

Research Article

Synchronization of Chaotic Fractional-Order WINDMI Systems via Linear State Error Feedback Control

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We propose a fractional-order WINDMI system, as a generalization of an integer-order system developed by Sprott (2003). The considered synchronization scheme consists of identical master and slave fractional-order WINDMI systems coupled by linear state error variables. Based on the stability theory of nonlinear fractional-order systems, linear state error feedback control technique is applied to achieve chaos synchronization, and a linear control law is derived analytically to achieve synchronization of the chaotic fractional-order WINDMI system. Numerical simulations validate the main results of this work.

1. Introduction

The solar-wind-driven magnetosphere-ionosphere is a complex driven-damped dynamical system which exhibits a variety of dynamical states that include low-level steady plasma convection, episodic releases of geotail stored plasma energy into the ionosphere known broadly as substorms, and states of continuous strong unloading [1]. In 1998, Horton and Doxas [2] firstly proposed the WINDMI system, a six-dimensional nonlinear dynamics model, which was derived for the basic energy components of the night-side magnetotail coupled to the ionosphere by the region-1 currents. Smith et al. [3] explored the dynamical range of the WINDMI model. Horton et al. [1] introduced reductions to derive a new minimal three-dimensional WINDMI model. Sprott [4] further simplified the integer-order WINDMI

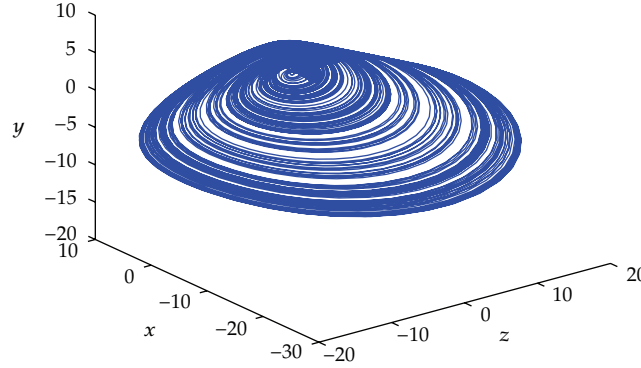


Figure 1: Phase portrait of system (1.1) versus $a = 0.7$ and $b = 2.5$.

model as follows:

$$\begin{aligned}\frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= z, \\ \frac{dz}{dt} &= -az - y + b - e^x,\end{aligned}\tag{1.1}$$

where x , y , and z are variables and a , b are positive constants. System (1.1) has a chaotic attractor when the usual parameters are $a = 0.7$ and $b = 2.5$, as shown in Figure 1 (see [4]).

Fractional calculus, a generalization of differentiation and integration to an arbitrary order, is an old mathematical topic with over 300-year-old history [5]. Fractional-order differential equations can be used to describe many systems in interdisciplinary fields, but it was not used in science and engineering for many years because there exist many difficulties, such as the absence of impactful solution methods and numerical simulation schemes for fractional differential equations. However, during the past several years, with the development of computer simulation technology [6, 7], fractional calculus has been realizing the utility and applicability to various branches of science and engineering, such as fractional viscoelastic fluids [8], fractional diffusion processes [9, 10], fractional-order viscoelastic material models [11], fractional-order HIV Model [12, 13], and fractional-order controllers [14–16]. Undoubtedly, fractional calculus will be applied into more and more areas of classical and modern analysis.

Chaos synchronization plays a very important role in the theory and applications. Synchronization of fractional-order chaotic systems was first presented by Deng and Li [17]. As an active research area, chaos synchronization with fractional calculus has received increasing attention in recent years due to its potentials in both theory and applications [18–25]. On the other hand, as compared with sliding mode control, standard PID feedback control, and so on, the advantage of linear state error feedback control is that it is linear and easier to implement for chaos synchronization.

The remainder of this paper is organized as follows. In Section 2, the WINDMI system is generalized from integer to noninteger order. Then, chaos synchronization via linear

feedback control is studied in Section 3. Numerical simulations are presented in Section 4, and finally conclusions in Section 5 close the paper.

2. System Description

There are several definitions of fractional derivatives. In this paper, we will adopt the Riemann-Liouville fractional derivative definition as follows.

Definition 2.1 (see [26]). The q th-order fractional derivative of function $f(t)$ with respect to t and the terminal value 0 is given by

$$\frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{q-m+1}} d\tau, \quad (2.1)$$

where $\Gamma(\cdot)$ is a Gamma function, m is an integer and satisfies $m-1 < q < m \in \mathbb{Z}^+$.

The integer derivative of a function has relationship with only its nearby points while the fractional derivative takes into account nonlocal characteristics like “infinite memory” [26, 27]. As a result, a model described with fractional derivative possesses memory, which may help to have a better understanding to the importance of assembling large substorm databases with a large range of event sizes. Here, we introduce fractional calculus into system (1.1). The new system is described with fractional derivative as follows:

$$\begin{aligned} \frac{d^{q_1} x}{dt^{q_1}} &= y, \\ \frac{d^{q_2} y}{dt^{q_2}} &= z, \\ \frac{d^{q_3} z}{dt^{q_3}} &= -az - y + b - e^x, \end{aligned} \quad (2.2)$$

in which x , y , and z are variables and a, b are positive constants; $q = (q_1, q_2, q_3)$ is subject to $0 < q_1, q_2, q_3 < 1$. If $q = (1, 1, 1)$, system (2.2) degenerates into system (1.1).

In order to observe the synchronization behavior in two identical fractional-order WINDMI systems, we set a drive-response configuration with a drive system given by the fractional-order WINDMI systems (with three state variables denoted by the subscript m) and with a response system (with three variables denoted by the subscript s) as follows.

The drive system is described by

$$\begin{aligned} \frac{d^{q_1} x_m}{dt^{q_1}} &= y_m, \\ \frac{d^{q_2} y_m}{dt^{q_2}} &= z_m, \\ \frac{d^{q_3} z_m}{dt^{q_3}} &= -az_m - y_m + b - e^{x_m}, \end{aligned} \quad (2.3)$$

and the response system is given by

$$\begin{aligned}\frac{d^{q_1} x_s}{dt^{q_1}} &= y_s + u_1, \\ \frac{d^{q_2} y_s}{dt^{q_2}} &= z_s + u_2, \\ \frac{d^{q_3} z_s}{dt^{q_3}} &= -az_s - y_s + b - e^{x_s},\end{aligned}\tag{2.4}$$

where u_1 and u_2 are the linear state error feedback controllers. Then, the rest of our task is to design suitable linear controllers that can synchronize systems (2.3) and (2.4).

3. Synchronization Scheme

Matignon [28] defined the internal and external stability properties of linear fractional-order differential systems of finite dimension and derived the necessary and sufficient conditions.

Theorem 3.1 (see [28]). *Consider that the following n -dimensional linear fractional-order autonomous system:*

$$\frac{d^q x}{dt^q} = Ax, \quad x(0) = x_0,\tag{3.1}$$

with $0 < q < 1$, $x \in R^n$, and $A \in R^{n \times n}$, is asymptotically stable if and only if $|\arg(\lambda)| > q\pi/2$ is satisfied for all eigenvalues λ of matrix A . Also system (3.1) is stable if and only if $|\arg(\lambda)| \geq q\pi/2$ is satisfied for all eigenvalues λ of matrix A .

Theorem 3.2 (see [29]). *Consider the following autonomous n -dimensional nonlinear fractional differential equation:*

$$\frac{d^q X(t)}{dt^q} = F(X(t)), \quad X(0) = X_0 = (x_{10}, x_{20}, \dots, x_{n0})^T, \quad m-1 < q < m \in Z^+.\tag{3.2}$$

Letting $\hat{X} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)^T$ be an equilibrium of system (3.2), that is, $d^q \hat{X}/dt^q = F(\hat{X}) = 0$, and letting $A = (\partial F/\partial X)|_{X=\hat{X}}$ be the Jacobian matrix at the point \hat{X} , then the point \hat{X} is asymptotically stable when $|\arg(\text{eig}(A))| > q_m\pi/2$, where $q_m = \max_{1 \leq i \leq n} \{q_i\}$.

From Theorem 3.2, the following corollary holds.

Corollary 3.3. *Letting $\hat{X} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)^T$ be an equilibrium of system (3.2), that is, $d^q \hat{X}/dt^q = F(\hat{X}) = 0$, and letting $A = (\partial F/\partial X)|_{X=\hat{X}}$ be the Jacobian matrix at the point \hat{X} , then the point \hat{X} is locally asymptotically stable if A is an upper or lower triangular matrix and all eigenvalues of A are negative real numbers.*

Theorem 3.4. *The drive system (2.3) and the response system (2.4) will approach global synchronization for any initial condition with the following control law:*

$$u_1 = c_1(x_s - x_m) - (y_s - y_m), \quad u_2 = c_2(y_s - y_m) - (z_s - z_m), \quad (3.3)$$

where $c_1 < 0$ and $c_2 < 0$.

Proof. Set the synchronization error variables as follows:

$$\begin{aligned} e_1 &= x_s - x_m, \\ e_2 &= y_s - y_m, \\ e_3 &= z_s - z_m. \end{aligned} \quad (3.4)$$

By subtracting (2.3) from (2.4) and using (3.4), the synchronization error system can be obtained as follows:

$$\begin{aligned} \frac{d^{q_1} e_1}{dt^{q_1}} &= e_2 + u_1, \\ \frac{d^{q_2} e_2}{dt^{q_2}} &= e_3 + u_2, \\ \frac{d^{q_3} e_3}{dt^{q_3}} &= -ae_3 - e_2 + e^{x_m}(1 - e^{e_1}), \end{aligned} \quad (3.5)$$

where $a > 0$, x_m is the state variable of system (2.3), u_1 and u_2 are linear state error feedback controllers.

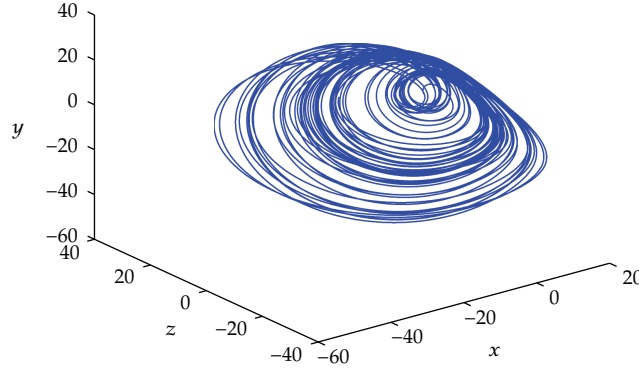
When $u_1 = c_1(x_s - x_m) - (y_s - y_m)$ and $u_2 = c_2(y_s - y_m) - (z_s - z_m)$, system (3.5) can be rewritten as

$$\begin{aligned} \frac{d^{q_1} e_1}{dt^{q_1}} &= c_1 e_1, \\ \frac{d^{q_2} e_2}{dt^{q_2}} &= c_2 e_2, \\ \frac{d^{q_3} e_3}{dt^{q_3}} &= -ae_3 - e_2 + e^{x_m}(1 - e^{e_1}). \end{aligned} \quad (3.6)$$

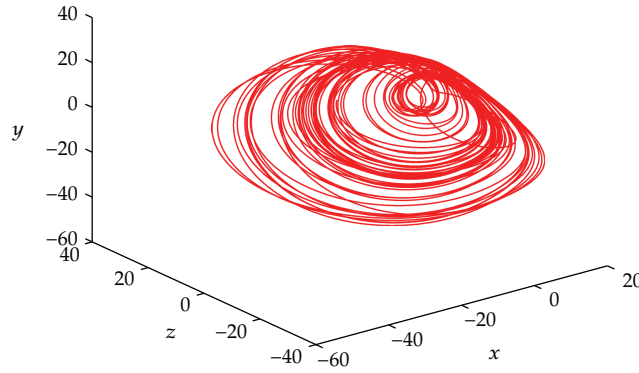
System (3.6) has only one equilibrium point at $E_0 = (0, 0, 0)$. Its Jacobian matrix evaluated at equilibrium point E_0 is given by

$$J(E_0) = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ -e^{x_m+e_1} & -1 & -a \end{pmatrix} = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ -e^{x_m} & -1 & -a \end{pmatrix}. \quad (3.7)$$

Obviously, (3.7) is a lower triangular matrix, and x_m is a state variable of drive system (2.3). Thus, from Corollary 3.3, one can get that x_m has no effect on the stability of system



(a) Drive system (2.3)



(b) Response system (2.4)

Figure 2: Chaotic attractors of drive system (2.3) and response system (2.4) versus $q_1 = 0.77$, $q_2 = 0.98$, $q_3 = 0.8$, $a = 0.8$, $b = 6.4$, $c_1 = -1$, and $c_2 = -1$.

(3.6), and if $c_1 < 0$ and $c_2 < 0$, then system (3.6) is asymptotically stable; that is, the drive system (2.3) and the response system (2.4) are synchronized finally.

The theorem is proved. \square

Evidently, the advantage of the linear feedback controllers proposed in Theorem 3.4 is that they are robust, linear, and have lower dimensions than that of the states; moreover, they are easier to be designed and implemented for chaos synchronization than standard PID feedback controllers, sliding mode controllers, nonlinear feedback controllers, and so on.

4. Numerical Simulations

4.1. Discretization Scheme

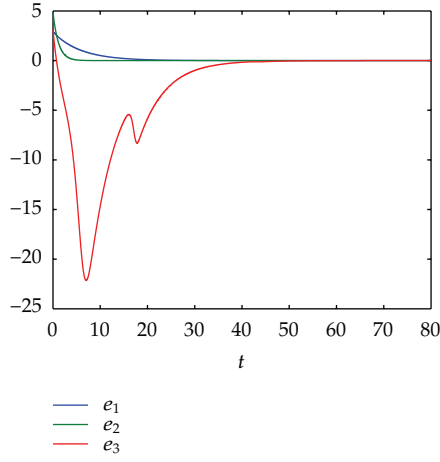
Based on the Adams-Bashforth-Moulton predictor-corrector scheme [6, 30], we can build the numerical calculation formula for the proposed synchronization scheme as follows.

Letting (x, y, z) and $(\hat{x}, \hat{y}, \hat{z})$ represent (x_m, y_m, z_m) and (x_s, y_s, z_s) , respectively. Setting $h = T/N$, $t_n = nh$, $n = 0, 1, \dots, N \in \mathbb{Z}^+$, the drive system (2.3) and the response system (2.4) can be discretized as

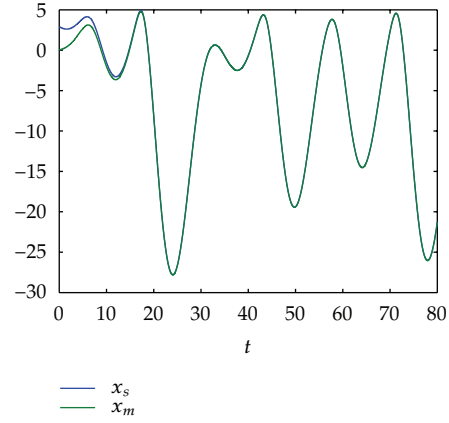
$$\begin{aligned}
x_{n+1} &= x_0 + \frac{h^{q_1}}{\Gamma(q_1 + 2)} y_{n+1}^p + \frac{h^{q_1}}{\Gamma(q_1 + 2)} \sum_{j=0}^n \alpha_{1,j,n+1} y_j, \\
y_{n+1} &= y_0 + \frac{h^{q_2}}{\Gamma(q_2 + 2)} z_{n+1}^p + \frac{h^{q_2}}{\Gamma(q_2 + 2)} \sum_{j=0}^n \alpha_{2,j,n+1} z_j, \\
z_{n+1} &= z_0 + \frac{h^{q_3}}{\Gamma(q_3 + 2)} \left(-e^{x_{n+1}^p} - y_{n+1}^p - a z_{n+1}^p + b \right) + \frac{h^{q_3}}{\Gamma(q_3 + 2)} \sum_{j=0}^n \alpha_{3,j,n+1} \left(-e^{x_j} - y_j - a z_j + b \right), \\
\hat{x}_{n+1} &= \hat{x}_0 + \frac{h^{q_1}}{\Gamma(q_1 + 2)} \left(y_{n+1}^p + c_1 \left(\hat{x}_{n+1}^p - x_{n+1}^p \right) \right) + \frac{h^{q_1}}{\Gamma(q_1 + 2)} \sum_{j=0}^n \alpha_{1,j,n+1} \left(y_j + c_1 \left(\hat{x}_j - x_j \right) \right), \\
\hat{y}_{n+1} &= \hat{y}_0 + \frac{h^{q_2}}{\Gamma(q_2 + 2)} \left(z_{n+1}^p + c_2 \left(\hat{y}_{n+1}^p - y_{n+1}^p \right) \right) + \frac{h^{q_2}}{\Gamma(q_2 + 2)} \sum_{j=0}^n \alpha_{2,j,n+1} \left(z_j + c_2 \left(\hat{y}_j - y_j \right) \right), \\
\hat{z}_{n+1} &= \hat{z}_0 + \frac{h^{q_3}}{\Gamma(q_3 + 2)} \left(-e^{\hat{x}_{n+1}^p} - \hat{y}_{n+1}^p - a \hat{z}_{n+1}^p + b \right) + \frac{h^{q_3}}{\Gamma(q_3 + 2)} \sum_{j=0}^n \alpha_{3,j,n+1} \left(-e^{\hat{x}_j} - \hat{y}_j - a \hat{z}_j + b \right),
\end{aligned} \tag{4.1}$$

where

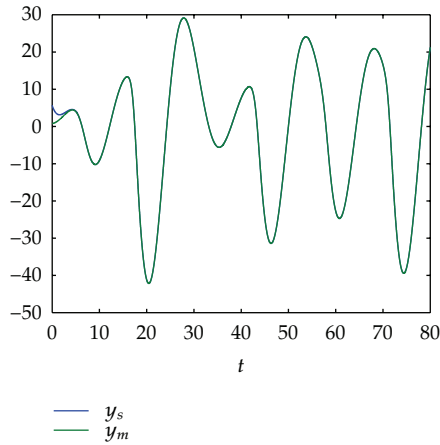
$$\begin{aligned}
x_{n+1}^p &= x_0 + \frac{1}{\Gamma(q_1)} \sum_{j=0}^n \beta_{1,j,n+1} y_j, \\
y_{n+1}^p &= y_0 + \frac{1}{\Gamma(q_2)} \sum_{j=0}^n \beta_{2,j,n+1} z_j, \\
z_{n+1}^p &= z_0 + \frac{1}{\Gamma(q_3)} \sum_{j=0}^n \beta_{3,j,n+1} \left(-e^{x_j} - y_j - a z_j + b \right), \\
\hat{x}_{n+1}^p &= \hat{x}_0 + \frac{1}{\Gamma(q_1)} \sum_{j=0}^n \beta_{1,j,n+1} \left(y_j + c_1 \left(\hat{x}_j - x_j \right) \right), \\
\hat{y}_{n+1}^p &= \hat{y}_0 + \frac{1}{\Gamma(q_2)} \sum_{j=0}^n \beta_{2,j,n+1} \left(z_j + c_2 \left(\hat{y}_j - y_j \right) \right), \\
\hat{z}_{n+1}^p &= \hat{z}_0 + \frac{1}{\Gamma(q_3)} \sum_{j=0}^n \beta_{3,j,n+1} \left(-e^{\hat{x}_j} - \hat{y}_j - a \hat{z}_j + b \right),
\end{aligned}$$



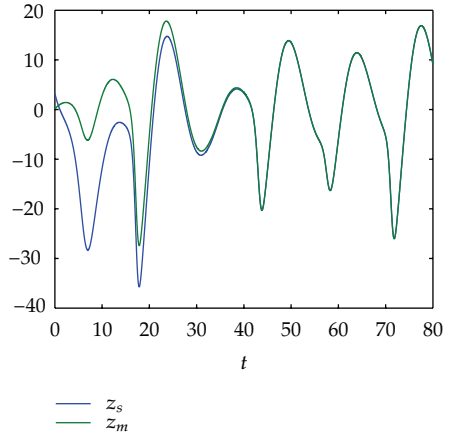
(a) Synchronization errors between drive system (2.3) and response system (2.4)



(b) Time evolutions of x_s and x_m



(c) Time evolutions of y_s and y_m



(d) Time evolutions of z_s and z_m

Figure 3: Synchronization errors between drive system (2.3) and response system (2.4) versus $q_1 = 0.77$, $q_2 = 0.98$, $q_3 = 0.8$, $a = 0.8$, $b = 6.4$, $c_1 = -1$, and $c_2 = -1$.

$$\alpha_{i,j,n+1} = \begin{cases} n^{q_i+1} - (n - q_i)(n + 1)^{q_i}, & j = 0, \\ (n - j + 2)^{q_i+1} + (n - j)^{q_i+1} - 2(n - j + 1)^{q_i+1}, & 1 \leq j \leq n, i = 1, 2, 3, \\ 1, & j = n + 1, \end{cases}$$

$$\beta_{i,j,n+1} = \frac{h^{q_i}}{q_i} ((n - j + 1)^{q_i} - (n - j)^{q_i}), \quad 0 \leq j \leq n, i = 1, 2, 3.$$

(4.2)

4.2. Numerical Results

Based on the above mentioned discretization scheme, the drive system (2.3) and the response system (2.4) are integrated numerically with the fractional orders $q_1 = 0.77$, $q_2 = 0.98$, $q_3 = 0.8$ and using the initial values $x_m(0) = 0$, $y_m(0) = 0.8$, $z_m(0) = 0$, $x_s(0) = 4$, $y_s(0) = 4.8$, and $z_s(0) = 4$. Let $a = 0.8$, $b = 6.4$, $c_1 = -1$, $c_2 = -1$; the chaotic attractor of the drive system (2.3) is shown in Figure 2(a), and the chaotic attractor of the response system (2.4) is shown in Figure 2(b). From Figures 3(a)–3(d), it is clear that the synchronization is achieved for all these values.

5. Conclusion

In this paper, we introduce fractional-order calculus into the WINDMI system. Chaos synchronization of identical master and slave fractional-order WINDMI systems is studied by utilizing linear state error feedback control technique. Based on the stability theory of nonlinear fractional-order systems, linear feedback control law for chaos synchronization has been investigated. Numerical simulations are given to verify the effectiveness of the proposed synchronization scheme.

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