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Research Article

Pseudo-Steady-State Productivity Formula for a Partially Penetrating Vertical Well in a Box-Shaped Reservoir

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For a bounded reservoir with no flow boundaries, the pseudo-steady-state flow regime is common at long-producing times. Taking a partially penetrating well as a uniform line sink in three dimensional space, by the orthogonal decomposition of Dirac function and using Green's function to three-dimensional Laplace equation with homogeneous Neumann boundary condition, this paper presents step-by-step derivations of a pseudo-steady-state productivity formula for a partially penetrating vertical well arbitrarily located in a closed anisotropic box-shaped drainage volume. A formula for calculating pseudo skin factor due to partial penetration is derived in detailed steps. A convenient expression is presented for calculating the shape factor of an isotropic rectangle reservoir with a single fully penetrating vertical well, for arbitrary aspect ratio of the rectangle, and for arbitrary position of the well within the rectangle.

1. Introduction

Well productivity is one of primary concerns in oil field development and provides the basis for oil field development strategy. To determine the economical feasibility of drilling a well, the engineers need reliable methods to estimate its expected productivity. Well productivity is often evaluated using the productivity index, which is defined as the production rate per unit pressure drawdown. Petroleum engineers often relate the productivity evaluation to the long-time performance behavior of a well, that is, the behavior during pseudo-steady-state or steady-state flow.

For a bounded reservoir with no flow boundaries, the pseudo-steady-state flow regime is common at long producing times. In these reservoirs, also called volumetric reservoirs, there can be no flow across the impermeable outer boundary, such as a sealing fault, and fluid production must come from the expansion and pressure decline of the reservoir. This condition of no flow boundary is also encountered in a well that is offset on four sides.

Flow enters the pseudo-steady-state regime when the pressure transient reaches all boundaries after drawdown for a sufficiently long-time. During this period, the rate of pressure decline is almost identical at all points in the reservoir and wellbore. Therefore, the difference between the average reservoir pressure and pressure in the wellbore approaches a constant with respect to time. Pseudo-steady-state productivity index is defined as the production rate divided by the difference of average reservoir pressure and wellbore pressure, hence the productivity index is basically constant [1, 2].

In many oil reservoirs the producing wells are completed as partially penetrating wells. If a vertical well partially penetrates the formation, the streamlines converge and the area for flow decreases in the vicinity of the wellbore, which results in added resistance, that is, a pseudoskin factor. Only semianalytical and semi-empirical expressions are available in the literature to calculate pseudoskin factor due to partial penetration.

Rarely do wells drain ideally shaped drainage areas. Even if they are assigned regular geographic drainage areas, they become distorted after production commences, either because of the presence of natural boundaries or because of lopsided production rates in adjoining wells. The drainage area is then shaped by the assigned production share of a particular well. An oil reservoir often has irregular shape, but a rectangular shape is often used to approximate an irregular shape by petroleum engineers, so it is important to study well performance in a rectangular or box-shaped reservoir [1, 2].

2. Literature Review

The pseudo-steady-state productivity formula of a fully penetrating vertical well which is located at the center of a closed isotropic circular reservoir is [3, page 63]

$$Q_w = F_D \frac{2\pi KH(P_a - P_w)/(\mu B)}{\ln(R_e/R_w) - 3/4}, \tag{2.1}$$

where P_a is average reservoir pressure in the circular drainage area, P_w is flowing wellbore pressure, K is permeability, H is payzone thickness, μ is oil viscosity, B is oil formation volume factor, R_e is radius of circular drainage area, R_w is wellbore radius, and F_D is the factor which allows the use of field units and practical SI units, and it can be found in [3, page 52, Table 5.1].

Formula (2.1) is only applicable for a fully penetrating vertical well at the center of a circular drainage area with impermeable outer boundary.

If a vertical well is partially penetrate the formation, the streamlines converge and the area for flow decreases in the region around the wellbore, and this added resistance is included by introducing the pseudoskin factor, S_{ps}. Thus, (2.1) may be rewritten to include the pseudoskin factor due to partial penetration as [4, page 92]:

$$Q_w = F_D \frac{2\pi KH(P_a - P_w)/(\mu B)}{\ln(R_e/R_w) - 3/4 + S_{\text{ps}}}. \tag{2.2}$$

S_{ps} can be calculated by semianalytical and semiempirical expressions presented by Brons, Marting, Papatzacos, and Bervaldier [5–7].

Assume that the well-drilled length is equal to the well producing length, (i.e., perforated interval,) $L_p = L$, and define partial penetration factor η:

$$\eta = \frac{L_p}{H} = \frac{L}{H}. \tag{2.3}$$

Pseudoskin factor formula given by Brons and Marting is [5]

$$S_{ps} = \left(\frac{1}{\eta} - 1\right)\left[\ln(h_D) - G(\eta)\right], \tag{2.4}$$

where

$$h_D = \left(\frac{H}{R_w}\right)\left(\frac{K_h}{K_v}\right)^{1/2}, \tag{2.5}$$

$$G(\eta) = 2.948 - 7.363\eta + 11.45\eta^2 - 4.675\eta^3. \tag{2.6}$$

Pseudoskin factor formula given by Papatzacos is [6]

$$S_{ps} = \left(\frac{1}{\eta} - 1\right)\ln\left(\frac{\pi h_D}{2}\right) + \left(\frac{1}{\eta}\right)\ln\left[\left(\frac{\eta}{2+\eta}\right)\left(\frac{\Psi_1 - 1}{\Psi_2 - 1}\right)^{1/2}\right], \tag{2.7}$$

where h_D has the same meaning as in (2.5), and

$$\Psi_1 = \frac{H}{h_1 + 0.25L_p}, \qquad \Psi_2 = \frac{H}{h_1 + 0.75L_p}, \tag{2.8}$$

and h_1 is the distance from the top of the reservoir to the top of the open interval.

Pseudoskin factor formula given by Bervaldier is [7]

$$S_{ps} = \left(\frac{1}{\eta} - 1\right)\left[\frac{\ln(L_p/R_w)}{(1 - R_w/L_p)} - 1\right]. \tag{2.9}$$

It must be pointed out that the well location in the reservoir has no effect on S_{ps} calculated by (2.4), (2.7), and (2.9).

By solving-three-dimensional Laplace equation with homogeneous Dirichlet boundary condition, Lu et al. presented formulas to calculate S_{ps} in steady state [8].

To account for irregular drainage shapes or asymmetrical positioning of a well within its drainage area, a series of shape factors was developed by Dietz [9]. Formula (2.1) can be generalized for any shape into the following formula:

$$Q_w = F_D \frac{2\pi KH(P_a - P_w)/(\mu B)}{(1/2)\ln\left[2.2458A/\left(C_A R_w^2\right)\right]}, \tag{2.10}$$

where C_A is shape factor, and A is drainage area.

Dietz evaluated shape factor C_A for various geometries, in particular, for rectangles of various aspect ratios with single well in various locations. He obtained his results graphically, from the straight line portion of various pressure build-up curves. Earlougher et al. [10] carried out summations of exponential integrals to obtain dimensionless pressure drops at various points within a square drainage area and then used superposition of various square shapes to obtain pressure drops for rectangular shapes. The linear portions of the pressure drop curves so obtained, corresponding to pseudo-steady-state, were then used to obtain shape factors for various rectangles.

The methods used by Dietz and Earlougher et al. are limited to rectangles whose sides are integral ratios, and the well must be located at some special positions within the rectangle.

Lu and Tiab presented formulas to calculate productivity index and pseudoskin factor in pseudo-steady-state for a partially penetrating vertical well in a box-shaped reservoir, they also presented a convenient expression for calculating the shape factor of an isotropic rectangle reservoir [1, 2]. But in [1, 2], they did not provide detail derivation steps of their formulas.

The primary goal of this paper is to present step-by-step derivations of the pseudo-steady-state productivity formula and pseudoskin factor formula for a partially penetrating vertical well in an anisotropic box-shaped reservoir, which were given in [1, 2]. A similar procedure in [8] is given in this paper, point sink solution is first derived by the orthogonal decomposition of Dirac function and Green's function to Laplace equation with homogeneous Neumann boundary condition, then using the principle of superposition, point sink solution is integrated along the well length, uniform line sink solution is obtained, and rearrange the resulting solution, pseudo-steady-state productivity formula and shape factor formula are obtained. A convenient expression is derived for calculating the shape factor of an isotropic rectangle reservoir with a single fully penetrating vertical well, for arbitrary aspect ratio of the rectangle and for arbitrary position of the well within the rectangle.

3. Partially Penetrating Vertical Well Model

Figure 1 is a schematic of a partially penetrating well. A partially penetrating vertical well of length L drains a box-shaped reservoir with height H, length (x direction) a, and width (y direction) b. The well is parallel to the z direction with a length $L \le H$, and we assume $b \ge a$.

The following assumptions are made.

(1) The reservoir is homogeneous, anisotropic, and has constant K_x, K_y, K_z permeabilities, thickness H, and porosity ϕ. All the boundaries of the box-shaped drainage volume are sealed.

(2) The reservoir pressure is initially constant. At time $t = 0$, pressure is uniformly distributed in the reservoir, equal to the initial pressure P_i.

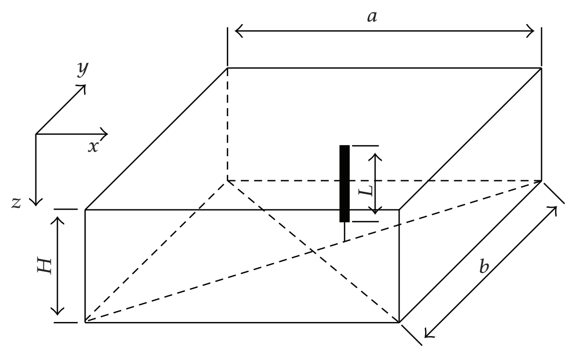

Figure 1: Partially Penetrating Vertical Well Model.

(3) The production occurs through a partially penetrating vertical well of radius R_w, represented in the model by a uniform line sink.

(4) A single phase fluid, of small and constant compressibility C_f, constant viscosity μ, and formation volume factor B, flows from the reservoir to the well at a constant rate Q_w. Fluids properties are independent of pressure.

(5) No gravity effect is considered. Any additional pressure drops caused by formation damage, stimulation, or perforation are ignored, we only consider pseudoskin factor due to partial penetration.

The partially penetrating vertical well is taken as a uniform line sink in three dimensional space. The coordinates of the two end points of the uniform link sink are $(x', y', 0)$ and (x', y', L). We suppose the point (x', y', z') is on the well line, and its point convergence intensity is q.

By the orthogonal decomposition of Dirac function and using Green's function to Laplace equation with homogeneous Dirichlet boundary condition, Lu et al. obtained point sink solution and uniform line sink solution to steady-state productivity equation of a partially penetrating vertical well in a circular cylinder reservoir [8]. For a box-shaped reservoir and a circular cylinder reservoir, the Laplace equation of a point sink is the same, in order to obtain the pressure at point (x, y, z) caused by the point (x', y', z'), we have to obtain the basic solution of the following Laplace equation:

$$K_x \frac{\partial^2 P}{\partial x^2} + K_y \frac{\partial^2 P}{\partial y^2} + K_z \frac{\partial^2 P}{\partial z^2} = \phi \mu C_t \frac{\partial P}{\partial t} + \mu q B \delta(x - x') \delta(y - y') \delta(z - z'), \tag{3.1}$$

in the box-shaped drainage volume:

$$\Omega = (0, a) \times (0, b) \times (0, H), \tag{3.2}$$

and we always assume

$$b \geq a \gg H, \tag{3.3}$$

and $\delta(x - x')$, $\delta(y - y')$, $\delta(z - z')$ are Dirac functions.

All the boundaries of the box-shaped drainage volume are sealed, that is,

$$\left.\frac{\partial P}{\partial N}\right|_{\Gamma} = 0, \tag{3.4}$$

where $\partial P/\partial N|_{\Gamma}$ is the exterior normal derivative of pressure on the surface of box-shaped drainage volume $\Gamma = \partial\Omega$.

The reservoir pressure is initially constant

$$P|_{t=0} = P_i. \tag{3.5}$$

Define average permeability:

$$K_a = \left(K_x K_y K_z\right)^{1/3}. \tag{3.6}$$

In order to simplify (3.1), we take the following dimensionless transforms:

$$\begin{gathered}
x_D = \left(\frac{x}{L}\right)\left(\frac{K_a}{K_x}\right)^{1/2}, \qquad y_D = \left(\frac{y}{L}\right)\left(\frac{K_a}{K_y}\right)^{1/2}, \qquad z_D = \left(\frac{z}{L}\right)\left(\frac{K_a}{K_z}\right)^{1/2}, \\
a_D = \left(\frac{a}{L}\right)\left(\frac{K_a}{K_x}\right)^{1/2}, \qquad b_D = \left(\frac{b}{L}\right)\left(\frac{K_a}{K_y}\right)^{1/2}, \\
L_D = \left(\frac{K_a}{K_z}\right)^{1/2}, \qquad H_D = \left(\frac{H}{L}\right)\left(\frac{K_a}{K_z}\right)^{1/2}, \\
t_D = \frac{K_a t}{\phi\mu C_t L^2}.
\end{gathered} \tag{3.7}$$

The dimensionless wellbore radius is [8]

$$R_{wD} = \frac{\left(K_z/\sqrt{K_x K_y}\right)^{1/6}\left[\left(K_x/K_y\right)^{1/4} + \left(K_y/K_x\right)^{1/4}\right] R_w}{2L}. \tag{3.8}$$

Assume that q is the point convergence intensity at the point sink (x', y', z'), the partially penetrating well is a uniform line sink, the total productivity of the well is Q_w, and there holds [8]

$$q = \frac{Q_w}{L_{pD}} = \frac{Q_w}{L_D}. \tag{3.9}$$

Dimensionless pressures are defined by

$$P_D = \frac{K_a L(P_i - P)}{\mu q B}, \tag{3.10}$$

$$P_{wD} = \frac{K_a L(P_i - P_w)}{\mu q B}. \tag{3.11}$$

Then (3.1) becomes

$$\frac{\partial P_D}{\partial t_D} - \left(\frac{\partial^2 P_D}{\partial x_D^2} + \frac{\partial^2 P_D}{\partial y_D^2} + \frac{\partial^2 P_D}{\partial z_D^2}\right) = \delta(x_D - x'_D)\delta(y_D - y'_D)\delta(z_D - z'_D), \tag{3.12}$$

in the dimensionless box-shaped drainage volume

$$\Omega_D = (0, a_D) \times (0, b_D) \times (0, H_D), \tag{3.13}$$

with boundary condition

$$\left.\frac{\partial P_D}{\partial N_D}\right|_{\Gamma_D} = 0, \tag{3.14}$$

and initial condition

$$P_D|_{t_D=0} = 0. \tag{3.15}$$

4. Point Sink Solution

For convenience in the following reference, we use dimensionless transforms given by (3.7)–(3.10), every variable, drainage domain, initial and boundary conditions should be taken as dimensionless, but we drop the subscript D.

Consequently, (3.12) is expressed as

$$\frac{\partial P}{\partial t} - \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2}\right) = \delta(x - x')\delta(y - y')\delta(z - z'). \tag{4.1}$$

Rewrite (3.14) below

$$\left.\frac{\partial P}{\partial N}\right|_{\Gamma} = 0, \tag{4.2}$$

and (3.15) becomes

$$P|_{t=0} = 0. \tag{4.3}$$

We want to solve (4.1) under the boundary condition (4.2) and initial condition (4.3), and to obtain point sink solution when the time t is so long that the pseudo-steady-state is reached.

If the boundary condition is (4.2), there exists the following complete normalized orthogonal system $\{g_{lmn}(x,y,z)\}$ [11, 12]:

$$g_{lmn}(x,y,z) = \sqrt{\frac{1}{abHd_l d_m d_n}}\cos\left(\frac{l\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi z}{H}\right), \tag{4.4}$$

where l, m, n are nonnegative numbers, and

$$d_l = \begin{cases} 1 & \text{if } l = 0, \\ \dfrac{1}{2} & \text{if } l > 0, \end{cases} \tag{4.5}$$

and d_m, d_n have similar definitions.

According to the complete normalized orthogonal systems of the Laplace equation's basic solution, Dirac function has the following expression for homogeneous Neumann boundary condition ([13, 14]):

$$\delta(x-x')\delta(y-y')\delta(z-z') = \sum_{l,m,n=0}^{\infty} g_{lmn}(x,y,z)g_{lmn}(x',y',z'). \tag{4.6}$$

In order to simplify the following derivations, we define the following notation:

$$\sum_{l,m,n=0}^{\infty} F_{lmn}(x,y,z) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_{lmn}(x,y,z), \tag{4.7}$$

which means in any function $F(x,y,z)$, the subscripts l, m, n of any variable must count from 0 to infinite.

And define

$$\sum_{l+m+n>0} F_{lmn}(x,y,z) = \sum_{l\geq 0}^{\infty}\sum_{m\geq 0}^{\infty}\sum_{n\geq 0}^{\infty} F_{lmn}(x,y,z) \quad (l+m+n>0), \tag{4.8}$$

which means in any function $F(x,y,z)$, the subscripts l, m, n of any variable must be no less than zero, and at least one of the three subscripts l, m, n must be positive to guarantee $l+m+n>0$. And the upper limit of the subscripts l, m, n is infinite.

Let

$$P(t,x,y,z;x',y',z') = \sum_{l,m,n=0}^{\infty} e_{lmn}(t)g_{lmn}(x,y,z), \tag{4.9}$$

where $e_{lmn}(t)$ are undetermined coefficients.

Substituting (4.9) into left-hand side of (4.1), and substituting (4.6) into right-hand side of (4.1), we obtain

$$\sum_{l,m,n=0}^{\infty}\left\{\frac{\partial e_{lmn}(t)}{\partial t}g_{lmn}(x,y,z)-e_{lmn}(t)\Delta\left[g_{lmn}(x,y,z)\right]\right\}$$
$$=\sum_{l,m,n=0}^{\infty}\left\{\frac{\partial e_{lmn}(t)}{\partial t}+e_{lmn}(t)\lambda_{lmn}\right\}g_{lmn}(x,y,z) \tag{4.10}$$
$$=\sum_{l,m,n=0}^{\infty}g_{lmn}(x',y',z')g_{lmn}(x,y,z),$$

where Δ is the three-dimensional Laplace operator

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}, \tag{4.11}$$

$$\lambda_{lmn}=\left(\frac{l\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2+\left(\frac{n\pi}{H}\right)^2. \tag{4.12}$$

From (4.3) and (4.9),

$$e_{lmn}(0)=0, \tag{4.13}$$

compare the coefficients of $g_{lmn}(x,y,z)$ at both sides of (4.10), we obtain

$$\frac{\partial e_{lmn}(t)}{\partial t}+\lambda_{lmn}e_{lmn}(t)=g_{lmn}(x',y',z'), \tag{4.14}$$

because $\lambda_{000}=0$, from (4.14),

$$\begin{aligned}e_{000}(t)&=g_{000}(x',y',z')t\\&=\frac{t}{\sqrt{abH}}.\end{aligned} \tag{4.15}$$

When $\lambda_{lmn}\neq 0$ $(l+m+n>0)$, solve (4.14),

$$e_{lmn}(t)=\frac{\left[1-\exp(-\lambda_{lmn}t)\right]g_{lmn}(x',y',z')}{\lambda_{lmn}}. \tag{4.16}$$

Substitute (4.15) and (4.16) into (4.9) and obtain

$$
\begin{aligned}
P(t,x,y,z;x',y',z') &= \sum_{l,m,n=0}^{\infty} e_{lmn}(t) g_{lmn}(x,y,z) \\
&= \left(\frac{t}{\sqrt{abH}}\right) g_{000}(x,y,z) \\
&\quad + \sum_{l+m+n>0} \frac{\left[1-\exp(-\lambda_{lmn}t)\right] g_{lmn}(x',y',z') g_{lmn}(x,y,z)}{\lambda_{lmn}} \\
&= \frac{t}{abH} + \sum_{l+m+n>0} \frac{g_{lmn}(x',y',z') g_{lmn}(x,y,z)}{\lambda_{lmn}} \\
&\quad - \sum_{l+m+n>0} \frac{\exp(-\lambda_{lmn}t) g_{lmn}(x',y',z') g_{lmn}(x,y,z)}{\lambda_{lmn}}.
\end{aligned}
\tag{4.17}
$$

Define

$$
I_1 = \frac{t}{abH}, \tag{4.18}
$$

$$
\begin{aligned}
I_2 &= \Psi(x,y,z;x',y',z') \\
&= \sum_{l+m+n>0} \frac{g_{lmn}(x',y',z') g_{lmn}(x,y,z)}{\lambda_{lmn}},
\end{aligned}
\tag{4.19}
$$

$$
I_3 = \sum_{l+m+n>0} \frac{\exp(-\lambda_{lmn}t) g_{lmn}(x',y',z') g_{lmn}(x,y,z)}{\lambda_{lmn}}, \tag{4.20}
$$

then

$$
P(t,x,y,z;x',y',z') = I_1 + I_2 - I_3. \tag{4.21}
$$

Recall (4.19), the average value of Ψ throughout of the total volume of the box-shaped reservoir is

$$
\begin{aligned}
\Psi_{a,v} &= \left(\frac{1}{V}\right) \int_{\Omega} \Psi(x,y,z) dV \\
&= \left(\frac{1}{V}\right) \int_0^a \int_0^b \int_0^H \Psi(x,y,z;x',y',z') dx\, dy\, dz \\
&= \left(\frac{1}{V}\right) \left(\frac{g_{lmn}(x',y',z')}{\lambda_{lmn}}\right) \int_0^a \int_0^b \int_0^H \sum_{l+m+n>0} g_{lmn}(x,y,z) dx\, dy\, dz.
\end{aligned}
\tag{4.22}
$$

Note that $l+m+n>0$ implies that at least one of l, m, n must be greater than 0, without losing generality, we may assume

$$l > 0, \tag{4.23}$$

then

$$\int_0^a \cos\left(\frac{l\pi x}{a}\right) dx = 0. \tag{4.24}$$

So,

$$\int_0^a \int_0^b \int_0^H \sum_{l+m+n>0} g_{lmn}(x,y,z)\,dx\,dy\,dz = 0, \tag{4.25}$$

consequently,

$$\Psi_{a,v} = 0. \tag{4.26}$$

If time t is sufficiently long, pseudo-steady-state is reached, I_3 decreases by exponential law, I_3 will vanish, that is,

$$I_3 \approx 0, \tag{4.27}$$

then

$$P(t,x,y,z;x',y',z') = \frac{t}{abH} + \Psi(x,y,z;x',y',z'). \tag{4.28}$$

Substituting (4.28) into (4.1), we have

$$\frac{1}{abH} - \Delta\Psi = \delta(x-x')\delta(y-y')\delta(z-z'). \tag{4.29}$$

Define

$$\begin{aligned} f(x,y,z) &= -\Delta\Psi \\ &= -\left(\frac{1}{abH}\right) + \delta(x-x')\delta(y-y')\delta(z-z'), \end{aligned} \tag{4.30}$$

note that Ψ is equal to I_2 in (4.19), and

$$\frac{\partial\Psi}{\partial N} = 0, \quad \text{on } \Gamma. \tag{4.31}$$

From Green's Formula [15],

$$0 = \int_{\Gamma} \frac{\partial \Psi}{\partial N} dS = \int_{\Omega} \Delta \Psi dV = -\int_{\Omega} f(x, y, z) dV, \tag{4.32}$$

that is,

$$\int_{\Omega} f(x, y, z) dV = 0, \tag{4.33}$$

where V is volume of drainage domain Ω.

Define the following notation of internal product of functions $f(x, y, z)$ and $g(x, y, z)$:

$$\lfloor f(x, y, z), g(x, y, z) \rfloor = \int_{\Omega} f(x, y, z) g(x, y, z) dx\, dy\, dz = \int_{\Omega} f(x, y, z) g(x, y, z) dV, \tag{4.34}$$

where $\lfloor f, g \rfloor$ means the internal product of functions f and g.

From (4.33), we know that the internal product of $f(x, y, z)$ and constant number 1 is zero

$$\lfloor f(x, y, z), 1 \rfloor = 0, \tag{4.35}$$

and it is easy to prove

$$\lfloor f(x, y, z), g_{000} \rfloor = 0, \tag{4.36}$$

where g_{000} means g_{lmn} when $l = m = n = 0$.

Thus, $f(x, y, z)$ can be decomposed as [13, 14]:

$$\begin{aligned} f(x, y, z) &= \sum_{l,m,n=0}^{\infty} \lfloor f, g_{lmn}(x', y', z') \rfloor g_{lmn}(x, y, z) \\ &= \sum_{l+m+n>0} \lfloor f, g_{lmn}(x', y', z') \rfloor g_{lmn}(x, y, z) \\ &= \sum_{l+m+n>0} \lfloor \delta(x - x')\delta(y - y')\delta(z - z'), g_{lmn}(x', y', z') \rfloor g_{lmn}(x, y, z) \\ &= \sum_{l+m+n>0} g_{lmn}(x', y', z') g_{lmn}(x, y, z). \end{aligned} \tag{4.37}$$

The drainage volume is

$$V = abH. \tag{4.38}$$

Recall (4.28), the average pressure throughout the reservoir is

$$P_{a,v} = \left(\frac{1}{V}\right)\int_{\Omega} P(x,y,z)dx\,dy\,dz = \frac{t}{abH} + \Psi_{a,v}. \tag{4.39}$$

The wellbore pressure at point (x_w, y_w, z_w) is

$$P_w = \frac{t}{abH} + \Psi_w, \tag{4.40}$$

where Ψ_w is the value of Ψ at wellbore point (x_w, y_w, z_w).

Combining (4.39) and (4.40) gives

$$P_{a,v} - P_w = \Psi_{a,v} - \Psi_w, \tag{4.41}$$

which implies $P_{a,v} - P_w$ is independent of time.

5. Uniform Line Sink Solution

For convenience, in the following reference, every variable, drainage domain, initial and boundary conditions should be taken as dimensionless, but we drop the subscript D.

The producing portion of the partially penetrating well is between point $(x', y', 0)$ and point (x', y', L), recall (4.4) and (4.19), in order to obtain uniform line sink solution, we integrate Ψ with respect to z' from 0 to L, then

$$\begin{aligned} J(x,y,z;x',y',z';l,m,n) &= \int_0^L \Psi(x,y,z;x',y',z')dz' \\ &= \sum_{l+m+n>0} \Im_{lmn}(x,y,z;x',y',z';l,m,n), \end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
&\sum_{l+m+n>0} \mathfrak{I}_{lmn}(x,y,z;x',y',z';l,m,n) \\
&\quad= \sum_{l+m+n>0} \left(\frac{1}{abHd_l d_m d_n \lambda_{lmn}}\right) \cos\left(\frac{l\pi x}{a}\right) \\
&\qquad\times \cos\left(\frac{m\pi y}{b}\right) \cos\left(\frac{n\pi z}{H}\right) \\
&\qquad\times \cos\left(\frac{m\pi y'}{b}\right) \cos\left(\frac{l\pi x'}{a}\right) \int_0^L \cos\left(\frac{n\pi z'}{H}\right) dz' \\
&\quad= \sum_{l+m+n>0} \left(\frac{1}{abHd_l d_m d_n \lambda_{lmn}}\right) \cos\left(\frac{l\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) \cos\left(\frac{n\pi z}{H}\right) \\
&\qquad\times \begin{cases} \left(\dfrac{H}{\pi n}\right) \cos\left(\dfrac{l\pi x'}{a}\right) \cos\left(\dfrac{m\pi y'}{b}\right) \sin\left(\dfrac{n\pi L}{H}\right) & \text{if } l \neq 0, \\ L \cos\left(\dfrac{m\pi y'}{b}\right) \cos\left(\dfrac{l\pi x'}{a}\right) & \text{if } l = 0. \end{cases}
\end{aligned} \tag{5.2}$$

Define

$$C = \{(l,m,n) : l+m+n > 0\}, \tag{5.3}$$

$$C_1 = \{(l,m,n) : l = m = 0, n > 0\}, \tag{5.4}$$

$$C_2 = \{(l,m,n) : l = 0, m > 0, n \geq 0\}, \tag{5.5}$$

$$C_3 = \{(l,m,n) : l > 0, m \geq 0, n \geq 0\}, \tag{5.6}$$

then it is easy to prove

$$\begin{gathered} C = C_1 \cup C_2 \cup C_3, \\ C_1 \cap C_2 = \emptyset, \qquad C_2 \cap C_3 = \emptyset, \qquad C_3 \cap C_1 = \emptyset. \end{gathered} \tag{5.7}$$

Recall (5.1) and (5.2), and use (5.3)–(75), $J(x,y,z;x',y',z';l,m,n)$ can be decomposed as

$$\begin{aligned} J &= \sum_{l+m+n>0} \mathfrak{I}_{lmn}(x,y,z;x',y',z';l,m,n) \\ &= \sum_{n=1}^{\infty} \mathfrak{I}_{00n} + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \mathfrak{I}_{0mn} + \sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathfrak{I}_{lmn}. \end{aligned} \tag{5.8}$$

Define the following notations:

$$J_z = \sum_{n=1}^{\infty} \Im_{00n}, \tag{5.9}$$

$$J_{yz} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \Im_{0mn}, \tag{5.10}$$

$$J_{xyz} = \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Im_{lmn}, \tag{5.11}$$

so

$$J = J_z + J_{yz} + J_{xyz}, \tag{5.12}$$

and the average value of J at wellbore can be written as

$$J_{a,w} = (J_{z,a})_w + (J_{yz,a})_w + (J_{xyz,a})_w. \tag{5.13}$$

Rearrange (4.12) and obtain

$$\lambda_{lmn} = \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{H}\right)^2 = \left(\frac{\pi}{H}\right)^2 \left(n^2 + \mu_{lm}^2\right), \tag{5.14}$$

where

$$\begin{gathered}
\mu_{lm}^2 = \left(\frac{lH}{a}\right)^2 + \left(\frac{mH}{b}\right)^2 = \left(\frac{H}{b}\right)^2 \left[m^2 + \left(\frac{lb}{a}\right)^2\right], \\
\mu_{l0} = \frac{lH}{a}, \\
\lambda_{lm0} = \left(\frac{\pi}{H}\right)^2 \mu_{lm}^2, \\
\lambda_{0mn} = \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{H}\right)^2 = \left(\frac{\pi}{H}\right)^2 \left[n^2 + \left(\frac{mH}{b}\right)^2\right], \\
\lambda_{00n} = \frac{n^2\pi^2}{H^2}.
\end{gathered} \tag{5.15}$$

There hold [16, page 47]

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} = \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12} \quad (0 \le x \le 2\pi), \tag{5.16}$$

$$\sum_{n=1}^{\infty} \frac{1 - \cos(nx)}{n^4} = \frac{\pi^2 x^2}{12} - \frac{\pi x^3}{12} + \frac{x^4}{48} \quad (0 \le x \le 2\pi). \tag{5.17}$$

Recall (5.4) and (5.9), J_z is for the case $l = m = 0$, $n > 0$, and at wellbore of the off-center well,

$$
\begin{aligned}
& y = y' \neq 0, \qquad x' \neq 0, \qquad x = x' + R_w, \qquad 0 \leq z = z' \leq L, \\
(J_z)_w &= \sum_{n=1}^{\infty} \left(\frac{1}{abHd_n\lambda_{00n}} \right) \cos\left(\frac{n\pi z}{H} \right) \int_0^L \cos\left(\frac{n\pi z'}{H} \right) dz' \\
&= \left(\frac{2}{abH} \right) \sum_{n=1}^{\infty} \left(\frac{H^2}{\pi^2 n^2} \right) \cos\left(\frac{n\pi z}{H} \right) \left(\frac{H}{n\pi} \right) \sin\left(\frac{n\pi L}{H} \right) \\
&= \left(\frac{2H^2}{ab\pi^3} \right) \sum_{n=1}^{\infty} \left(\frac{1}{n^3} \right) \sin\left(\frac{n\pi L}{H} \right) \cos\left(\frac{n\pi z}{H} \right).
\end{aligned}
\tag{5.18}
$$

The average value of $(J_z)_w$ along the well length is

$$
\begin{aligned}
(J_{z,a})_w &= \left(\frac{1}{L} \right) \int_0^L J_z dz \\
&= \left(\frac{1}{L} \right) \sum_{n=1}^{\infty} \left(\frac{2H^2}{\pi^3 abn^3} \right) \sin\left(\frac{n\pi L}{H} \right) \int_0^L \cos\left(\frac{n\pi z}{H} \right) dz \\
&= \sum_{n=1}^{\infty} \left(\frac{2H^2}{\pi^3 abLn^3} \right) \sin\left(\frac{n\pi L}{H} \right) \left[\left(\frac{H}{n\pi} \right) \sin\left(\frac{n\pi L}{H} \right) \right] \\
&= \sum_{n=1}^{\infty} \left(\frac{2H^3}{\pi^4 abLn^4} \right) \sin^2\left(\frac{n\pi L}{H} \right) \\
&= \left(\frac{H^3}{\pi^4 abL} \right) \sum_{n=1}^{\infty} \left(\frac{1}{n^4} \right) \left[1 - \cos\left(\frac{2n\pi L}{H} \right) \right] \\
&= \left(\frac{H^3}{\pi^4 abL} \right) \left(\frac{2\pi L}{H} \right)^2 \left[\frac{\pi^2}{12} - \frac{\pi}{12} \left(\frac{2\pi L}{H} \right) + \frac{1}{48} \left(\frac{2\pi L}{H} \right)^2 \right] \\
&= \left(\frac{4HL}{ab} \right) \left(\frac{1}{12} - \frac{L}{6H} + \frac{L^2}{12H^2} \right) \\
&= \left(\frac{2HL}{3ab} \right) \left(\frac{1}{2} - \frac{L}{H} + \frac{L^2}{2H^2} \right),
\end{aligned}
\tag{5.19}
$$

where we have used (5.17).

For a fully penetrating well, $L = H$, then

$$
(J_{z,a})_w = 0. \tag{5.20}
$$

Recall (5.5) and (5.10), J_{yz} is for the case $l = 0, m > 0, n \geq 0$, and at wellbore of the off-center well,

$$
\begin{aligned}
& y = y' \neq 0, \qquad x' \neq 0, \qquad x = x' + R_w, \qquad 0 \leq z = z' \leq L, \\
(J_{yz})_w &= \left(\frac{1}{abH}\right)\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left(\frac{1}{d_m d_n \lambda_{0mn}}\right)\cos^2\left(\frac{m\pi y'}{b}\right)\cos\left(\frac{n\pi z}{H}\right)\int_0^L \cos\left(\frac{n\pi z'}{H}\right)dz' \\
&= \left(\frac{2}{abH}\right)\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{\cos^2(m\pi y'/b)\cos(n\pi z/H)}{\pi^2 d_n\left[(n/H)^2 + (m/b)^2\right]}\int_0^L \cos\left(\frac{n\pi z'}{H}\right)dz'\right\} \\
&= \left(\frac{2}{abH}\right)\sum_{m=1}^{\infty}\left\{\sum_{n=1}^{\infty}\frac{2(H/n\pi)\cos(n\pi z/H)\sin(n\pi L/H)\cos^2(m\pi y'/b)}{\pi^2\left[(n/H)^2 + (m/b)^2\right]}\right. \\
&\qquad\qquad \left. +\cos^2\left(\frac{m\pi y'}{b}\right)\left(\frac{b^2 L}{\pi^2 m^2}\right)\right\} \\
&= \left(\frac{2H^3}{\pi^3 abH}\right)\sum_{m=1}^{\infty}\left\{\sum_{n=1}^{\infty}\frac{2\cos(n\pi z/H)\sin(n\pi L/H)\cos^2(m\pi y'/b)}{n\left[n^2 + (mH/b)^2\right]}\right. \\
&\qquad\qquad \left. +\cos^2\left(\frac{m\pi y'}{b}\right)\left(\frac{b^2 L\pi}{H^3 m^2}\right)\right\} \\
&= \left(\frac{2H^2}{\pi^3 ab}\right)\left(\frac{\pi L b^2}{H^3}\right)\sum_{m=1}^{\infty}\left(\frac{1}{m^2}\right)\cos^2\left(\frac{m\pi y'}{b}\right) \\
&\quad + \left(\frac{2H^2}{\pi^3 ab}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{2\cos(n\pi z/H)\sin(n\pi L/H)\cos^2(m\pi y'/b)}{n\left[n^2 + (mH/b)^2\right]}\right\} \\
&= \left(\frac{2bL}{\pi^2 aH}\right)\sum_{m=1}^{\infty}\left(\frac{1}{m^2}\right)\cos^2\left(\frac{m\pi y'}{b}\right) \\
&\quad + \left(\frac{2H^2}{\pi^3 ab}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{2\cos(n\pi z/H)\sin(n\pi L/H)\cos^2(m\pi y'/b)}{n\left[n^2 + (mH/b)^2\right]}\right\},
\end{aligned}
\tag{5.21}
$$

where we use the following formulas [16, page 47]:

$$
\sum_{m=1}^{\infty}\left(\frac{1}{m^2}\right)\cos(mx) = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} \quad (0 \leq x \leq 2\pi), \tag{5.22}
$$

$$
\sum_{m=1}^{\infty}\left(\frac{1}{m^2}\right)\cos^2(mx) = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{2} \quad (0 \leq x \leq \pi). \tag{5.23}
$$

The average value of $(J_{yz})_w$ along the well length is

$$
\begin{aligned}
(J_{yz,a})_w &= \left(\frac{1}{L}\right)\int_0^L J_{yz}dz \\
&= \left(\frac{2bL}{aH}\right)\left(\frac{1}{6}-\frac{y'}{2b}+\frac{y'^2}{2b^2}\right) \\
&\quad+\left(\frac{2H^2}{abL\pi^3}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{2\sin(n\pi L/H)\cos^2(m\pi y'/b)}{n\left[n^2+(mH/b)^2\right]}\int_0^L\cos\left(\frac{n\pi z}{H}\right)dz\right\} \\
&= \left(\frac{2bL}{aH}\right)\left(\frac{1}{6}-\frac{y'}{2b}+\frac{y'^2}{2b^2}\right) \\
&\quad+\left(\frac{2H^2}{abL\pi^3}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{2H\sin^2(n\pi L/H)\cos^2(m\pi y'/b)}{\pi n^2\left[n^2+(mH/b)^2\right]}\right\} \\
&= \left(\frac{2bL}{aH}\right)\left(\frac{1}{6}-\frac{y'}{2b}+\frac{y'^2}{2b^2}\right) \\
&\quad+\left(\frac{2H^3}{abL\pi^4}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{[1-\cos(2n\pi L/H)]\cos^2(m\pi y'/b)}{n^2\left[n^2+(mH/b)^2\right]}\right\} \\
&= \left(\frac{2bL}{aH}\right)\left(\frac{1}{6}-\frac{y'}{2b}+\frac{y'^2}{2b^2}\right)+\left(\frac{2H^3}{abL\pi^4}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{b}{mH}\right)^2\cos^2\left(\frac{m\pi y'}{b}\right) \\
&\quad\times\left[1-\cos\left(\frac{2n\pi L}{H}\right),\frac{1}{n^2}-\frac{1}{n^2+(mH/b)^2}\right] \\
&= \left(\frac{2bL}{aH}\right)\left(\frac{1}{6}-\frac{y'}{2b}+\frac{y^{2'}}{2b^2}\right)+\left(\frac{H^3}{2abL\pi^4}\right)\sum_{m=1}^{\infty}\left(\frac{b}{mH}\right)^2\cos^2\left(\frac{m\pi y'}{b}\right) \\
&\quad\times\sum_{n=1}^{\infty}\left[\frac{1}{n^2}-\frac{\cos(2n\pi L/H)}{n^2}-\frac{1}{n^2+(mH/b)^2}+\frac{\cos(2n\pi L/H)}{n^2+(mH/b)^2}\right] \\
&= \left(\frac{2bL}{aH}\right)\left(\frac{1}{6}-\frac{y'}{2b}+\frac{y'^2}{2b^2}\right)+\left(\frac{2H^3}{abL\pi^4}\right)\sum_{m=1}^{\infty}\left(\frac{b}{mH}\right)^2\cos^2\left(\frac{m\pi y'}{b}\right) \\
&\quad\times\left\{\frac{\pi^2}{6}-\left[\frac{\pi^2}{6}-\frac{\pi}{2}\left(\frac{2\pi L}{H}\right)+\frac{1}{4}\left(\frac{2\pi L}{H}\right)^2\right]\right. \\
&\qquad-\left[\left(\frac{b\pi}{2mH}\right)\coth\left(\frac{mH\pi}{b}\right)-\frac{1}{2}\left(\frac{b}{mH}\right)^2\right] \\
&\qquad\left.+\left[\left(\frac{b\pi}{2mH}\right)\frac{\cosh[(mH\pi/b)(1-2L/H)]}{\sinh(mH\pi/b)}-\frac{1}{2}\left(\frac{b}{mH}\right)^2\right]\right\},
\end{aligned}
\tag{5.24}
$$

where we use the following formulas [16, page 47]:

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2+\beta^2} = \left(\frac{\pi}{2\beta}\right)\left\{\frac{\cosh[\beta(\pi-x)]}{\sinh(\beta\pi)}\right\} - \frac{1}{2\beta^2} \quad (0 \le x \le 2\pi), \tag{5.25}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+\beta^2} = \left(\frac{\pi}{2\beta}\right)\coth(\beta\pi) - \frac{1}{2\beta^2} \quad (0 \le x \le 2\pi), \tag{5.26}$$

and we may simplify (5.24) further

$$\begin{aligned}
(J_{yz,a})_w &= \left(\frac{2bL}{aH}\right)\left(\frac{1}{6} - \frac{y'}{2b} + \frac{y'^2}{2b^2}\right) + \left(\frac{2H^3}{abL\pi^4}\right)\sum_{m=1}^{\infty}\cos^2\left(\frac{m\pi y'}{b}\right)\left(\frac{b}{mH}\right)^2 \\
&\quad \times \left\{\frac{\pi^2 L}{H} - \frac{\pi^2 L^2}{H^2} + \left(\frac{b\pi}{2mH}\right)\left\{\frac{\cosh[(mH\pi/b)(1-2L/H)]}{\sinh(mH\pi/b)} - \coth\left(\frac{mH\pi}{b}\right)\right\}\right\} \\
&= \left(\frac{2bL}{aH}\right)\left(\frac{1}{6} - \frac{y'}{2b} + \frac{y'^2}{2b^2}\right) + \left(\frac{2H^3}{abL\pi^4}\right)\sum_{m=1}^{\infty}\cos^2\left(\frac{m\pi y'}{b}\right)\left(\frac{1}{m^2}\right) \\
&\quad \times \left\{\frac{\pi^2 L b^2}{H^3} - \frac{\pi^2 L^2 b^2}{H^4} + \left(\frac{b^3\pi}{2mH^3}\right)\right. \\
&\qquad \left. \times \left\{\frac{\cosh[(mH\pi/b)(1-2L/H)]}{\sinh(mH\pi/b)} - \coth\left(\frac{mH\pi}{b}\right)\right\}\right\} \\
&= \left(\frac{2bL}{aH}\right)\left(\frac{1}{6} - \frac{y'}{2b} + \frac{y'^2}{2b^2}\right) + \left(\frac{2b}{a\pi^2}\right)\left(1 - \frac{L}{H}\right)\left(\frac{\pi^2}{6} - \frac{\pi^2 y'}{2b} + \frac{\pi^2 y^{2'}}{2b^2}\right) \\
&\quad + \left(\frac{b^2}{aL\pi^3}\right)\sum_{m=1}^{\infty}\cos^2\left(\frac{m\pi y'}{b}\right)\left(\frac{1}{m^3}\right) \\
&\quad \times \left\{\frac{\cosh[(mH\pi/b)(1-2L/H)]}{\sinh(mH\pi/b)} - \coth\left(\frac{mH\pi}{b}\right)\right\} \\
&= \left(\frac{2b}{a}\right)\left(\frac{1}{6} - \frac{y'}{2b} + \frac{y^{2'}}{2b^2}\right) \\
&\quad + \left(\frac{b^2}{aL\pi^3}\right)\sum_{m=1}^{\infty}\left[\frac{\cos^2(m\pi y'/b)}{m^3}\right]\left\{\frac{\cosh[(mH\pi/b)(1-2L/H)]}{\sinh(mH\pi/b)} - \coth\left(\frac{mH\pi}{b}\right)\right\}.
\end{aligned} \tag{5.27}$$

For a fully penetrating well, $L = H$, then

$$(J_{yz,a})_w = \left(\frac{2b}{a}\right)\left(\frac{1}{6} - \frac{y'}{2b} + \frac{y'^2}{2b^2}\right). \tag{5.28}$$

Define

$$f(x) = \sinh[\alpha(1-x)]\sinh(\alpha x), \tag{5.29}$$

since the derivative of $f(x)$ is

$$\begin{aligned} f'(x) &= \alpha\cosh(\alpha x)\sinh[\alpha(1-x)] - \alpha\cosh[\alpha(1-x)]\sinh(\alpha x) \\ &= \alpha\sinh[\alpha(1-2x)], \end{aligned} \tag{5.30}$$

consequently,

$$f'\left(\frac{1}{2}\right) = 0. \tag{5.31}$$

When $x = 0$ and $x = 1$,

$$f(0) = f(1) = 0. \tag{5.32}$$

When $x = 1/2$, $f(x)$ reaches maximum value, let

$$x = \frac{L}{H}, \tag{5.33}$$

and the producing length L is a variable, define

$$\begin{aligned} F(L) &= \frac{\cosh\left[\beta\pi(1-2L/H)\right] - \cosh(\beta\pi)}{\sinh(\beta\pi)} \\ &= \frac{-2\sinh\left[\beta\pi(1-L/H)\right]\sinh\left[\beta\pi L/(H)\right]}{\sinh(\beta\pi)}, \end{aligned} \tag{5.34}$$

thus when $L = H/2$, $|F(L)|$ reaches maximum value,

$$\begin{aligned} |F(L)|_{\max} &= \left|F\left(\frac{H}{2}\right)\right| \\ &= \frac{2\sinh^2(\beta\pi/2)}{\sinh(\beta\pi)} \\ &= \frac{2\sinh^2(\beta\pi/2)}{2\sinh(\beta\pi/2)\cosh(\beta\pi/2)} \\ &= \frac{\sinh(\beta\pi/2)}{\cosh(\beta\pi/2)} < 1, \end{aligned} \tag{5.35}$$

so $F(L)$ is a bounded function, let

$$\beta = \frac{mH}{b}, \tag{5.36}$$

then

$$\begin{aligned}
(J_{yz,a})_w &= \left(\frac{2b}{a}\right)\left(\frac{1}{6} - \frac{y'}{2b} + \frac{y^{2'}}{2b^2}\right) + \left(\frac{b^2}{aL\pi^3}\right) \\
&\quad \times \sum_{m=1}^{\infty}\left[\frac{\cos^2(m\pi y'/b)}{m^3}\right]\left\{\frac{\cosh[(mH\pi/b)(1-2L/H)]}{\sinh(mH\pi/b)} - \coth\left(\frac{mH\pi}{b}\right)\right\} \\
&= \left(\frac{2b}{a}\right)\left(\frac{1}{6} - \frac{y'}{2b} + \frac{y^{2'}}{2b^2}\right) + \left(\frac{b^2}{aL\pi^3}\right) \\
&\quad \times \sum_{m=1}^{\infty}\left[\frac{\cos^2(m\pi y'/b)}{m^3}\right]\left\{\frac{-2\sinh[(mH\pi/b)(1-L/H)]\sinh(mL\pi/b)}{\sinh(mH\pi/b)}\right\} \\
&\approx \left(\frac{2b}{a}\right)\left(\frac{1}{6} - \frac{y'}{2b} + \frac{y^{2'}}{2b^2}\right) + \left(\frac{b^2}{aL\pi^3}\right) \\
&\quad \times \sum_{m=1}^{M}\left[\frac{\cos^2(m\pi y'/b)}{m^3}\right]\left\{\frac{-2\sinh[(mH\pi/b)(1-L/H)]\sinh(mL\pi/b)}{\sinh(mH\pi/b)}\right\}.
\end{aligned} \tag{5.37}$$

Since $0 < L/H < 1$, from (5.34) and (5.35), there holds

$$\begin{aligned}
&\sum_{m=101}^{\infty}\left|\left[\frac{\cos^2(m\pi y'/b)}{m^3}\right]\left\{\frac{-2\sinh[(mH\pi/b)(1-L/H)]\sinh(mL\pi/b)}{\sinh(mH\pi/b)}\right\}\right| \\
&\quad \le \sum_{m=101}^{\infty}\frac{1}{m^3} \\
&\quad = \zeta(3) - \sum_{m=1}^{100}\frac{1}{m^3} \\
&\quad = 4.9502 \times 10^{-5},
\end{aligned} \tag{5.38}$$

where $\zeta(3)$ is *Riemann*-ζ function:

$$\zeta(3) = \sum_{m=1}^{\infty}\frac{1}{m^3} = 1.202057, \tag{5.39}$$

thus

$$\sum_{m=1}^{\infty}\left(\frac{1}{m^3}\right)\left\{\frac{2\sinh[(mH\pi/b)(1-L/H)]\sinh(mL\pi/b)}{\sinh(mH\pi/b)}\right\}$$
$$\approx\sum_{m=1}^{100}\left(\frac{1}{m^3}\right)\left\{\frac{2\sinh[(mH\pi/b)(1-L/H)]\sinh(mL\pi/b)}{\sinh(mH\pi/b)}\right\}. \tag{5.40}$$

So, in (5.37), $M = 100$ is sufficient to reach engineering accuracy.

Recall (5.6) and (5.11), J_{xyz} is for the case $l > 0, m \ge 0, n \ge 0$, and at wellbore of the off-center well,

$$y = y' \neq 0, \qquad x' \neq 0, \qquad x = x' + R_w, \qquad 0 \le z = z' \le L, \tag{5.41}$$

then

$$\begin{aligned}(J_{xyz})_w &= \left(\frac{1}{abH}\right)\\ &\quad\times\sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\left[\frac{\cos(n\pi z/H)\cos(l\pi x'/a)\cos[(l\pi(x'+R_w))/a]\cos^2(m\pi y'/b)}{d_l d_m d_n \lambda_{lmn}}\right]\right.\\ &\qquad\left.\times\int_0^L \cos\left(\frac{n\pi z'}{H}\right)dz'\right\}\\ &= \left(\frac{1}{abH}\right)\sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right]\cos^2\left(\frac{m\pi y'}{b}\right)\\ &\quad\times\left\{\sum_{n=1}^{\infty}\left[\frac{4(H/n\pi)\sin(n\pi L/H)\cos(n\pi z/H)}{d_m\lambda_{lmn}}\right]+\frac{2L}{d_m\lambda_{lm0}}\right\}.\end{aligned} \tag{5.42}$$

The average value of $(J_{xyz})_w$ along the well length is

$$\begin{aligned}(J_{xyz,a})_w &= \left(\frac{1}{abH}\right)\sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right]\cos^2\left(\frac{m\pi y'}{b}\right)\\ &\quad\times\left\{\sum_{n=1}^{\infty}\left[\frac{4(H/n\pi)\sin(n\pi L/H)\int_0^L\cos(n\pi z/H)dz}{d_m\lambda_{lmn}L}\right]+\frac{2L}{d_m\lambda_{lm0}}\right\}\end{aligned}$$

$$= \left(\frac{1}{abH}\right)\sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right]\cos^2\left(\frac{m\pi y'}{b}\right)$$

$$\times\left\{\sum_{n=1}^{\infty}\left[\frac{4(H/n\pi)^2\sin^2(n\pi L/H)}{d_m\lambda_{lmn}L}\right]+\frac{2L}{d_m\lambda_{lm0}}\right\}$$

$$= \left(\frac{H^4}{abH\pi^4}\right)\sum_{l=1}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right]\sum_{m=0}^{\infty}\cos^2\left(\frac{m\pi y'}{b}\right)$$

$$\times\left\{\sum_{n=1}^{\infty}\frac{2[1-\cos(2n\pi L/H)]}{d_m n^2(n^2+\mu_{lm}^2)L}+\frac{2\pi^2 L}{d_m H^2\mu_{lm}^2}\right\}$$

$$= \left(\frac{H^3}{ab\pi^4}\right)\sum_{l=1}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right]\left\{\sum_{m=0}^{\infty}\left(\frac{2}{d_m}\right)\cos^2\left(\frac{m\pi y'}{b}\right)\right.$$

$$\left.\times\left\{\sum_{n=1}^{\infty}\left(\frac{1}{\mu_{lm}^2 L}\right)\left[1-\cos\left(\frac{2n\pi L}{H}\right)\right]\times\left(\frac{1}{n^2}-\frac{1}{n^2+\mu_{lm}^2}\right)+\frac{\pi^2 L}{H^2\mu_{lm}^2}\right\}\right.$$

$$= \left(\frac{H^3}{ab\pi^4}\right)\sum_{l=1}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right]\left\{\sum_{m=0}^{\infty}\left(\frac{2}{d_m\mu_{lm}^2 L}\right)\cos^2\left(\frac{m\pi y'}{b}\right)\right.$$

$$\left.\times\left\{\sum_{n=1}^{\infty}\left[\frac{1}{n^2}-\frac{\cos(2n\pi L/H)}{n^2}-\frac{1}{n^2+\mu_{lm}^2}+\frac{\cos(2n\pi L/H)}{n^2+\mu_{lm}^2}\right]\right\}+\left(\frac{\pi^2L^2}{H^2}\right)\right\}$$

$$= \left(\frac{H^3}{ab\pi^4}\right)\sum_{l=1}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right]\sum_{m=0}^{\infty}\left(\frac{2}{d_m\mu_{lm}^2 L}\right)\cos^2\left(\frac{m\pi y'}{b}\right)$$

$$\times\left\{\frac{\pi^2}{6}-\left[\frac{\pi^2}{6}-\left(\frac{\pi}{2}\right)\left(\frac{2\pi L}{H}\right)+\left(\frac{1}{4}\right)\left(\frac{2\pi L}{H}\right)^2\right]\right.$$

$$-\left[\left(\frac{\pi}{2\mu_{lm}}\right)\coth(\mu_{lm}\pi)-\left(\frac{1}{2\mu_{lm}^2}\right)\right]$$

$$\left.+\left[\left(\frac{\pi}{2\mu_{lm}}\right)\frac{\cosh[\mu_{lm}\pi(1-2L/H)]}{\sinh(\mu_{lm}\pi)}-\frac{1}{2\mu_{lm}^2}\right]\right\}+\left(\frac{\pi^2L^2}{H^2}\right)\Bigg\}, \tag{5.43}$$

where we use (5.22) and (5.25).

Let $x = 0$, recast (5.26), we obtain

$$\begin{gathered}
\frac{1}{\beta^2} + 2\sum_{n=1}^{\infty}\frac{1}{n^2+\beta^2} = \left(\frac{\pi}{\beta}\right)\coth(\beta\pi), \\
\sum_{n=0}^{\infty}\frac{1}{(n^2+\beta^2)d_n} = \left(\frac{\pi}{\beta}\right)\coth(\beta\pi).
\end{gathered} \tag{5.44}$$

So,

$$\begin{aligned}
(J_{xyz,a})_w &= \left(\frac{H^3}{ab\pi^4}\right)\sum_{l=1}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right]\sum_{m=0}^{\infty}\left(\frac{2}{d_m\mu_{lm}^2 L}\right)\cos^2\left(\frac{m\pi y'}{b}\right) \\
&\quad\times\left\{\left(\frac{\pi^2 L}{H} - \frac{\pi^2 L^2}{H^2} + \frac{\pi^2 L^2}{H^2}\right) + \left(\frac{\pi}{2\mu_{lm}}\right)\left[\frac{\cosh[\mu_{lm}\pi(1-2L/H)]}{\sinh(\mu_{lm}\pi)} - \coth(\mu_{lm}\pi)\right]\right\} \\
&= \left(\frac{H^3}{ab\pi^4}\right)\sum_{l=1}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right] \\
&\quad\times\left\{\sum_{m=0}^{\infty}\left(\frac{\pi}{d_m\mu_{lm}^3 L}\right)\cos^2\left(\frac{m\pi y'}{b}\right)\times\left\{\frac{\cosh[\mu_{lm}\pi(1-2L/H)]}{\sinh(\mu_{lm}\pi)} - \coth(\mu_{lm}\pi)\right\}\right. \\
&\qquad\left. + \left(\frac{2}{L}\right)\left(\frac{\pi^2 L}{H}\right)\sum_{m=0}^{\infty}\frac{\cos^2(m\pi y'/b)}{d_m\mu_{lm}^2}\right\} \\
&= \left(\frac{H^3}{ab\pi^4}\right)\sum_{l=1}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right] \\
&\quad\times\left\{\sum_{m=0}^{\infty}\left(\frac{\pi}{d_m\mu_{lm}^3 L}\right)\cos^2\left(\frac{m\pi y'}{b}\right)\times\left\{\frac{\cosh[\mu_{lm}\pi(1-2L/H)]}{\sinh(\mu_{lm}\pi)} - \coth(\mu_{lm}\pi)\right\}\right. \\
&\qquad\left. + \left(\frac{ab\pi^3}{H^3 l}\right)\coth\left(\frac{lb\pi}{a}\right) + \left(\frac{\pi^2}{H}\right)\sum_{m=0}^{\infty}\frac{\cos(2m\pi y'/b)}{d_m\mu_{lm}^2}\right\} \\
&= \left(\frac{H^3}{abL\pi^3}\right)\sum_{l=1}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right] \\
&\quad\times\left\{\sum_{m=0}^{\infty}\left(\frac{1}{d_m\mu_{lm}^3}\right)\cos^2\left(\frac{m\pi y'}{b}\right)\times\left\{\frac{\cosh[\mu_{lm}\pi(1-2L/H)]}{\sinh(\mu_{lm}\pi)} - \coth(\mu_{lm}\pi)\right\}\right\} \\
&\quad+\left(\frac{H^3}{ab\pi^4}\right)\sum_{l=1}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}, \left(\frac{ab\pi^3}{H^3 l}\right)\coth\left(\frac{lb\pi}{a}\right)\right. \\
&\qquad\qquad\left. + \left(\frac{\pi^2}{H}\right)\sum_{m=0}^{\infty}\frac{\cos(2m\pi y'/b)}{d_m\mu_{lm}^2}\right].
\end{aligned} \tag{5.45}$$

Since

$$\begin{aligned}
&\left(\frac{\pi^2}{H}\right)\sum_{m=0}^{\infty}\left[\frac{\cos(2m\pi y'/b)}{d_m\mu_{lm}^2}\right] \\
&\quad=\left(\frac{b^2}{H^3}\right)\sum_{m=0}^{\infty}\left\{\frac{\cos(2m\pi y'/b)}{d_m\left[m^2+(bl/a)^2\right]}\right\} \\
&\quad=\left(\frac{b^2}{H^3}\right)\left(\frac{\pi a}{bl}\right)\left\{\frac{\cosh[\pi bl(1-2y'/b)/a]}{\sinh(\pi bl/a)}\right\} \\
&\quad\approx\left(\frac{b}{H^3}\right)\left(\frac{\pi a}{l}\right)\exp\left(-\frac{2\pi y' l}{a}\right), \\
&\left(\frac{H^3}{ab\pi^4}\right)\left|\sum_{l=1}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right]\left(\frac{b}{H^3}\right)\left(\frac{\pi a}{l}\right)\exp\left(-\frac{2\pi y' l}{a}\right)\right| \\
&\quad\le\left(\frac{1}{\pi^3}\right)\ln[1-\exp(-2\pi y'/a)] \\
&\quad\approx 0,
\end{aligned} \tag{5.46}$$

thus

$$\begin{aligned}
(J_{xyz,a})_w &\approx \left(\frac{H^3}{abL\pi^3}\right)\sum_{l=1}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right]\sum_{m=0}^{\infty}\left(\frac{1}{d_m\mu_{lm}^3}\right)\cos^2\left(\frac{m\pi y'}{b}\right) \\
&\quad\times\left\{\frac{\cosh[\mu_{lm}\pi(1-2L/H)]}{\sinh(\mu_{lm}\pi)}-\coth(\mu_{lm}\pi)\right\} \\
&\quad+\left(\frac{1}{\pi}\right)\sum_{l=1}^{\infty}\left(\frac{1}{l}\right)\coth\left(\frac{lb\pi}{a}\right)\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right] \\
&\approx \left(\frac{H^3}{abL\pi^3}\right)\sum_{l=1}^{\infty}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right]\sum_{m=0}^{\infty}\left(\frac{1}{d_m\mu_{lm}^3}\right)\cos^2\left(\frac{m\pi y'}{b}\right) \\
&\quad\times\left\{\frac{\cosh[\mu_{lm}\pi(1-2L/H)]}{\sinh(\mu_{lm}\pi)}-\coth(\mu_{lm}\pi)\right\} \\
&\qquad-\left(\frac{1}{2\pi}\right)\ln\left\{4\sin\left(\frac{\pi R_w}{2a}\right)\sin\left[\frac{\pi(2x'+R_w)}{2a}\right]\right\},
\end{aligned} \tag{5.47}$$

where we use the following formula [16, page 46]:

$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}=-\ln\left[2\sin\left(\frac{x}{2}\right)\right], \tag{5.48}$$

and the following simplifications:

$$\coth\left(\frac{lb\pi}{a}\right) \approx 1,$$
$$\sum_{l=1}^{\infty}\left(\frac{1}{l}\right)\coth\left(\frac{lb\pi}{a}\right)\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right]$$
$$\approx \sum_{l=1}^{\infty}\left(\frac{1}{l}\right)\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right] \tag{5.49}$$
$$= -\frac{1}{2}\ln\left\{4\sin\left(\frac{\pi R_w}{2a}\right)\sin\left[\frac{\pi(2x'+R_w)}{2a}\right]\right\}.$$

For a fully penetrating well, $L = H$, (5.47) is simplified as

$$\left(J_{xyz,a}\right)_w = -\left(\frac{1}{2\pi}\right)\ln\left\{4\sin\left(\frac{\pi R_w}{2a}\right)\sin\left[\frac{\pi(2x'+R_w)}{2a}\right]\right\}. \tag{5.50}$$

Recall (5.13), then

$$J_{a,w} = \left(J_{z,a}\right)_w + \left(J_{yz,a}\right)_w + \left(J_{xyz,a}\right)_w$$
$$= \left(\frac{2HL}{3ab}\right)\left(\frac{1}{2} - \frac{L}{H} + \frac{L^2}{2H^2}\right) + \left(\frac{2b}{a}\right)\left(\frac{1}{6} - \frac{y'}{2b} + \frac{y^{2'}}{2b^2}\right)$$
$$+ \left(\frac{b^2}{aL\pi^3}\right)\sum_{m=1}^{M}\left[\frac{\cos^2(m\pi y'/b)}{m^3}\right]\left\{\frac{\cosh[(mH\pi/b)(1-2L/H)]}{\sinh(mH\pi/b)} - \coth\left(\frac{mH\pi}{b}\right)\right\}$$
$$+ \left(\frac{H^3}{abL\pi^3}\right)\left\{\sum_{l=1}^{N}\cos\left(\frac{l\pi x'}{a}\right)\cos\left[\frac{l\pi(x'+R_w)}{a}\right] \times \sum_{m=0}^{M}\left(\frac{1}{d_m\mu_{lm}^3}\right)\cos^2\left(\frac{m\pi y'}{b}\right)\right.$$
$$\left.\times\left\{\frac{\cosh[\mu_{lm}\pi(1-2L/H)]}{\sinh(\mu_{lm}\pi)} - \coth(\mu_{lm}\pi)\right\}\right\}$$
$$- \left(\frac{1}{2\pi}\right)\ln\left\{4\sin\left(\frac{\pi R_{\mathrm{w}}}{2a}\right)\sin\left[\frac{\pi(2x'+R_w)}{2a}\right]\right\}. \tag{5.51}$$

Recall (4.28) and (4.40), the average wellbore pressure along the uniform line sink is

$$P_{a,w} = \frac{t}{abH} + J_{a,w}, \tag{5.52}$$

then (4.41) becomes

$$P_{a,v} - P_{a,w} = J_{a,v} - J_{a,w}, \tag{5.53}$$

which implies $P_{a,v} - P_{a,w}$ is independent of time.

6. Productivity Formula and Shape Factor Formula

Note that (5.53) is in dimensionless form, that is,

$$P_{a,vD} - P_{a,wD} = J_{a,vD} - J_{a,wD}. \tag{6.1}$$

Formulas (4.26), (4.41), (5.1), (5.2), and (5.53) are in dimensionless forms, recall (4.26) and obtain

$$\Psi_{a,vD} = 0, \qquad J_{a,vD} = 0, \tag{6.2}$$

which implies

$$P_{a,vD} - P_{a,wD} = -J_{a,wD} = -\left[\frac{K_a L(P_a - P_w)}{\mu q B}\right]. \tag{6.3}$$

In order to simplify the above formulas, let

$$Y_e = a, \qquad X_e = b, \qquad Y_w = x', \qquad X_w = y', \tag{6.4}$$

then

$$Y_{eD} = a_D, \qquad X_{eD} = b_D, \qquad Y_{wD} = x'_D, \qquad X_{wD} = y'_D. \tag{6.5}$$

Combining (3.6), (3.9), (5.13), (6.3), the pseudo-steady-state productivity formula for a partially penetrating vertical well in an anisotropic closed box-shaped reservoir is obtained

$$Q_w = F_D \frac{2\pi\left(K_x K_y\right)^{1/2} H(P_a - P_w)/(\mu B)}{\Lambda + S_{\text{ps}}}, \tag{6.6}$$

where P_a is average reservoir pressure throughout the box-shaped drainage volume, P_w is average wellbore pressure, and

$$\Lambda = \left(\frac{4\pi X_{eD}}{\eta Y_{eD}}\right)\left(\frac{1}{6} - \frac{X_{wD}}{2X_{eD}} + \frac{X_{wD}^2}{2X_{eD}^2}\right) - \frac{\ln\{4|\sin[\pi(2Y_{wD} + R_{wD})/2Y_{eD}]|\sin[\pi R_{wD}/2Y_{eD}]\}}{\eta} \tag{6.7}$$

$$\begin{aligned} S_{\text{ps}} = {} & \left(\frac{4\pi H_D L_D}{3\eta X_{eD} Y_{eD}}\right)\left(\frac{1}{2} - \eta + \frac{\eta^2}{2}\right) + \left(\frac{2X_{eD}^2}{\pi^2 \eta Y_{eD} L_D}\right)\sum_{m=1}^{M}\left[\frac{\cos^2(m\pi X_{wD}/X_{eD})}{m^3}\right] \\ & \times \left\{\frac{\cosh[(m\pi H_D/X_{eD})(1-2\eta)]}{\sinh(m\pi H_D/X_{eD})} - \coth\left(\frac{m\pi H_D}{X_{eD}}\right)\right\} \\ & + \left\{\left(\frac{2H_D^3}{\pi^2 \eta X_{eD} Y_{eD} L_D}\right)\sum_{l=1}^{N}\cos\left(\frac{l\pi Y_{wD}}{Y_{eD}}\right)\cos\left[\frac{l\pi(Y_{wD} + R_{wD})}{Y_{eD}}\right]\right. \\ & \left.\times\sum_{m=0}^{M}\left(\frac{1}{d_m \mu_{lm}^3}\right)\cos^2\left(\frac{m\pi X_{wD}}{X_{eD}}\right)\left\{\frac{\cosh[\mu_{lm}\pi(1-2\eta)]}{\sinh(\mu_{lm}\pi)} - \coth(\mu_{lm}\pi)\right\}\right\}, \end{aligned} \tag{6.8}$$

where η is partial penetration factor defined in (2.3), S_{ps} is pseudoskin factor due to partial penetration, and

$$\mu_{lm} = \left[\left(\frac{lH}{a}\right)^2 + \left(\frac{mH}{b}\right)^2\right]^{1/2}. \tag{6.9}$$

In the above equations, $M = N = 100$ is sufficient to reach engineering accuracy.

For a fully penetrating well, $L = H$, then (6.8) reduces to

$$S_{\text{ps}} = 0. \tag{6.10}$$

If a fully penetrating vertical well located in a closed isotropic rectangular reservoir,

$$S_{\text{ps}} = 0, \qquad L_p = H, \qquad K_x = K_y = K_z = K. \tag{6.11}$$

Then (6.6) reduces to

$$Q_w = \frac{2\pi KH(P_a - P_w)/(\mu B)}{\Theta - \ln\{4\sin[\pi R_w/(2Y_e)]\sin(\pi Y_w/Y_e)\}}, \tag{6.12}$$

where

$$\Theta = \left(\frac{4\pi X_e}{Y_e}\right)\left[\frac{1}{6} - \frac{1}{2}\left(\frac{X_w}{X_e}\right) + \frac{1}{2}\left(\frac{X_w}{X_e}\right)^2\right]. \tag{6.13}$$

Note that for a rectangle, its area is $A = X_e Y_e$, recall (2.10), equate (2.10) to (6.12),

$$\frac{2\pi KH(P_a - P_w)/(\mu B)}{\Theta - \ln\{4\sin[\pi R_w/(2Y_e)]\sin(\pi Y_w/Y_e)\}} = \frac{2\pi KH(P_a - P_w)/(\mu B)}{(1/2)\ln\left[2.2458X_e Y_e/(C_A R_w^2)\right]}. \tag{6.14}$$

A new expression to calculate the Dietz shape factor is obtained by solving C_A in (6.14),

$$C_A = \frac{88.6657 f_1 \sin^2(\pi f_3)}{\exp(f_4)}, \tag{6.15}$$

where

$$f_1 = \frac{X_e}{Y_e}, \qquad f_2 = \frac{X_w}{X_e}, \qquad f_3 = \frac{Y_w}{Y_e}, \tag{6.16}$$

$$f_4 = (8\pi f_1)\left(\frac{1}{6} - \frac{f_2}{2} + \frac{f_2^2}{2}\right). \tag{6.17}$$

Formula (6.6) is recommended to calculate productivity index in pseudo-steady-state, because it does not require the shape factor, and it is applicable to an off-center partially penetrating vertical well in pseudo-steady-state arbitrarily located in an anisotropic box-shaped reservoir.

So, the step-by-step derivations of pseudo-steady-state productivity formula and shape factor formula which were published in [1, 2] have been given in the above sections.

7. Examples and Discussions

The following examples are given to calculate well productivity index, pseudoskin factor due to partial penetration, and shape factor.

Example One

Use (6.6) to calculate productivity index of a partially penetrating vertical well in pseudo-steady-state in a closed box-shaped anisotropic reservoir. The wellbore, reservoir, and fluid properties data practical *SI* units are given in Table 1.

Table 1: Wellbore, Reservoir, and Fluid Properties Data.

Reservoir length, X_e	800 m
Reservoir width, Y_e	200 m
Payzone thickness, H	20 m
Well location in x direction, X_w	100 m
Well location in y direction, Y_w	50 m
Producing well length, L_p	10 m
Wellbore radius, R_w	0.1 m
Permeability in x direction, K_x	0.1 μm^2
Permeability in y direction, K_y	0.4 μm^2
Permeability in z direction, K_z	0.025 μm^2
Oil viscosity, μ	5.0 mPa.s
Formation volume factor, B	1.25 Rm^3/Sm^3

Solution. The average permeability is

$$K_a = (0.1 \times 0.4 \times 0.025)^{1/3} = 0.1 \left(\mu\text{m}^2\right). \tag{7.1}$$

Using dimensionless transforms given by (3.7) through (3.10), we obtain

$$\begin{gathered}
X_{eD} = \left(\frac{800}{10}\right) \times \left(\frac{0.1}{0.1}\right)^{1/2} = 80.0, \qquad Y_{eD} = \left(\frac{200}{10}\right) \times \left(\frac{0.1}{0.4}\right)^{1/2} = 10.0, \\
X_{wD} = \left(\frac{100}{10}\right) \times \left(\frac{0.1}{0.1}\right)^{1/2} = 10.0, \qquad Y_{wD} = \left(\frac{50}{10}\right) \times \left(\frac{0.1}{0.4}\right)^{1/2} = 2.5, \\
L_D = \left(\frac{0.1}{0.025}\right)^{1/2} = 2.0, \qquad H_D = \left(\frac{20}{10}\right) \times \left(\frac{0.1}{0.025}\right)^{1/2} = 4.0, \\
R_{wD} = \left(\frac{0.025}{(0.1 \times 0.4)^{1/2}}\right)^{1/6} \times \left(\left(\frac{0.1}{0.4}\right)^{1/4} + \left(\frac{0.4}{0.1}\right)^{1/4}\right) \times \frac{0.1}{(2 \times 10)} = 0.0075 \\
\eta = \frac{10.0}{20.0} = 0.5, \quad 1 - 2\eta = 0, \\
\mu_{lm} = \left[\left(\frac{4l}{10}\right)^2 + \left(\frac{4m}{80}\right)^2\right]^{1/2} = \left(\frac{4l^2}{25} + \frac{m^2}{400}\right)^{1/2}.
\end{gathered} \tag{7.2}$$

Recalling (6.8), pseudoskin factor due to partial penetration can be expressed as

$$S_{\text{ps}} = \Psi_1 + \Psi_2 + \Psi_3, \tag{7.3}$$

where

$$\begin{aligned}
\Psi_1 &= \left(\frac{4\pi H_D L_D}{3\eta X_{eD} Y_{eD}}\right)\left(\frac{1}{2} - \eta + \frac{\eta^2}{2}\right), \\
\Psi_2 &= \left(\frac{2X_{eD}^2}{\pi^2 \eta Y_{eD} L_D}\right)\sum_{m=1}^{100}\left[\frac{1}{m^3}\cos^2\left(\frac{m\pi X_{wD}}{X_{eD}}\right)\right] \\
&\quad \times \left\{\frac{\cosh\left[(mH_D\pi/X_{eD})(1-2\eta)\right]}{\sinh(mH_D\pi/X_{eD})} - \coth\left(\frac{mH_D\pi}{X_{eD}}\right)\right\}, \\
\Psi_3 &= \left(\frac{2H_D^3}{\pi^2 \eta X_{eD} Y_{eD} L_D}\right)\sum_{l=1}^{100}\cos\left(\frac{l\pi Y_{wD}}{Y_{eD}}\right)\cos\left(\frac{l\pi (Y_{wD}+R_{wD})}{Y_{eD}}\right) \\
&\quad \times \left\{\sum_{m=0}^{100}\left(\frac{1}{d_m \mu_{lm}^3}\right)\cos^2\left(\frac{m\pi X_{wD}}{X_{eD}}\right)\left\{\frac{\cosh\left[\mu_{lm}\pi(1-2\eta)\right]}{\sinh(\mu_{lm}\pi)} - \coth(\mu_{lm}\pi)\right\}\right\}.
\end{aligned} \tag{7.4}$$

Consequently,

$$\begin{aligned}
\Psi_1 &= \left(\frac{4\times\pi\times 4.0\times 2.0}{3\times 0.5\times 80\times 10}\right)\times\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{2\times 2^2}\right) \\
&= 0.01047, \\
\Psi_2 &= \left(\frac{2\times 80^2}{\pi^2\times 0.5\times 10\times 2.0}\right) \\
&\quad \times \left(\sum_{m=1}^{100}\left(\frac{1}{m^3}\times\cos^2\left(\frac{10}{80}\times\pi\times m\right)\right.\right. \\
&\qquad \left.\left.\times\left(\frac{\cosh(0)}{\sinh(4.0\times\pi\times m/80.0)} - \coth\left(\frac{4.0}{80.0}\times\pi\times m\right)\right)\right)\right) \\
&= -10.92,
\end{aligned}$$

$$
\begin{aligned}
\Psi_3 &= \left(\frac{2\times 4.0^3}{\pi^2\times 0.5\times 80\times 10\times 2}\right)\times\sum_{l=1}^{100}\cos^2\left(\frac{l\times\pi\times 2.5}{10}\right)\\
&\quad\times\left(\sum_{m=1}^{100}\left(\frac{2}{\left(4\times l^2/25+m^2/400\right)^{3/2}}\right)\times\cos^2\left(\frac{m\times\pi\times 10}{80}\right)\right.\\
&\quad\times\left(\frac{\cosh(0)}{\sinh\left(\pi\times\sqrt{4\times l^2/25+m^2/400}\right)}-\coth\left(\pi\times\sqrt{4\times\frac{l^2}{25}+\frac{m^2}{400}}\right)\right)\\
&\quad\left.+\left(\frac{125}{8\times l^3}\right)\times\left(\frac{\cosh(0)}{\sinh(\pi\times 2\times l/5)}-\coth\left(\pi\times 2\times\frac{l}{5}\right)\right)\right)\\
&= -1.086,\\
S_{\mathrm{ps}} &= 0.01047+(-10.92)+(-1.086)\approx -12.00.
\end{aligned}
\tag{7.5}
$$

Recalling (6.7), Λ is calculated by

$$
\begin{aligned}
\Lambda &= \left(\frac{4\times\pi\times 8}{0.5}\right)\times\left(\frac{1}{6}-\frac{1}{2\times 8}+\frac{1}{2\times 8^2}\right)\\
&\quad-\frac{\ln(4\times\sin(\pi\times 0.0075/2\times 10)\times\sin(\pi\times(0.0075+2\times 2.5)/2\times 10))}{0.5}\\
&= 27.52.
\end{aligned}
\tag{7.6}
$$

We use (6.6), $F_D = 86.4$ for practical SI units, the productivity index (the production rate per unit pressure drawdown) in pseudo-steady-state of the given well is

$$
PI = \frac{86.4\times 2\times\pi\times(0.1\times 0.4)^{1/2}\times 20/(5\times 1.25)}{27.52+(-12.00)} = 22.39\ \left(\mathrm{Sm}^3/\mathrm{D}/\mathrm{MPa}\right).
\tag{7.7}
$$

Example Two

Using the formulas given by Brons and Marting, Papatzacos, Bervaldier, calculate pseudoskin factor of the well in Example One.

Solution. If we use Brons and Marting's pseudoskin factor formula, then

$$K_h = (K_x K_y)^{1/2} = (0.1 \times 0.4)^{1/2} = 0.2\,(\mu m^2),$$
$$h_D = \left(\frac{H}{R_w}\right)\left(\frac{K_h}{K_v}\right)^{1/2} = \left(\frac{20}{0.1}\right) \times \left(\frac{0.2}{0.025}\right)^{1/2} = 565.685,$$
$$\begin{aligned} G(\eta) &= 2.948 - 7.363\eta + 11.45\eta^2 - 4.675\eta^3 \\ &= 2.948 - 7.363 \times 0.5 + 11.45 \times 0.5^2 - 4.675 \times 0.5^3 \\ &= 1.545, \end{aligned} \tag{7.8}$$

thus from (2.4), we have

$$\begin{aligned} S_{\text{ps}} &= \left(\frac{1}{\eta} - 1\right)[\ln(h_D) - G(\eta)] \\ &= \left(\frac{1}{0.5} - 1\right)[\ln(565.685) - 1.545] \\ &= 4.793. \end{aligned} \tag{7.9}$$

If we use Papatzacos's pseudoskin factor formula, then

$$\begin{aligned} h_1 &= 0, \\ \Psi_1 &= \frac{H}{h_1 + 0.25L_p} = \frac{20}{0 + 0.25 \times 10} = 8.0, \\ \Psi_2 &= \frac{H}{h_1 + 0.75L_p} = \frac{20}{0 + 0.75 \times 10} = 2.667, \end{aligned} \tag{7.10}$$

thus from (2.7), we have

$$\begin{aligned} S_{\text{ps}} &= \left(\frac{1}{\eta} - 1\right)\ln\left(\frac{\pi h_D}{2}\right) + \left(\frac{1}{\eta}\right)\ln\left[\left(\frac{\eta}{2+\eta}\right)\left(\frac{\Psi_1 - 1}{\Psi_2 - 1}\right)^{1/2}\right] \\ &= \left(\frac{1}{0.5} - 1\right)\ln\left(\frac{\pi \times 565.685}{2}\right) + \left(\frac{1}{0.5}\right)\ln\left[\left(\frac{0.5}{2+0.5}\right)\left(\frac{8.0 - 1}{2.667 - 1}\right)^{1/2}\right] \\ &= 5.006. \end{aligned} \tag{7.11}$$

If we use Bervaldier's pseudoskin factor formula, then

$$L_p = 10\,(\text{m}), \qquad R_w = 0.1\,(\text{m}), \tag{7.12}$$

thus from (2.9), we have

$$\begin{aligned} S_{\mathrm{ps}} &= \left(\frac{1}{\eta} - 1\right)\left[\frac{\ln(L_p/R_w)}{1 - R_w/L_p} - 1\right] \\ &= \left(\frac{1}{0.5} - 1\right)\left[\frac{\ln(10/0.1)}{1 - 0.1/10} - 1\right] \\ &= 3.652. \end{aligned} \tag{7.13}$$

But the pseudoskin factor in Example One calculated by (6.8) is

$$S_{\mathrm{ps}} = -12.0. \tag{7.14}$$

Formulas (2.4), (2.7), and (2.9) cannot account for the effect of well location inside a finite drainage volume on S_{ps}. But (6.8) is applicable to a well arbitrarily located in a box-shaped reservoir, S_{ps} is a function of well location parameters X_w and Y_w, and S_{ps} is also a function of reservoir size parameters X_e and Y_e. This is the reason why significant differences exist between S_{ps} calculated by (6.8) and S_{ps} calculated by (2.4), (2.7), and (2.9).

Example Three

A fully penetrating vertical well is located at the center of an isotropic rectangular reservoir with $X_e/Y_e = 4$, calculate the shape factor and compare with the corresponding shape factors given by Dietz and Earlougher et al.

Solution. Since the well is located at the center of the rectangular reservoir with $X_e/Y_e = 4$, use (6.16),

$$f_1 = \frac{X_e}{Y_e} = 4, \qquad f_2 = \frac{X_w}{X_e} = 0.5, \qquad f_3 = \frac{Y_w}{Y_e} = 0.5, \tag{7.15}$$

then

$$f_4 = (8\pi \times 4) \times \left(\frac{1}{6} - \frac{0.5}{2} + \frac{0.5^2}{2}\right) = 4.1888. \tag{7.16}$$

Use (6.15), the shape factor is

$$\begin{aligned} C_A &= \frac{88.6657 \times 4 \times \sin^2(\pi \times 0.5)}{\exp(4.1888)} \\ &= 5.3783. \end{aligned} \tag{7.17}$$

The corresponding shape factor given by Dietz [9] is $C_A = 5.38$, and $C_A = 5.3790$ given by Earlougher et al. [10]. Thus, there does not exist significant difference between the shape factor values calculated by our proposed formula and given by Dietz and Earlougher et al., which indicates that our proposed formula is reliable and reasonable accurate.

More examples are given in [1, 2] to calculate productivity index and pseudoskin factor due to partial penetration by using the proposed formulas, the values of shape factors obtained by the methods of Dietz, Earlougher, and the proposed shape factor formula are compared. The proposed formulas are shown to be reliable and reasonable accurate by the examples in [1, 2], because the proposed equations are derived by solving analytically the involved three-dimensional Laplace equation, they are a fast analytical tool to evaluate well performance in pseudo-steady-state.

8. Summary and Conclusions

The summary and conclusions of this paper are given below.

(1) A pseudo-steady-state productivity formula for an off-center partially penetrating vertical well in a closed box-shaped reservoir is presented.

(2) A formula for calculating pseudoskin factor due to partial penetration is presented; the pseudoskin factor of a vertical well in a box-shaped reservoir is a function of well location and reservoir size.

(3) The proposed formulas are reliable and reasonable accurate, because the proposed formulas are derived by the orthogonal decomposition of Dirac function and Green's function to Laplace equation with homogeneous Neumann boundary condition, they are a fast analytical tool to evaluate well performance in pseudo-steady-state.

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