1}$, respectively.

The direct transient heat source initial boundary value problem consists in finding $u(t, x)$ with $(t, x) \in Q$ given a boundary input $g(t, x)$ with $(t, x) \in \overline{\Sigma}$, an initial input

$u_0(x)$ with $(t,x) \in \Omega_0$ and a source distribution $f(t,x)$ with $(t,x) \in Q$ that verifies the problem:

$$\left(P_{u_0,g,f}\right) \quad \begin{cases} \partial_t u - \Delta u = f & \text{in } Q, \\ u = u_0 & \text{in } \Omega_0, \\ u = g & \text{on } \Sigma, \end{cases} \tag{2.2}$$

and Dirichlet data compatibility condition, $u_0 = g$ at the time-space cylinder corner $\Gamma_0$.

It is well known that this direct problem is well posed with a classical unique solution, Isakov introduced by [2], for initial and boundary data and coefficients in spaces of continuous and Holder continuous functions. For Hilbert space framework we need to introduce, following Lions and Magenes [16], anisotropic Sobolev spaces. For $r, s > 0$

$$H^{r,s}(Q) := L^2(I; H^r(\Omega)) \cap H^s\left(I; L^2(\Omega)\right) \tag{2.3}$$

and the associated lateral boundary spaces

$$H^{r,s}(\Sigma) := L^2(I; H^r(\Gamma)) \cap H^s\left(I; L^2(\Gamma)\right). \tag{2.4}$$

Here $H^r(\Omega)$ and $H^r(\Gamma)$, $r \geq 0$ are the Hilbert family of Sobolev space in the $L^2$ theory, and the space $L^2(I;X)$ denotes the class of functions that are strongly measurable on $I = [0,T]$ with range in $X$ with the following Hilbert norm:

$$\|v\|_{L^2(I;X)} = \left(\int_I \|v\|_X^2 dt\right)^{1/2} < \infty. \tag{2.5}$$

The normal space with null lateral boundary trace will be

$$H^{r,s}_{0,\bullet}(Q) := L^2\left(I; H^r_0(\Omega)\right) \cap H^s\left(I; L^2(\Omega)\right) \subset H^{r,s}(Q). \tag{2.6}$$

A comprehensive presentation of these spaces in the contest of boundary integral operators related with the heat equation and the heat potential can be found in Costabel [17]. The adjoint transient heat problem has a straightforward definition

$$\left(P^*_{v_T,g,f}\right) \quad \begin{cases} -\partial_t v - \Delta v = f & \text{in } Q, \\ v = v_T & \text{in } \Omega_T, \\ v = g & \text{on } \Sigma, \end{cases} \tag{2.7}$$

and Dirichlet data compatibility condition, $v_T = g$, at the time-space cylinder corner $\Gamma_T$. The time reversal operator

$$\kappa_T : H^{r,s}(Q) \longrightarrow H^{r,s}(Q), \qquad v(t,x) \longmapsto \kappa[v](t,x) = v(T-t,x) \tag{2.8}$$

can be used to change changes of variables $u^*(t,x) = v(T-t,x)$ and convert the adjoint problem into an equivalent direct problem. The following theorems, lemmas and proposition are important in the well posing of the direct and inverse problems related with the source reconstruction that we are investigating. We adapt the notation in order to make easy the use of these results and address the proofs to the related part in the book Lions and Magenes [16].

**Theorem 2.1** (first trace theorem). *For $u \in H^{r,s}(Q)$ with $r > 1/2$ and $s \geq 0$. We can define the continuous and linear mappings:*

(1) $\gamma[u] := u|_\Sigma$ *from* $H^{r,s}(Q) \to H^{r-1/2,s-s/2r}(\Sigma)$ *and*

(2) $\gamma_1[u] := \partial_\nu u|_\Sigma$ *from* $H^{r,s}(Q) \to H^{r-3/2,s-3s/2r}(\Sigma)$.

*Also, for $r \geq 0$ and $s > 1/2$ the continuous mapping*

(1) $\gamma_0[u] := u(0,x)$ *from* $H^{r,s}(Q) \to H^{r-r/2s}(\Omega_0)$ *and*

(2) $\gamma_T[u] := u(T,x)$ *from* $H^{r,s}(Q) \to H^{r-r/2s}(\Omega_T)$.

*Proof.* See Theorem 2.2 on Section 4 of Lions and Magenes [16]. □

*Remark 2.2* (nonsurjectivity of traces). The following mappings are not onto:

(1) $\gamma_0, \gamma_T : H^{r,s}(Q) \to H^{r-r/2s}(\Omega_0)$ if $r > 1/2$;

(2) $\gamma : H^{r,s}(Q) \to H^{r-1/2,s-s/2r}(\Sigma)$ if $s > 1/2$;

(3) $\gamma_1 : H^{r,s}(Q) \to H^{r-3/2,s-3s/2r}(\Sigma)$ if $s > 3/2$.

Two important spaces for applications are $H^{2,1}(Q)$ and $H^{1,1/2}(Q)$. For the first, only the normal trace $\gamma_1$ is surjective. For the second, both the traces on lateral boundary are surjective, that is, $\gamma$ and $\gamma_1$. In both cases the traces on initial and final time are not onto.

Since we need traces for treating nonhomogeneous problems such as (2.2), $P_{u_0,g,f}$ with its associated inverse source problem, we will introduce the following definitions.

*Definition 2.3* (Cauchy datum). By Cauchy datum associated with Problem (2.2) we mean the functions:

$$(u_0, g, u_T, g_\nu) \in H^{r-r/2s}(\Omega_0) \times H^{r-1/2,s-s/2r}(\Sigma) \times H^{r-r/2s}(\Omega_T) \times H^{r-3/2,s-3s/2r}(\Sigma). \tag{2.9}$$

*Definition 2.4* (consistent Cauchy datum). By consistent Cauchy datum associated with Problem (2.2) we mean the functions:

$$(u_0, g, u_T, g_\nu) \in (\gamma_0, \gamma, \gamma_T, \gamma_1)[H^{r,s}], \tag{2.10}$$

and by the first trace Theorem 2.1, we have that

$$(\gamma_0, \gamma, \gamma_T, \gamma_1)[H^{r,s}] \subset H^{r-r/2s}(\Omega_0) \times H^{r-1/2,s-s/2r}(\Sigma) \times H^{r-r/2s}(\Omega_T) \times H^{r-3/2,s-3s/2r}(\Sigma). \tag{2.11}$$

Now, since the consistent datum are in the range of the trace operators, the nonhomogeneous problem will be always well posed. From now on, by Cauchy data we will always mean consistent Cauchy data.

The well posedness of the problem holds for regular boundaries and may be resumed as follows.

**Theorem 2.5** (regular data $r \geq 0$). *Consider the problem $P_{u_0,g,f}$.*
*The regular nonhomogeneous problem with consistent data*

$$(u_0, g, f) \in H^{2(r+1/2)}(\Omega_0) \times H^{2r+3/2,r+3/4}(\Sigma) \times H^{2r,r}(Q), \quad r \geq 0, \ 2r, r \notin \mathbb{Z} + \frac{1}{2} \tag{2.12}$$

*and appropriated compatibility relations in corner $\Gamma_0$ admit a unique solution $u \in H^{2r+2,r+1}(Q)$. The associated mapping $G : (u_0, g, f) \to u$ is continuous. Note that for these initial time and Dirichlet data the final time and normal trace are*

$$(\gamma_T[u], \gamma_1[u]) \in H^{2(r+1/2)}(\Omega_T) \times H^{2r+1/2,r+1/4}(\Sigma). \tag{2.13}$$

*Proof.* See Theorem 6.2 in Section 4 of Lions and Magenes [16]. □

**Theorem 2.6** (distributional data $-1 < r < 0$). *Consider the problem $P_{u_0,g,f}$.*
*The distributional nonhomogeneous problem with consistent data*

$$(u_0, g, f) \in \mathscr{H}^{2(r+1/2)}(\Omega_0) \times \mathscr{H}^{2r+3/2,r+3/4}(\Sigma) \times \mathscr{H}^{2r,r}(Q), \quad -1 < r < 0, \tag{2.14}$$

*and appropriated compatibility relations in corner $\Gamma_0$ admit a unique solution $u \in H^{2r+2,r+1}(Q)$. The associated mapping $G : (u_0, g, f) \to u$ is continuous. Note that for these initial time and Dirichlet data the final time and normal trace are*

$$(\gamma_T[u], \gamma_1[u]) \in \mathscr{H}^{2(r+1/2)}(\Omega_T) \times \mathscr{H}^{2r+1/2,r+1/4}(\Sigma). \tag{2.15}$$

*Here we have defined*

$$\mathscr{H}^{\beta}(\Omega) = \begin{cases} H^{\beta}(\Omega) & \text{if } \beta \geq 0, \\ \left(H^{-\beta}(\Omega)\right)' & \text{if } \beta \leq 0, \end{cases}$$
$$\mathscr{H}^{\alpha,\frac{\alpha}{2}}(\Omega) = \begin{cases} H^{\alpha,\alpha/2}(\Omega) & \text{if } \alpha \geq 0, \\ \left(H^{-\alpha,-\alpha/2}(\Omega)\right)' & \text{if } \alpha \leq 0. \end{cases} \tag{2.16}$$

*Proof.* See Subsection 15.1 on Section 4 of Lions and Magenes [16]. □

*Definition 2.7* (the dual operator domain $X^{r+1}(Q)$). If $r \geq 0$ with $2r$ and $r \notin \mathbb{Z} + 1/2$,

$$X^{r+1}(Q) = \left\{ v \in H^{2(r+1),r+1}(Q),\ \gamma[v] = 0,\ \gamma_T[v] = 0,\ -\partial_t v - \Delta v \in H^{2r,r}_{0,0}(Q) \right\}, \tag{2.17}$$

where $H^{2r,r}_{0,0}(Q)$ is the closure of $\mathfrak{D}(Q)$ in $H^{2r,r}(Q)$.

**Lemma 2.8** (adjoint isomorphis of order $r+1$). *$X^{r+1}(Q)$ with the graph norm*

$$\|v\|_{X^{r+1}(Q)} = \left( \|v\|^2_{H^{2(r+1),r+1}(Q)} + \|(-\partial_t - \Delta)v\|^2_{H^{2r,r}(Q)} \right)^{1/2} \tag{2.18}$$

*is a Hilbert space. For $r \geq 0$ and with $2r$ and $r \notin \mathbb{Z} + 1/2$, the adjoint operator $-\partial_t - \Delta$ is an isomorphism of $X^{r+1}(Q)$ onto $H^{2r,r}_{0,0}(Q)$.*

*Proof.* See Section 7.2 on Section 4 of Lions and Magenes [16]. □

*Definition 2.9* (the space $\Xi^{2(r+1),r+1}(Q)$). By definition $X^{r+1}(Q) \subset H^{2(r+1),r+1}(Q)$. We need a space $\Xi^{2(r+1),r+1}(Q)$ with the following properties:

(1) the space $\mathfrak{D}(Q)$ is dense in $\Xi^{2(r+1),r+1}(Q)$ for $r > -1$;

(2) $X^{r+1}(Q) \subset H^{2(r+1),r+1}(Q) \subset \Xi^{2(r+1),r+1}(Q)$ for $r \geq 0$, $2r$ and $r \notin \mathbb{Z} + 1/2$;

(3) it is least space with these two properties.

This space is independent of properties of the boundary operators and is defined in Section 9.1 of Lions and Magenes [16] by using functions equivalent to the distance to the boundary and interpolation Theorems.

*Definition 2.10* (the dual space $\Xi^{-2(r+1),-(r+1)}(Q)$). The dual space

$$\Xi^{-2(r+1),-(r+1)}(Q) = \left(\Xi^{2(r+1),r+1}(Q)\right)' \tag{2.19}$$

for $r \geq -1$ is a distribution-space-appropriated definition of continuous linear forms on $X^{r+1}(Q)$ and consequently for source definition.

**Theorem 2.11** (distributional data $r \leq -1$). *Consider the problem $P_{u_0,g,f}$.*

*The distributional nonhomogeneous problem with consistent data*

$$(u_0, g, f) \in \Xi^{2(r+1/2)}(\Omega_0) \times H^{2r+3/2,r+3/4}(\Sigma) \times \Xi^{2r,r}(Q), \quad r \leq -1,\ 2r, r \notin \mathbb{Z} + \frac{1}{2} \tag{2.20}$$

*and appropriated compatibility relations in corner $\Gamma_0$ admit a unique solution $u \in H^{2r+2,r+1}(Q)$. The associated mapping $G : (u_0, g, f) \rightarrow u$ is continuous. Note that for these initial time and Dirichlet data the normal trace is*

$$(\gamma_T[u], \gamma_1[u]) \in \Xi^{2(r+1/2)}(\Omega_T) \times H^{2r+1/2,r+1/4}(\Sigma). \tag{2.21}$$

*Proof.* See Theorem 12.1 of Lions and Magenes [16]. □

The following theorems represent an effort based on domain and range operator modification found on Chapter 4 of the book Lions and Magenes [16], in order to make traces mappings onto. We include them to illustrate the complexity involved. Since our main commitment is with the inverse source problem, we consider it more efficient to simplify the process to assure Cauchy data consistence by restricting its range in the first trace Theorem 2.1.

*Definition 2.12* (the domain space $D^{-r}_{\partial_t-\Delta}(Q)$). One has

$$D^{-r}_{\partial_t-\Delta}(Q) = \left\{u \in H^{-2r,-r}(Q), (\partial_t - \Delta u) \in \Xi^{-2(r+1),-(r+1)}(Q)\right\} \tag{2.22}$$

with $r \geq 0$, $2r$ and $r \notin \mathbb{Z} + 1/2$. Provided the graph norm

$$\|v\|_{D^{-r}_{\partial_t-\Delta}(Q)} = \left(\|v\|^2_{H^{-2r,-r}(Q)} + \|(\partial_t - \Delta)v\|^2_{\Xi^{-2(r+1),-(r+1)}(Q)}\right)^{1/2}, \tag{2.23}$$

it is a Hilbert space.

**Theorem 2.13.** *The mappings*

$$\begin{gathered}\gamma : D^{-r}_{\partial_t-\Delta}(Q) \longrightarrow H^{-2r-1/2,-r-1/4}(\Sigma), \\ \gamma_1 : D^{-r}_{\partial_t-\Delta}(Q) \longrightarrow H^{-2r-3/2,-r-3/4}(\Sigma)\end{gathered} \tag{2.24}$$

*for every $r \geq 0$ with $2r$ and $r \notin \mathbb{Z} + 1/2$ are continuous and linear. Further these mappings are onto if their domains and range are restricted to*

$$\gamma : \mathcal{V}(Q) \longrightarrow H_{0,\bullet}^{2r+1/2,r+1/4}(\Sigma),$$
$$\gamma_1 : \mathcal{V}(Q) \longrightarrow H_{0,\bullet}^{2r+3/2,r+3/4}(\Sigma), \tag{2.25}$$

*where* $\mathcal{V} = \{v \in H_{0,\bullet}^{2(r+1),r+1}(Q), (-\partial_t - \Delta)v \in H^{2(r-1),r-1}(Q)\}$.

**Theorem 2.14.** *The mappings*

$$\gamma_0 : D_{\partial_t - \Delta}^{-r}(Q) \longrightarrow H^{-2r+1/2}(\Omega_0),$$
$$\gamma_T : D_{\partial_t - \Delta}^{-r}(Q) \longrightarrow H^{-2r+1/2}(\Omega_T) \tag{2.26}$$

*for every $r \geq 0$ with $2r$ and $r \notin \mathbb{Z} + 1/2$ are continuous and linear. Further these mappings are onto if their domains and range are restricted to*

$$\gamma_0 : \mathcal{V}_1(Q) \longrightarrow H_0^{2r+1/2}(\Omega_0),$$
$$\gamma_T : \mathcal{V}_1(Q) \longrightarrow H_0^{2r+3/2}(\Omega_T), \tag{2.27}$$

*where* $\mathcal{V}_1 = \{v \in H^{2(r+1),r+1}(Q), (-\partial_t - \Delta)v \in H_{0,0}^{2(r-1),r-1}(Q)\}$.

*Proof.* See Theorem 10.4 and Lemma 10.2 in Section 4 of Lions and Magenes [16]. □

*Remark 2.15.* The data in the lateral and bottom $d$-dimensional surfaces of time-space cylinder can be considered as Dirichlet prescribed boundary data in the extended boundary $\overline{\Omega_0 \cup \Sigma}$ of the transient heat equation problem. This set is adjoint by the extended boundary $\overline{\Omega_T \cap \Sigma}$ for the adjoint problem. Since the transient heat problem has one derivative in time and two derivatives in space, the problem and its adjoint are posed with the following set of Cauchy data: prescribed only Dirichlet on the bottom $\Omega_0$ and on the top $\Omega_T$ of the cylinder and both Dirichlet and Neumann data at the lateral cylinder surface $\Sigma$. If there exists data compatibility at corners $\Gamma_0$ and $\Gamma_T$, it will give to the transient heat problem a character similar to the Poisson Laplace equation.

**Lemma 2.16** (solution operator). *The solution operator is a continuous but not injective linear operator associated with the Direct Problem* (2.2) $P_{u_0,g,f}$

(1) *for regular data with $r \geq 0$ and if $2r$ and $r \notin \mathbb{Z} + 1/2$*

$$S : H^{2(r+1/2)}(\Omega) \times H^{2r+3/2,r+3/4}(\Sigma) \times H^{2r,r}(Q) \longrightarrow H^{2r+2,r+1}(Q), \tag{2.28}$$

(2) *for distributional data with $-1 < r < 0$*

$$S : \mathscr{H}^{2(r+1/2)}(\Omega) \times \mathscr{H}^{2r+3/2,r+3/4}(\Sigma) \times \mathscr{H}^{2r,r}(Q) \longrightarrow \mathscr{H}^{2r+2,r+1}(Q), \tag{2.29}$$

(3) *for distributional data with* $r \leq -1$ *and if* $2r$ *and* $r \notin \mathbb{Z} + 1/2$

$$S : \Xi^{2(r+1/2)}(\Omega) \times H^{2r+3/2,r+3/4}(\Sigma) \times \Xi^{2r,r}(Q) \longrightarrow H^{2r+2,r+1}(Q), \tag{2.30}$$

*defined by*

$$S(u_0, g, f) = u \tag{2.31}$$

*when* $u \in H^{2r+2,r+1}(Q)$ *is solution of Problem* (2.2) *with initial data* $(u_0, g) = (u|_{\Omega_0}, u|_{\Sigma})$.

*Proof.* It is a combination of results found in Theorem 6.2, Subsection 15.1 and Theorem 12.1 on Chapter 4 of Lions and Magenes [16]. Note that operator $S$ is continuous, but not necessarily one to one. □

*Definition 2.17* (extended dirichlet-to-neumann map). We call The extended Dirichlet-to-Neumann map for the Problem (2.2)

(1) for regular data with $r \geq 0$ and $2r$ and $r \notin \mathbb{Z} + 1/2$:

$$\Lambda^f_{\Omega,\Sigma} : H^{2r+1}(\Omega_0) \times H^{2r+3/2,r+3/4}(\Sigma) \longrightarrow H^{2r+1}(\Omega_T) \times H^{2r+1/2,r+1/4}(\Sigma), \tag{2.32}$$

(2) for distributional data with $-1 < r < 0$

$$\Lambda^f_{\Omega,\Sigma} : \mathscr{H}^{2r+1}(\Omega_0) \times \mathscr{H}^{2r+3/2,r+3/4}(\Sigma) \longrightarrow \mathscr{H}^{2r+1}(\Omega_T) \times \mathscr{H}^{2r+1/2,r+1/4}(\Sigma), \tag{2.33}$$

(3) for distributional data with $r \leq -1$ and $2r$ and $r \notin \mathbb{Z} + 1/2$:

$$\Lambda^f_{\Omega,\Sigma} : \Xi^{2r+1}(\Omega_0) \times H^{2r+3/2,r+3/4}(\Sigma) \longrightarrow \Xi^{2r+1}(\Omega_T) \times H^{2r+1/2,r+1/4}(\Sigma), \tag{2.34}$$

defined by

$$\Lambda^f_{\Omega,\Sigma}[(u_0, g)] = (u|_{\Omega_T}, \partial_\nu u|_{\partial\Omega}) \tag{2.35}$$

when $u \in H^{2r+2,r+1}(Q)$ is solution of Problem (2.2) with initial data $(u_0, g) = (u|_{\Omega_0}, u|_{\Sigma})$.

Note that this operator can be viewed as a combination of the standard Dirichlet to Neumann map in the spatial boundary with the input-to-output map in the time boundary, that is, in initial and final interval times, found in control theory.

*Definition 2.18* (dirichlet green function). By the Dirichlet Green's function $G(t, x, \tau, \zeta)$ for the Problem (2.2) we mean its solution with source $\delta(x - \zeta, t - \tau)$, $(t, x, \tau, \zeta) \in Q \times Q$, and homogeneous Dirichlet data on the extended boundary $\Omega_T \times \Sigma$, that is, $G(t, x, \tau, \zeta) = 0$ for $(t, x)$ on $\Omega_T \times \Sigma$.

*Remark 2.19.* For regular data the Green's function exist, Costabel [17], and we can show that

$$u(t,x) = \int_{\Omega_0} u_0(\zeta)G(t,x,0,\zeta)d\zeta + \int_{\Sigma} g(\tau,\zeta)\frac{\partial G(t,x,\tau,\zeta)}{\partial \nu_{(\tau,\zeta)}}d\sigma_{(\tau,\zeta)} + \int_{Q} f(\tau,\zeta)G(t,x,\tau,\zeta)d\zeta d\tau \tag{2.36}$$

for $(t,x) \in \overline{Q}$ is an explicit solution $S$ to Problem (2.2). By using problem's (2.2) linearity we formally decompose the solution in three parts

$$\begin{gathered} u = S\left[u_0, g, f\right] := S[u_0, 0, 0] + S\left[0, g, 0\right] + S\left[0, 0, f\right], \\ S\left[u_0, g, f\right] : H^{2(r+1/2)}(\Omega) \times H^{2r+3/2, r+3/4}(\Sigma) \times H^{2r,r}(Q) \longrightarrow H^{2r+2,r+1}(Q), \end{gathered} \tag{2.37}$$

where

$$S[u_0, 0, 0] : H^{2(r+1/2)}(\Omega) \longrightarrow H^{2r+2,r+1}(Q) \tag{2.38}$$

is the homogeneous Dirichlet zero source initial value auxiliary problem solution and

$$S\left[0, g, 0\right] : H^{2r+3/2, r+3/4}(\Sigma) \longrightarrow H^{2r+2,r+1}(Q) \tag{2.39}$$

is the zero source zero initial value auxiliary Dirichlet problem solution and

$$S\left[0, 0, f\right] : H^{2r,r}(Q) \longrightarrow H^{2r+2,r+1}(Q) \tag{2.40}$$

is the zero data auxiliary Dirichlet auxiliary problem.

**Lemma 2.20** (composition of trace and solution). *The extended Dirichlet-to-Neumann map is a composition of the final time trace and the lateral boundary normal trace with the Solution operator:*

$$\Lambda^f_{\Omega,\Sigma}[u_0, g] = (\gamma_T, \gamma_1)S\left[u_0, g, f\right] = \left(\gamma_T \circ S, \gamma_1 \circ S\right)\left[u_0, g, f\right] = \left(u_T, g^{\nu}\right). \tag{2.41}$$

*Proof.* Consequence of the definition. □

## 3. Inverse Transient Heat Equation Source Problem

The inverse source problem that we address consists in the recovery of the source $f$, knowing the extended Dirichlet-to-Neumann map $\Lambda^f_{\Omega,\Sigma}$. When $r = 0$, the data are regular, the Green's function exists and $f \in L^2(Q)$. Let us investigate this situation, and then, we will show that the unique information available in this inverse problem is given only by one measurement, say, the bottom and top Dirichlet data and lateral cylinder Cauchy boundary data. The inverse problem $\mathrm{IP}^{f}_{(u_0,g),(u_T,g^\nu)}$ is to find $f \in L^2(Q)$ such that

$$\left(\mathrm{IP}^{f}_{(u_0,g),(u_T,g^\nu)}\right) \quad (u_T, g^\nu) = \Lambda^{f}_{\Omega,\Sigma}(u_0, g) \tag{3.1}$$

for all given data pair $(u_0, g) \times (u_T, g^\nu)$ corresponding to different solutions to the direct problem. By taking the normal trace at lateral cylinder boundary of the solution (2.36), we obtain that

$$\gamma_1[u] = \left.\frac{\partial u}{\partial \nu_{(t,x)}}\right|_{\Sigma} = \Lambda^{f}_{\bullet,\Sigma}[(u_0, g)] = \gamma_1[S[u_0,0,0]] + \gamma_1[S[0,g,0]] + \gamma_1[S[0,0,f]] \tag{3.2}$$

or

$$\begin{aligned}\gamma_1[u](t,x) = {}& \int_{\Omega_0} u_0(\zeta) \left.\frac{\partial G(t,x,0,\zeta)}{\partial \nu_{(t,x)}}\right|_{\Sigma} d\zeta - \int_{\Sigma} g(\tau,\zeta) \left.\frac{\partial^2 G(t,x,\tau,\zeta)}{\partial \nu_{(t,x)}\partial \nu_{(\tau,\zeta)}}\right|_{\Sigma} d\sigma_{(\tau,\zeta)} \\ &+ \int_{Q} f(\tau,\zeta) \left.\frac{\partial G(t,x,\tau,\zeta)}{\partial \nu_{(t,x)}}\right|_{\Sigma} d\zeta d\tau\end{aligned} \tag{3.3}$$

is an explicit expression of the lateral part of the extended Dirichlet-to-Neumann map $\Lambda^{f}_{\bullet,\Sigma}$. mapping for $f \in L^2(Q)$

$$\Lambda^{f}_{\bullet,\Sigma} : H^1(\Omega) \times H^{3/2,3/4}(\Sigma) \longrightarrow H^{1/2,1/4}(\Sigma). \tag{3.4}$$

Note that it appears decomposed in its partial traces

$$\Lambda^{f}_{\bullet,\Sigma}[(u_0, g)] := \Lambda^{0}_{\bullet,\Sigma}[(u_0, 0)] + \Lambda^{0}_{\bullet,\Sigma}[(0, g)] + \Lambda^{f}_{\bullet,\Sigma}[(0,0)], \tag{3.5}$$

where

$$\Lambda^{0}_{\bullet,\Sigma}[(\cdot, 0)] : H^1(\Omega_0) \longrightarrow H^{1/2,1/4}(\Sigma) \tag{3.6}$$

is the zero source auxiliary initial value zero Dirichlet problem solution lateral normal trace and

$$\Lambda^{0}_{\bullet,\Sigma}[(0, \cdot)] : H^{3/2,3/4}(\Sigma) \longrightarrow H^{1/2,1/4}(\Sigma) \tag{3.7}$$

is the zero source nonhomogeneous Dirichlet zero initial auxiliary problem solution lateral normal trace and

$$\Lambda^{f}_{\bullet,\Sigma}[(0,0)] \in H^{1/2,1/4}(\Sigma) \tag{3.8}$$

is the homogeneous Dirichlet zero initial source auxiliary problem solution lateral normal trace.

By taking the final time trace at cylinder top boundary of the solution (2.36) we obtain that

$$\gamma_T[u] = u(T,\cdot) = \Lambda^f_{\Omega,\bullet}[(u_0,g)] = \gamma_T[S[u_0,0,f]] + \gamma_T[S[0,g,0]] + \gamma_T[G[0,0,f]] \tag{3.9}$$

or

$$\begin{aligned}\gamma_T[u](t,x) = &\int_{\Omega_0} u_0(\zeta)G(T,x,0,\zeta)d\zeta - \int_\Sigma g(\tau,\zeta)\frac{\partial G(T,x,\tau,\zeta)}{\partial\nu_{(\tau,\zeta)}}d\sigma_{(\tau,\zeta)} \\ &+ \int_Q f(\tau,\zeta)G(T,x,\tau,\zeta)d\zeta d\tau\end{aligned} \tag{3.10}$$

is an explicit expression of the final part of the input-to-output map

$$\Lambda^f_{\Omega,\bullet} : H^1(\Omega_0)\times H^{3/2,3/4}(\Sigma) \longrightarrow H^1(\Omega_T). \tag{3.11}$$

Note that its appears decomposed in its partial traces

$$\Lambda^f_{\Omega,\bullet}[(u_0,g)] := \Lambda^0_{\Omega,\bullet}[(u_0,0)] + \Lambda^0_{\Omega,\bullet}[(0,g)] + \Lambda^f_{\Omega,\bullet}[(0,0)], \tag{3.12}$$

where

$$\Lambda^0_{\Omega,\bullet}[(\cdot,0)] : H^1(\Omega_0) \longrightarrow H^1(\Omega_T) \tag{3.13}$$

is the zero source auxiliary initial value zero Dirichlet problem solution final trace,

$$\Lambda^0_{\bullet,\Sigma}[(0,\cdot)] : H^{3/2,3/4}(\Sigma) \longrightarrow H^1(\Omega_T) \tag{3.14}$$

is the zero source nonhomogeneous Dirichlet zero initial auxiliary problem solution final trace and

$$\Lambda^f_{\Omega,\bullet}[(0,0)] \in H^1(\Omega_T) \tag{3.15}$$

is the homogeneous Dirichlet zero initial source auxiliary problem solution final trace.

*Definition 3.1* (relative extended dirichlet-to-neumann map). Consider two problems $P_{u_0,g,f}$ and $P_{u_0,g,0}$, one with source $f$ and two other with zero source, but both with the same consistent initial time and Dirichlet data. By the Relative extended Dirichlet-to-Neumann map for $f \in L^2(Q)$ we mean the application:

$$\Lambda^f_{\Omega,\Sigma} - \Lambda^0_{\Omega,\Sigma} : H^1(\Omega_0)\times H^{3/2,3/4}(\Sigma) \longrightarrow H^1(\Omega_T)\times H^{1/2,1/4}(\Sigma). \tag{3.16}$$

Note that the consistence of data $(u_0, g)$ is necessary to existence of solution to the problems $P_{u_0,g,f}$ and $P_{u_0,g,0}$.

**Lemma 3.2.** *Let $u_j$, $j = 1, 2, 3, \ldots$ be different solutions of Problem* (2.2) *with the same source $f \in L^2(Q)$ and different initial time and Dirichlet data $(u_{0_j}, g_j)$, $j = 1, 2, 3, \ldots$, respectively. Then*

(i) *the relative extended Dirichlet-to-Newman operator $\Lambda^f_{\Omega,\Sigma} - \Lambda^0_{\Omega,\Sigma}$ is an operator whose functional value depends only on the source function $f \in L^2(Q)$, but is independent of the initial time and Dirichlet data $(u_0, g)$,*

(ii) *tor all solution of consistent data problems $P_{f,u_{0_j},g_j}$, $j = 1, 2, 3, \ldots$, with the same source, the source satisfies the systems of integral equations*

$$\int_Q f(\tau,\zeta)\left(G(T,x,\tau,\zeta), \frac{\partial G(t,x,\tau,\zeta)}{\partial \nu_{(t,x)}}\right) d\zeta d\tau = \left(\Lambda^f_{\Omega,\Sigma} - \Lambda^0_{\Omega,\Sigma}\right)\left[u_{0_j}, g_j\right] = \Lambda^f_{\Omega,\Sigma}[0,0] \tag{3.17}$$

*which depend only on the Relative extended Dirichlet-to-Neumann map.*

*Proof.* Both results (i) and (ii) are trivial consequences of (3.3) and (3.10). □

*Remark 3.3.* Note that in this case the unique information available for source reconstruction is given by only one measurement, that is, that final-Neumann boundary measurement

$$\left(u_T, \partial_{\nu_{(t,x)}} u\right) = \Lambda^f_{\Omega,\Sigma}[u_0, g] = \Lambda^f_{\Omega,\Sigma}[0,0] \in H^1(\Omega_T) \times H^{1/2,1/4}(\Sigma) \tag{3.18}$$

corresponding to some specific initial-Dirichlet data $(u_0, g)$, which may be assumed as zero without loss of generality.

### *3.1. Existence of Regular Solution*

The fact that only one Cauchy data can be used in a nonredundant source reconstruction suggests a method for solved inverse source problem that is based on the formulation of a high-order well posed Dirichlet equation problem. As observed by Friedman [18] and explored in this work, the parabolic transient heat equation behavior is similar to that of elliptic stationary equation. They coexist in a larger class of partial differential equations. In that class we found also the fourth-order in space and second order in time equation that is obtained if we apply simultaneously the transient heat operator (2.2) and its adjoint (2.7) to an auxiliary field related with the fields $u$ as in problem (3.20). We have thus existence of solution in an especial class analogous to the class introduced in Alves et al. [10] for the Helmholtz operator problem,

$$H^{r,s}_{\partial_t - \Delta} = \left\{ f \in H^{r,s}(Q) \mid (-\partial_t - \Delta) f = 0 \right\}. \tag{3.19}$$

**Proposition 3.4.** *Suppose that the available consistent Cauchy data to be used in the reconstruction is $(u_0, g) \times (u_T, g_\nu) \in (H^1(\Omega_0), H^{3/2,3/4}(\Sigma)) \times (H^1(\Omega_T), H^{1/2,1/4}(\Sigma))$, then there exist a solution to inverse source problem, problem* (3.1).

*Proof.* In order to determine the Relative extended Dirichlet-to-Neumann Map, we first use the data to solve the zero source auxiliary problem $P_{u_0,g,0}$, obtaining $\Lambda^0_{\Omega,\Sigma}(u_0,g) = (u^0_T, g^0_\nu)$. Since this datum is consistent, this solution exists. Then $\Lambda^0_{\Omega,\Sigma}(0,0) = \Lambda^f_{\Omega,\Sigma}(u_0,g) - \Lambda^0_{\Omega,\Sigma}(u_0,g)$. We can now consider the following fourth-order problem formulated, as it has been suggested in the introduction of this subsection:

$$\left(P^4_{0,u_0,u_T,g,g^\nu}\right) \quad \begin{cases} (-\partial_t - \Delta)(\partial_t - \Delta)v = (-\partial_t^2 + \Delta^2)v = 0 & \text{in } Q, \\ v = 0 & \text{in } \Omega_0, \\ v = \Lambda^0_{\Omega,\bullet}(0,0) & \text{in } \Omega_T, \\ v = 0 & \text{on } \Sigma, \\ \partial_\nu v = \Lambda^0_{\bullet,\Sigma}(0,0) & \text{on } \Sigma. \end{cases} \tag{3.20}$$

The fourth-order operator $(-\partial_t^2 + \Delta^2)$ can be separated with an unbounded sequence of parameters $\lambda$ into two elliptic operators, one being biharmonic type in space $(-\lambda + \Delta^2)$, and the other second order in time $(-\partial_t^2 + \lambda)$. Since the initial condition is zero, by finite energy method, the regularity of these operators is $H^1(I, L^2(\Omega))$ and $L^2(I; H^2(\Omega))$, respectively, and it is well posed and has a unique solution $v = S(u_0, g, u_T, g^\nu) \in H^{2,1}(Q)$. The inverse source solution will be $f = (-\partial_t - \Delta)v$. □

*Remark 3.5.* The fourth-order problem used to show existence has the same equation used for modeling the direct problem involving elastic vibration of beams or plates, $-\partial^2 u - c^2\Delta^2 u = F$, but with an unphysical imaginary parameter $c = 1i$. Obviously, as a direct problem equation (3.20) is unphysical, but as a fourth-order model for the inverse source reconstruction with source in the class (3.19), it is just a method for source reconstruction based on higher order equation. Analytical solutions for this equation in one-dimension and two-dimensions polar geometry can be determined by noting the following complete and dense sets, respectively:

(1) $\{X_n(x) = \cosh(\kappa_n x) - \cos(\kappa_n x) - ((\cosh(\kappa_n) - \cos(\kappa_n))/(\sinh(\kappa_n) - \sin(\kappa_n)))(\sinh(\kappa_n x) - \sin(\kappa_n x))\} \subset L^2(0,1)$ where $\kappa_n$, $n = 1, 2, \ldots$ are positive roots of the equation $\cosh(\kappa)\cos(\kappa) = 1$;

(2) $\{\phi_{mn}(r,\theta) = [I_m(\kappa_{mn} J_m(\kappa_{mn} r) - J_m(\kappa_{mn} I_m(\kappa_{mn} r)] \exp(1\,\mathrm{im}\,\theta)\} \subset L^2(B(0,1))$ where $\kappa_{mn}$, $m, n = 1, 2, \ldots$ are the positive roots of the systems of equations

$$J_m(\kappa) I'(\kappa) - I_m(\kappa) J'(\kappa) = 0, \quad m = 1, 2, \ldots. \tag{3.21}$$

Note that $m, n \to \infty$, $\kappa_n \to \infty$ and $\kappa_{m,n} \to \infty$. In the case of the hyperbolic behavior of the wave-type equation that appears in beams and plates, the data in time are initial and the initial derivative traces and can be adjusted in the non-homogeneous case (this is expected since the boundary control is exact, Alves et al. [19]). In our fourth-order problem that combines the heat equation and its adjoint, we have a two-point boundary at times $t = 0$ and $t = T$ that cannot be simultaneously controlled (this parabolic type problem is only null controllable, Tucsnak and George Weiss [20]) and it is not possible to adjust at the same time the two time boundaries. Since by Lemma 3.2. we can always avoid nonhomogeneous initial condition, the problem is well posed. So, we have here a heat equation control analogous to the harmonic control in the Laplacian Dirichlet operator source reconstruction from boundary Neumann data problem.

### 3.2. The Regular Data Class

*Definition 3.6* (regular class). We call a class of distributional consistent data $\mathcal{R}$ regular if its relative extended Dirichlet-to-Neumann map is regular

$$\mathcal{R} = \Big\{ (u_0, g), (u_T, g_\nu) \in \Big( \mathscr{H}^{2r+1}(\Omega_0) \times \mathscr{H}^{2r+3/2, r+3/4}(\Sigma) \Big) \times \Big( \mathscr{H}^{2r+1}(\Omega_T) \times \mathscr{H}^{2r+1/2, r+1/4}(\Sigma) \Big) \mid$$
$$-1 < r < 0; \Big( \Lambda^f_{\Omega,\Sigma} - \Lambda^0_{\Omega,\Sigma} \Big)[u_0, g] \in (\gamma_T, \gamma_1)\Big[ H^{2,1}(Q) \Big] \subset H^1(\Omega_T) \times H^{1/2,1/4}(\Sigma) \Big\} \tag{3.22}$$

or

$$\mathcal{R} = \Big\{ (u_0, g), (u_T, g_\nu) \in \Big( \Xi^{2r+1}(\Omega_0) \times \mathscr{H}^{2r+3/2, r+3/4}(\Sigma) \Big) \times \Big( \Xi^{2r+1}(\Omega_T) \times \mathscr{H}^{2r+1/2, r+1/4}(\Sigma) \Big) \mid$$
$$r < -1; \Big( \Lambda^f_{\Omega,\Sigma} - \Lambda^0_{\Omega,\Sigma} \Big)[u_0, g] \in (\gamma_T, \gamma_1)\Big[ H^{2,1}(Q) \Big] \subset H^1(\Omega_T) \times H^{1/2,1/4}(\Sigma) \Big\}. \tag{3.23}$$

**Theorem 3.7.** *An inverse problem* $IP^f_{(u_0,g),(u_T,g^\nu)}$ *admits a solution* $f \in L^2(Q)$ *only if* $(u_0, g), (u_T, g^\nu) \in \mathcal{R}$.

*Proof. Necessity*: Suppose the $IP^f_{(u_0,g),(u_T,g^\nu)}$ has a solution $f \in L^2(Q)$. Let us consider the following auxiliary problems: $P_{0,0,f}$ with solution $w_0 = S[0,0,f]$ and $P_{u_0,g,0}$ with solution $w_1 = S[u_0, g, f]$. Then, by the additivity principle for linear operators

$$\Lambda^f_{\Omega,\Sigma}[u_0, g] = \Lambda^f_{\Omega,\Sigma}[0,0] + \Lambda^0_{\Omega,\Sigma}[u_0, g] \Longrightarrow \Big( \Lambda^f_{\Omega,\Sigma} - \Lambda^0_{\Omega,\Sigma} \Big)[u_0, g] = \Lambda^f_{\Omega,\Sigma}[0,0]. \tag{3.24}$$

Since $f \in L^2(Q)$ implies by regularity that $w_0 \in H^{2,1}(Q)$, by the first trace Theorem 2.1. the data are in $\mathcal{R}$.

*Sufficiency*: suppose the data are in the regular class. Consider the fourth-order problem (3.20). As we have shown in Section 3.4, it is well posed with a solution $v \in H^{2,1}(Q)$ and the source $f = (-\partial - \Delta)v \in L^2(Q)$ is solution to the inverse problem with data in the class $\mathcal{R}$. □

### 3.3. The Reciprocity Gap Functional

The second Green formula

$$\int_Q ((\partial_t u - \Delta u)v - u(-\partial_t v - \Delta v))dx\,dt$$
$$= \int_\Sigma (\gamma[u]\gamma_1[v] - \gamma_1[u]\gamma[v])d\sigma_{(t,x)} - \int_{\Omega_T} \gamma_T[u]\gamma_T[v]dx + \int_{\Omega_0} \gamma_0[u]\gamma_0[v]dx \tag{3.25}$$

applied to problems $P_{u_0,g,f}$ with normal trace at the cylinder lateral boundary $\Sigma$, $\gamma_1[u] = g^\nu$, initial value at $\Omega_0$, $\gamma_0[u] = u_0$ and the adjoint problem $P^*_{u_T,\gamma[v],0}$ with $u_T$ time $T$ value and zero source yield the following expression for reciprocity gap functional in the transient heat equation context:

$$\int_Q fv\,dx\,dt = \int_\Sigma (g\gamma_1[v] - g^\nu\gamma[v])d\sigma_{(t,x)} - \int_{\Omega_T} u_T\gamma_T[v]dx + \int_{\Omega_0} u_0\gamma_0[v]dx \tag{3.26}$$

or, by using the extended Dirichlet-to-Neumann notation,

$$\int_Q fv\,dx\,dt = \int_\Sigma \left(g\gamma_1[v] - \Lambda^f_{\bullet,\Sigma}[u_0,g]\gamma[v]\right)d\sigma_{(t,x)} - \int_{\Omega_T} \Lambda^f_{\Omega,\bullet}[u_0,g]\gamma_T[v]dx + \int_{\Omega_0} u_0\gamma_0[v]dx \tag{3.27}$$

By subtracting the extended Dirichlet-to-Neumann map for the zero source problem $P_{u_0,g,0}$ with the same data, we obtain the following weak form for the systems (3.17):

$$\int_Q fv\,dx\,dt = -\int_{\Omega_T} \Lambda^f_{\Omega,\bullet}[0,0]\gamma_T[v]dx - \int_\Sigma \Lambda^f_{\bullet,\Sigma}[0,0]\gamma[v]d\sigma_{(t,x)} \tag{3.28}$$

We call (3.28) as the transient heat reciprocity gap equation. This is a weak variational equation that must be tested for all functions in

$$H^{2,1}_{-\partial_t-\Delta}(Q) = \left\{v \in H^{2,1}(Q) \mid -\partial_t v - \Delta v = 0\right\}. \tag{3.29}$$

### 3.4. Nonobservability

Let

$$\mathcal{R}_0 := \left\{(u_0,g),(u_T,g^\nu) \in \mathcal{R} \text{ such that } \Lambda^f_{\Omega,\Sigma}[0,0] = 0\right\}. \tag{3.30}$$

Let us consider the following set of sources:

$$\mathcal{F}_0 := \left\{f \in L^2(Q); f \text{ is a solution of } \mathrm{IP}^f_{(u_0,g),(u_T,g^\nu)}; (u_0,g),(u_T,g^\nu) \in \mathcal{R}_0\right\}. \tag{3.31}$$

Note that by the Transient Reciprocity Gap equation (3.28)

$$f \in \mathcal{F}_0 \Longrightarrow \int_Q f v \, dx \, dt = 0 \quad \forall v \in H^{2,1}_{-\partial_t - \Delta}(Q) \tag{3.32}$$

and $\mathcal{F}_0 \subset (H^{2,1}_{-\partial_t-\Delta}(Q))^{\perp}$ is a set of non observable sources.

*Remark 3.8* (ill-posed problem). Since the extended Dirichlet-to-Neumann map is a composition of the solution operator $S$ and the trace operator, this non uniqueness result is expected. Also, trace operator may not be onto, as we have pointed in Remark 2.2, and problems with existence may occur if the data are not consistent. Finally, we note that the integral systems (3.17) and the variational (3.28) are compact operators, so instability problems are expected in numerical solutions. In that bad situation, a generalized concept of inverse need to be adopted, Engl et al. [21].

### 3.5. Uniqueness of Solution in Classical Spaces

The source reconstruction of consistent boundary data is not possible, if we do not have additional information about the source. As an example, we may reconstruct a time independent source in the transient model or even a source in the source class (3.19). In this work, we are interested in the reconstruction of a source which is given by a characteristic function defined with a star-shaped support. This is a generalization of the classical results for reconstruction of star-shape sources in the Laplace Poisson Dirichlet Problem, Roberty and Alves [9], and in the Helmholtz Poisson Dirichlet Problem, Roberty and Rainha [15]. Isakov [2] treats a problem for reconstruction of an unknown domain moving in time with the prescribed exterior thermal potential. He proves the following theorem.

**Theorem 3.9.** *Consider $\omega$ with $\overline{\omega} \subset \Omega$. Let $Q_j \subset Q$, for $j = 1, 2$, with $\overline{Q_j} \subset \omega \times [0, T]$, with spatial boundaries $\Sigma_j := \partial_x Q_j$ of class $C^{1+\lambda}$ and satisfying:*

$$\text{the sets } Q_j \cap \{t = \tau\}, \quad \left(Q \setminus \left(\overline{Q_1} \cup \overline{Q_2}\right)\right) \cap \{t = \tau\}, \quad \left(Q \setminus \overline{Q_j} \cap \{t = \tau\}\right) \tag{3.33}$$

*are connected for $0 < \tau < T$. Let $\omega_j = \overset{o}{\overline{(\overline{Q_j} \cap \{t = 0\})}}$ and*

$$f \in C^{\lambda, \lambda/2}\left(\overline{Q}\right), \quad f > 0 \text{ on } \overline{Q}. \tag{3.34}$$

*If solutions $u_j$ to the problems:*

$$\begin{gathered} \partial_t u_j - \Delta u_j = f \chi(Q_j) \quad \text{on } Q, \\ u_j = 0 \quad \text{in } \Omega_0 \end{gathered} \tag{3.35}$$

*coincides on $Q \setminus \omega \times (0, T)$, then $Q_1 = Q_2$.*

In his proof, he first demonstrates in two auxiliary lemmas that:

(1) in the initial time the two source support, $Q_1 \cup \{t = 0\} = Q_2 \cup \{t = 0\}$, must coincide and that

(2) there is a time $t_0 > 0$ such that the relative potential associated with the sources

$$u = U\left(\cdot, f\chi(Q_2 \cap \omega_1 \times (-T, 0])\right) - U\left(\cdot, f\chi(Q_1 \cap \omega_1 \times (-T, 0])\right) \tag{3.36}$$

satisfies $uf(\chi(q_2) - \chi(Q_1)) \leq 0$ on $\Re^n \times (0, t_0)$

and he uses these results to prove that the hypotheses that the source may be unequal are contradictory.

In this way, we expect that when the data are sufficiently regular, the moving characteristic source can be reconstructed. A common method for numerical solution of the transient heat equation is based on the construction of time marching schemes by the solution of a sequence of modified Helmholtz equations. In these schemes, at each time increment the source must be reconstructed. For the class of problems we also have demonstrated a uniqueness theorem in Roberty and Rainha [15]. We will return to this theorem after the presentation of this scheme.

## 4. The $\theta$-Scheme and the Modified Helmholtz Model for the Transient Heat Problem

One type of source that can be uniquely reconstructed from Neumann boundary measurements in a model based on Poisson equation with the Laplace operator $\Delta$ is star-shaped characteristic sources. This uniqueness resulted may be easily extended to a modified Helmholtz equation based model, Roberty and Rainha [15], and thus we present now an algorithm for moving transient source reconstruction in the heat equation based on this result. Let the source be given by

$$f(t, x) = \chi_{\omega(t)}(x) \quad \text{in } Q, \tag{4.1}$$

where $\omega(t)$, $t \in [0, T]$ is a representation of the star-shape source boundary. For one-dimensional problems it is a set with two points. For two or three-dimensional problems it is a moving Lipschitz parametric curve or surface in which the parameter has been omitted. With this in mind, we may rewrite the transient problem as

$$(P_{\chi_\omega}) \quad \begin{cases} \partial_t u - \Delta u = \chi_{\omega(t)}(x) & \text{in } Q, \\ -\Delta u_0 = \chi_{\omega(0)}(x) & \text{in } \Omega_0, \\ u = 0 & \text{on } \Sigma; \end{cases} \tag{4.2}$$

with transient Neumann history $\gamma_1[u] = \partial_\nu u = g^\nu$ in the lateral cylinder boundary.

The initial $u_0$ can be determined as solution of the Poisson problem $-\Delta u_0 = \chi_{\omega(0)}$, if the initial shape $\omega(0)$ is known. If it is not known, then we can use the Cauchy data,

$g(0)$ and $g^\nu(0)$, to solve the static inverse problem, by using the methodology in Roberty and Alves [9]. Consider a partition of the time interval $[0, T]$ into $N$ subintervals of length $\tau > 0$. Let $\{t_0, t_1, t_2, \ldots, t_n, t_{n+1}, \ldots, t_N\}$ be the knots of this partition, with $t_0 = 0$ *and* $t_N = T$. For $t_n < t < t_{n+1}$, $n = 0, 1, N-1$ we use the $\theta$-scheme approach for the discretization of (4.2). Define, for a function $h(t, x)$, a linear $\theta$ weighted approximation $\delta_\theta(h)(x)$ by

$$\delta_\theta(h)(x) = \theta h(t_{n+1}, x) + (1-\theta) h(t_n, x) \tag{4.3}$$

We start by approximating the time derivative $\partial_t u$ in (4.2), by a first order forward difference

$$\frac{\partial u}{\partial t}(t, x) \cong \frac{u(t_{n+1}, x) - u(t_n, x)}{\tau}, \quad x \in \Omega. \tag{4.4}$$

The diffusion and the characteristic source respectively by

$$\begin{aligned} \Delta u(t, x) &\cong \delta_\theta(\Delta u)(x), \quad x \in \Omega, \\ \chi_{\omega(t)}(x) &\cong \delta_\theta(\chi_\omega)(x), \quad x \in \Omega. \end{aligned} \tag{4.5}$$

By denoting $u^{n+1}(x)$ with $x \in \Omega$, the approximate solution at the time step $t_{n+1}$, the transient system (4.2) is approximated by the following sequence of stationary problems:

$$\left(H^{n+1}_{\chi_\omega}\right) \quad \begin{cases} -\Delta u^{n+1} + \lambda u^{n+1} = f_n + \chi_{\omega(t_{n+1})} & \text{in } \Omega, \\ u^{n+1} = 0 & \text{on } \Gamma, \\ \text{with } g^\nu(t_{n+1}) := \partial_\nu u^{n+1} & \text{on } \Gamma; \end{cases} \tag{4.6}$$

for $n = 0, 1, 2, \ldots, N$. Here $\lambda = 1/\tau\theta$ and

$$f_n = \frac{u^n + \tau(1-\theta)\Delta u^n + \tau\theta \chi_{\omega(t_n)}(x)}{\tau\theta}. \tag{4.7}$$

Note that $\Delta u^n + \chi_{\omega(t_n)} = \partial u^n / \partial t$ and that the initial Poisson problem determining the $u_0$ and $\chi_{\omega(0)}$ is

$$\left(H^0_{g,\chi_\omega}\right) \quad \begin{cases} -\Delta u^0 = \chi_{\omega(0)}(x) & \text{in } \Omega, \\ u^0 = 0 & \text{on } \Gamma, \\ \text{with } g^\nu(0) := \partial_\nu u^0 & \text{on } \Gamma. \end{cases} \tag{4.8}$$

The sequence of modified Helmholtz source inverse problem (4.6) starting with stationary problem (4.8) may be used to model a scheme for the reconstruction of star-shaped sources $\chi_{\omega(t_n)}(x)$, for a time knot sequence, showing its movement and deformation in the external domain $\Omega$. For this, we only need to know the transient Neumann data with zero Dirichlet datum on the external boundary $\Gamma = \partial\Omega$. Since we do not have experimental data, we will solve the direct problem with a different method, adding noise, and do an experimental data synthesis.

### *4.1. Iterative Source Reconstruction Scheme*

The source at time $t_n$ may be further calculated as

$$f_n = \frac{\lambda}{\theta} u^n - \frac{1-\theta}{\theta} f_{n-1}, \quad \text{for } n = 1, 2, \ldots \tag{4.9}$$

with $f_{-1} = 0$ and $f_0 = \lambda u^0$. Since the discretized direct problem (4.6) and (4.8) are linear, it may be decomposed into two subproblems separating the known part of the source from the part to be reconstructed, that is, $f_n$ and $\chi_{\omega(t_{n+1})}$. Let $y^{n+1}$, $n = -1, 0, 1, \ldots$ be a solution of

$$\left(H_{f_n}^{n+1}\right) \quad \begin{cases} -\Delta y^{n+1} + \lambda y^{n+1} = f_n & \text{in } \Omega, \\ y^{n+1} = 0 & \text{on } \Gamma, \\ \text{with } g_y^{\nu}(t_{n+1}) := \partial_{\nu} y^{n+1} & \text{on } \Gamma; \end{cases} \tag{4.10}$$

and let $w^{n+1}$, $n = -1, 0, 1, \ldots$ be solution of

$$\left(H_{\chi_\omega}^{n+1}\right) \quad \begin{cases} -\Delta w^{n+1} + \lambda w^{n+1} = \chi_{\omega(t_{n+1})}(x) & \text{in } \Omega, \\ w^{n+1} = 0 & \text{on } \Gamma, \\ \text{with } g_w^{\nu}(t_{n+1}) := \partial_{\nu} w^{n+1} & \text{on } \Gamma. \end{cases} \tag{4.11}$$

Then, by the superposition principle, the solution of (4.6) is $u^{n+1} = w^{n+1} + y^{n+1}$ and the Neumann data will be the sum of the decomposed parts $g^{\nu}(t_{n+1}) = g_w^{\nu}(t_{n+1}) + g_y^{\nu}(t_{n+1})$. The $Y$-problems (4.10) form a discrete sequence of problems with continuous source $f_n$ that may be solved before the time increment at $t_n$ begins. Its normal derivatives may be calculated and

$$g_y^{\nu}(t_{n+1}) := \partial_{\nu} y^{n+1} \quad \text{on } \Gamma, \tag{4.12}$$

subtracted from the synthetic transient Neumann data at knot $t_{n+1}$

$$g_w^{\nu}(t_{n+1}) = g^{\nu}(t_{n+1}) - g_y^{\nu}(t_{n+1}), \tag{4.13}$$

to produce the data for the modified Helmholtz (4.11) that will be used in the reconstruction of the source $\chi_{\omega(t_{n+1})}$ at time $t^{n+1}$. Note that by using the Reciprocity gap functional the characteristic star-shaped source may be reconstructed without solving the direct problem (4.11). By using the second Green's formula, this inverse problem is modeled with a nonlinear Fredholm integral equation of first kind.

In this modified Helmholtz equation formulation to the transient problem the uniqueness can be proved for each time interval.

**Theorem 4.1.** *Consider the direct problem* (4.11) *and its associated inverse problem with two sources* $\chi_{\omega_1(t_{n+1})}$ *and* $\chi_{\omega_2(t_{n+1})}$. *Let* $\omega_1(t_{n+1}), \omega_2(t_{n+1}) \subset \Omega$ *domains with* $C^2$ *boundary and* $\omega_1(t_{n+1}) \setminus \omega_2(t_{n+1})$, $\omega_2(t_{n+1}) \setminus \omega_1(t_{n+1})$, $\omega_1(t_{n+1}) \cap \omega_2(t_{n+1})$ *be connected. If the Cauchy data for the two problems are the same, then* $\omega_1(t_{n+1}) = \omega_2(t_{n+1})$.

*Proof.* The proof can be found in Roberty and Rainha [15]. It is based on contradictory consequences of the maximum principle and the initial hypotheses that the sources are unequal. □

## 5. Conclusions

We have presented a methodology for star-shape source reconstruction in the transient heat problem by using one set of Cauchy data history. With the adoption of an anisotropic Sobolev Hilbert mathematical framework, we can treat the problem with a methodology analogous to that used to study stationary elliptic problems. Therefore, by introducing a finite differences time $\theta$-scheme, we developed an algorithm based on a modified Helmholtz system, for which we have already studied the mathematically inverse source reconstruction problem.

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