Research Article

Application of Homotopy Analysis Method to the Unsteady Squeezing Flow of a Second-Grade Fluid between Circular Plates

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We investigated an axisymmetric unsteady two-dimensional flow of nonconducting, incompressible second grade fluid between two circular plates. The similarity transformation is applied to reduce governing partial differential equation (PDE) to a nonlinear ordinary differential equation (ODE) in dimensionless form. The resulting nonlinear boundary value problem is solved using homotopy analysis method and numerical method. The effects of appropriate dimensionless parameters on the velocity profiles are studied. The total resistance to the upper plate has been calculated.

1. Introduction

Squeezing flows are induced by externally normal stresses or vertical velocities by means of moving boundary. Squeezing flows have many applications in food industry, especially in chemical engineering. Some practical examples of squeezing flow include polymer processing, compression and injection molding. In addition, the lubrication system can also be modeled by squeezing flows. The study of squeezing flows has its origins in the 19th century and continues to receive considerable attention due to the practical applications in physical and biophysical areas. Stefan [1] published a classical paper on squeezing flow by using lubrication approximation. Such types of flow exist in lubrication when there is squeezing flow between two parallel plates. The tackiness of liquid adhesives also reflects squeeze film effects [2]. The squeeze film geometry has been studied extensively

since 1947. Other applications in the biomechanics area relate to squeezing flow between parallel plates and the alternation between contraction and expansion of the blood vessels. In addition, polymer extrusion processes are modeled using squeezing flow of viscous fluids [3]. The squeezing flow between parallel plates when the confining walls have a transverse motion has great importance in hydrodynamic lubrication theory Langlois [4] and Salbu [5] have analyzed isothermal compressible squeeze films neglecting inertial effects. Thorpe [6] presented an explicit solution of the squeeze flow problem taking inertial terms into account. Later, P. S. Gupta and A. S. Gupta [7] showed that the solution given by [6] fails to satisfy the boundary conditions. Gupta solution, however, is restricted only to small Reynolds number. Squeeze film between two plane annuli with fluid inertia effects has been studied by Elkouh [8]. Some numerical solutions of squeezing flow between parallel plates has been conducted by Verma [9] and later by Singh et al. [10]. In addition, Hamza [11] has considered suction and injection effects on flow between parallel plates reflecting squeezing flow. Further work on second grade fluid flow between parallel plates has been carried out by Rajagopal and Gupta [12] and by Dandapat and Gupta [13]. Both papers consider the problem of flow between rotating parallel plates. Barron and Wiley [14] have also considered this fluid model to extend the Newtonian flow theory for slender bodies in a dusty gas. This fluid model has also been considered even in flow of a dusty liquid. For example, Debnath and Ghosh [15, 16] used this fluid model to study the flow of dusty fluids in rotating and oscillating cases, in the presence of magnetic fields. Hamdan and Barron [17] used this fluid model to analyse the steady squeezing flow of incompressible dusty fluids between two cylindrical plates. The theoretical and experimental studies of squeezing flows have been conducted by many researchers [18–28]. In this work the unsteady 2-dimensional flow of nonconducting, incompressible second grade fluid between circular plates that are moving symmetrically about the line of axial symmetry and giving rise to the squeezing flow is studied. The unsteady equations of motion are reduced to a single nonlinear fifth-order ordinary differential after employing a similarity transformation. The resulting nonlinear boundary value problem is solved with homotopy analysis method (HAM) and numerical method.

Most scientific problems and phenomena are modeled by nonlinear ordinary or partial differential equations. In most cases, these problems do not have precise analytic solution. Recently, much attention has been devoted to the newly developed methods to construct approximate analytic solutions of nonlinear equations. The HAM is developed in 1992 by Liao [28–35] and Rashidi et al. [36–38]. Liao applied this method to solve many types of nonlinear equations in science and engineering and then, this method has been successfully applied to solve many other nonlinear evolution equations. In recent years, the application of HAM in nonlinear problems has been developed by scientists and engineers. We know all perturbation methods require small parameter in nonlinear equation and the approximate solutions of equation containing this parameter are expressed as series expansions in the small parameter. Selection of small parameter requires a special skill. A proper choice of small parameter gives acceptable results, while an improper choice may result in incorrect solutions. The HAM, does not require a small parameter in equation modeling phenomena. Briefly speaking, Liao's method is a universal one which can solve various kinds of nonlinear equations, thus many researchers applied this method to various linear and nonlinear problems. A substantial amount of research work has been invested in the study of linear and nonlinear systems of PDEs.
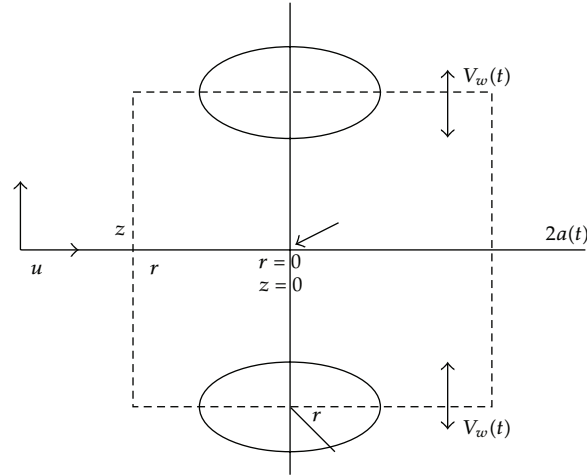
Figure 1: Geometry of the problem.

In Sections 2 and 3 description and mathematical formulation of the problem is presented. Sections 4 and 5 discussed the methods of solution of the problem. The results are discussed in Section 6 and Section 7 concludes finding of the present work.

2. Problem Description

The unsteady two-dimensional squeezing flow of nonconducting, incompressible second grade fluid between two circular plates is studied. The distance between the plates at any time t is $2a(t)$. The central axis of the channel is taken as the r-axis and z-axis is normal to it. It is assumed that the circular plates move symmetrically with respect to the central axis $z = 0$ and flow is axisymmetric about $r = 0$. The velocity components along the radial and axial directions are $u(r, z, t)$ and $w(r, z, t)$, respectively (Figure 1). The fluid is assumed to have density ρ and kinematics viscosity υ.

3. Mathematical Formulation

The basic equations governing the motion of a homogeneous incompressible second grade fluid neglecting the thermal effects are [37]

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(w) = 0, \tag{3.1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial r} + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right) + F_r,$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) + F_z, \tag{3.2}$$

where

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \tag{3.3}$$

where V is the velocity vector, P is the pressure, T is the stress tensor, μ is the coefficient of viscosity and α_1 and α_2 are material constants. The Rivlin-Ericksen tensors A_i are defined as

$$A_1 = \nabla V + (\nabla V)^T,$$

$$A_2 = \frac{dA_1}{dt} + A_1(\nabla V) + (\nabla V)^T A_1. \tag{3.4}$$

Now we formulate the equations of motion for unsteady two dimensional flow. Assuming that

$$V = [u(r,z,t), 0, w(r,z,t)], \tag{3.5}$$

and introducing the vorticity function $\Omega(r,z,t)$ and generalized pressure gradient $h(r,z,t)$ as

$$\Omega = \frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}, \tag{3.6}$$

$$h = \frac{\rho}{2}(u^2 + w^2) + p - \alpha_1\left(w\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) - u\frac{\partial}{\partial z}\right)\Omega - \frac{1}{4}(3\alpha_1 + 2\alpha_2)(A_1^2), \tag{3.7}$$

where

$$A_1^2 = \left[4\left(\frac{\partial u}{\partial r}\right)^2 + 4\left(\frac{\partial w}{\partial z}\right)^2 + 4\left(\frac{u}{r}\right)^2 + 2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2\right]. \tag{3.8}$$

We find that the equations of motion (3.1) and (3.2) reduce to

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{3.9}$$

$$\frac{\partial h}{\partial r} + \rho\left(\frac{\partial u}{\partial t} - w\Omega\right) = -\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\frac{\partial \Omega}{\partial t} - \alpha_1 w\left(\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\Omega - \frac{\Omega}{r^2}\right)$$

$$+(\alpha_1 + \alpha_2)\left(\frac{2}{r}\frac{\partial}{\partial z}(u\Omega) + \frac{\Omega^2}{r}\right), \tag{3.10}$$

$$\frac{\partial h}{\partial z} + \rho\left(\frac{\partial w}{\partial t} + u\Omega\right) = \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\left(\frac{\partial \Omega}{\partial r} + \frac{\Omega}{r}\right) + \alpha_1 u\left(\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\Omega - \frac{\Omega}{r^2}\right)$$

$$-(\alpha_1 + \alpha_2)\left(\frac{2}{r}\frac{\partial}{\partial z}(u\Omega)\right), \tag{3.11}$$

where $\nabla^2 = \partial^2/\partial r^2 + (1/r)(\partial/\partial r) + \partial^2/\partial z^2$ is Laplacian operator.

Eliminating the generalized pressure h between (3.10) and (3.11) to obtain

$$
\begin{aligned}
\rho\left(\frac{\partial \Omega}{\partial t} + \left(u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z} - \frac{u}{r}\right)\Omega\right) \\
= \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\left(\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\Omega - \frac{\Omega}{r^2}\right) \\
+ \alpha_1\left(u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z} - \frac{u}{r}\right)\left(\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\Omega - \frac{\Omega}{r^2}\right) \\
- 2\frac{(\alpha_1 + \alpha_2)}{r}\left(\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)u\Omega + \frac{1}{2}\frac{\partial}{\partial z}\left(\Omega^2\right)\right).
\end{aligned}
\tag{3.12}
$$

The boundary conditions on $u(r, z, t)$ and $w(r, z, t)$ are

$$
\begin{aligned}
u(r, z, t) = 0, \quad w(r, z, t) = v_w(t), \quad z = a, \\
w(r, z, t) = 0, \quad \frac{\partial u(r, z, t)}{\partial z} = 0, \quad z = 0,
\end{aligned}
\tag{3.13}
$$

where $v_w(t) = da/dt$ denotes the velocity of the plates. The first two conditions are due to the no-slip condition at the upper plate and the remaining two follow from the symmetry of the flow at $z = 0$.

If the dimensionless variable $\eta = z/a(t)$ is introduced, (3.6), (3.9), and (3.12) become

$$
\frac{\partial w}{\partial r} - \frac{\partial u}{a\partial \eta} = \Omega,
\tag{3.14}
$$

$$
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{a\partial \eta} = 0,
\tag{3.15}
$$

$$
\begin{aligned}
\rho\left(\frac{\partial \Omega}{\partial t} + \left(u\frac{\partial}{\partial r} + w\frac{\partial}{a\partial \eta} - \frac{u}{r}\right)\Omega\right) \\
= \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\left(\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\Omega - \frac{\Omega}{r^2}\right) \\
+ \alpha_1\left(u\frac{\partial}{\partial r} + w\frac{\partial}{a\partial \eta} - \frac{u}{r}\right)\left(\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\Omega - \frac{\Omega}{r^2}\right) \\
- 2\frac{(\alpha_1 + \alpha_2)}{r}\left(\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \eta^2}\right)u\Omega + \frac{1}{2a}\frac{\partial}{\partial \eta}\left(\Omega^2\right)\right).
\end{aligned}
\tag{3.16}
$$

The boundary conditions on $u(r, z, t)$ and $w(r, z, t)$ are

$$u(r, \eta, t) = 0, \quad w(r, \eta, t) = v_w(t), \quad \eta = 1, \tag{3.17}$$

$$w(r, \eta, t) = 0, \quad \frac{\partial u(r, \eta, t)}{\partial \eta} = 0, \quad \eta = 0. \tag{3.18}$$

If we define the velocity components as follows Singh et al. [10] and Birkhoff [39]

$$u = \frac{-r}{2a(t)} v_w(t) f'(\eta), \quad w = v_w(t) f(\eta). \tag{3.19}$$

It follows that (3.14) takes the form

$$\Omega = \frac{r}{2a(t)^2} v_w(t) f''(\eta), \tag{3.20}$$

substituting (3.18) and (3.19) in (3.15) and (3.16), we find that the continuity equation is identically satisfied and (3.16) becomes

$$\frac{av_w}{v} \left(f f''' - 2f'' - \eta f''' \right) + \frac{a(dv_w/dt)}{vv_w} f''$$
$$= f^{iv} + \frac{(\alpha_1 + \alpha_2)v_w}{\mu a} \left(2f'' f''' + f' f^{iv} \right) + \frac{\alpha_1 v_w}{\mu a} \left(f f^v - \eta f^v - 4f^{iv} + \frac{a(dv_w/dt)}{vv_w^2} f^{iv} \right), \tag{3.21}$$

where prime denotes differentiation with respect to η. The boundary conditions are determined from (3.17), (3.18), and (3.19) to be

$$\begin{aligned} f(1) = 1, \quad & f'(1) = 0, \\ f(0) = 0, \quad & f''(0) = 0. \end{aligned} \tag{3.22}$$

Thus for a similarity solution we define

$$\begin{aligned} \frac{av_w}{v} = R_e, \quad & \frac{a^2(dv_w/dt)}{vv_w} = R_e Q, \\ \frac{\alpha_1 v_w}{\mu a} = W_{e1}, \quad & \frac{\alpha_2 v_w}{\mu a} = W_{e2}, \end{aligned} \tag{3.23}$$

where R_e, Q and W_{e1} and W_{e2} are functions of t but for a similarity solution R_e, Q, W_{e1}, and W_{e2} are considered to be constants. After integrating the first equation of (3.23), we have

$$a(t) = (Kt + a_0)^{1/2}, \tag{3.24}$$

where K and a_0 are constants. When $K > 0$ and $a_0 > 0$, the plates move apart symmetrically with respect to $\eta = 0$. In addition, when $K < 0$ and $a_0 > 0$, the plates approach each other and squeezing flow exists with similar velocity profile as long as $(Kt + a_0) > 0$. From (3.23) and (3.24), it follows that $Q = -1$, and (3.21) becomes

$$f^{iv} + K^2 f'' = L((f - \eta) f''') + M((\eta - f) f^v) - N\left(2f'' f''' + f' f^{iv}\right), \tag{3.25}$$

where

$$K = \sqrt{\frac{3R_e}{1 - 5W_{e1}}},$$

$$L = \frac{R_e}{1 - 5W_{e1}},$$

$$M = \frac{W_{e1}}{1 - 5W_{e1}}, \tag{3.26}$$

$$N = \frac{W_{e1} + W_{e2}}{1 - 5W_{e1}}.$$

This nonlinear boundary value problem is solved by the homotopy analysis method as well as numerically in the following sections.

4. Basic Idea of the HAM

Let us consider the following differential equation

$$\mathcal{N}[f(\eta)] = 0, \tag{4.1}$$

where $\mathcal{N}$ is a nonlinear operator, η denotes independent variable, $f(\eta)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional HAM, Liao [30] constructs the so-called zero-order deformation equation

$$(1 - p)\mathcal{L}[\varphi(\eta; p) - f_0(\eta)] = p\hbar H(\eta)\mathcal{N}[\varphi(\eta; p)], \tag{4.2}$$

where $p \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, $H(\eta) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $f_0(\eta)$ is an initial guess of $f(\eta)$, $\varphi(\eta; p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in the HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\varphi(\eta; 0) = f_0(\eta), \qquad \varphi(\eta; 1) = u(\eta), \tag{4.3}$$

respectively. Thus as p increases from 0 to 1, the solution $\varphi(\eta; p)$ varies from the initial guess $f_0(\eta)$ to the solution $f(\eta)$. Expanding $\varphi(\eta; p)$ in Taylor series with respect to p, we have

$$\varphi(\eta; p) = f_0(\eta) + \sum_{m=1}^{+\infty} f_m(\eta) p^m, \tag{4.4}$$

where

$$f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \varphi(\eta; p)}{\partial p^m} \right|_{p=0}. \tag{4.5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series (4.4) converges at $p = 1$, then we have

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{+\infty} f(\eta), \tag{4.6}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [30]. As $\hbar = -1$ and $H(\eta) = 1$, (4.2) becomes

$$(1 - p)\mathcal{L}[\varphi(\eta; p) - f_0(\eta)] + p\mathcal{N}[\varphi(\eta; p)] = 0, \tag{4.7}$$

which is used mostly in the homotopy analysis method, where as the solution obtained directly, without using Taylor series [38].

According to the definition (4.5), the governing equation can be deduced from the zero-order deformation equation (4.2). Define the vector

$$\vec{f}_n(\eta) = \{f_0(\eta), f_1(\eta), \ldots, f_n(\eta)\}. \tag{4.8}$$

Differentiating (4.2) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar H(\eta) R_m\left(\vec{f}_{m-1}\right), \tag{4.9}$$

where

$$R_m\left(\vec{f}_{m-1}\right) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}[\varphi(\eta; p)]}{\partial p^{m-1}} \right|_{p=0},$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{4.10}$$

It should be emphasized that $f_m(\eta)$ for $m \geq 1$ is governed by the linear equation (4.9) with the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

5. Application

Consider (3.22) and (3.25) and let us solve them through the HAM. Following the HAM, we start with initial approximation $f_0(\eta) = -0.5\eta^3 + 1.5\eta$ and the linear operator

$$\mathcal{L}[f(\eta)] = \frac{\partial^4 \varphi(\eta; p)}{\partial \eta^4}, \tag{5.1}$$

and nonlinear operator is defined as

$$\mathcal{N}[f(\eta)] = -\frac{\partial^4 f}{\partial \eta^4} - K^2 \frac{\partial^2 f}{\partial \eta^2} + L\left((f-\eta)\frac{\partial^3 f}{\partial \eta^3}\right) + M\left((\eta-f)\frac{\partial^5 f}{\partial \eta^5}\right)$$
$$- N\left(2\frac{\partial^2 f}{\partial \eta^2}\frac{\partial^3 f}{\partial \eta^3} + \frac{\partial f}{\partial \eta}\frac{\partial^4 f}{\partial \eta^4}\right). \tag{5.2}$$

Using the above definition, with assumption $H(\eta) = 1$, we construct the zero-order deformation equation

$$(1-p)\mathcal{L}[\varphi(\eta; p) - f_0(\eta)] = p\hbar\mathcal{N}[\varphi(\eta; p)]. \tag{5.3}$$

Obviously, when $p = 0$ and $p = 1$,

$$\varphi(\eta; 0) = f_0(\eta), \qquad \varphi(\eta; 1) = f(\eta). \tag{5.4}$$

Differentiating the zero-order deformation equation (5.3) m times with respect to p, and finally dividing by $m!$, we have the mth-order deformation equation

$$\mathcal{L}[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar R_m(\vec{f}_{m-1}), \tag{5.5}$$

where

$$R(f_{m-1}) = -\frac{\partial^4 f_{m-1}(\eta)}{\partial \eta^4} - K^2 \frac{\partial^2 f_{m-1}(\eta)}{\partial \eta^2} - L\eta\frac{\partial^3 f_{m-1}(\eta)}{\partial \eta^3} + M\eta\frac{\partial^5 f_{m-1}(\eta)}{\partial \eta^5}$$
$$+ L\sum_{n=0}^{m-1} f_n(\eta)\frac{\partial^3 f_{m-1-n}(\eta)}{\partial \eta^3} - M\sum_{n=0}^{m-1} f_n(\eta)\frac{\partial^5 f_{m-1-n}(\eta)}{\partial \eta^5}$$
$$- 2N\sum_{n=0}^{m-1}\frac{\partial^2 f_n(\eta)}{\partial \eta^2}\frac{\partial^3 f_{m-1-n}(\eta)}{\partial \eta^3} - N\sum_{n=0}^{m-1}\frac{\partial f_n(\eta)}{\partial \eta}\frac{\partial^4 f_{m-1-n}(\eta)}{\partial \eta^4}. \tag{5.6}$$

Then

$$\mathcal{L}\left[f_m(\eta) - \chi_m f_{m-1}(\eta)\right] = \hbar R_m\left(\vec{f}_{m-1}\right),$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{5.7}$$

We choose auxiliary functions as follow:

$$\mathcal{H}(\eta) = 1. \tag{5.8}$$

Following the HAM and trying higher iterations with the unique and proper assignment of the results converge to the exact solution:

$$f(\eta) \approx f_0(\eta) + f_1(\eta) + f_2(\eta) + \cdots + f_m(\eta), \tag{5.9}$$

using the symbolic software Mathematica to solve the system of linear equations, (5.5), with the boundary conditions (3.22), and successively obtain

$$f_1(\eta) = -0.058567\hbar\eta + 0.115348\hbar\eta^3 - 0.054996\hbar\eta^5 - 0.001786\hbar\eta^7 \tag{5.10}$$

6. Results and Discussion

In this work the unsteady 2-dimensional flow of nonconducting, incompressible second grade fluid between two circular plates is considered. The squeezing flow is generated by moving the plates symmetrically with respect to the central region $z = 0$. The unsteady equations of motion are reduced to a single nonlinear 5th-order ordinary differential by using a similarity transformation. The resulting nonlinear boundary value problem is solved using the homotopy analysis method and numerical technique. The effect of the Reynolds number R_e on velocity profiles is depicted. In these profiles we fixed the nonNewtonian parameters $W_{e1} = 0.0$, $W_{e2} = 0.05$ and varied R_e as $R_e = 1.0, 1.5, 2.0$ (Figures 11 and 12). It is noted that the normal velocity increases as the Reynolds number increases. It is also observed that at a given time and for a fixed positive value of Reynolds number the normal velocity increases monotonically from $\eta = 0$ to $\eta = 1$ presents effect of Reynolds number on the longitudinal velocity. It is observed that this component of velocity increases near the central axis of the channel but deceases near the walls. The influence of nonNewtonian parameter W_{e1} on the normal and longitudinal velocity components is depicted. we varied $W_{e1} = 0.0, 0.05, 0.75$ fixing $W_{e2} = 0.05$, $R_e = 1.0$. This observation on the velocity components is similar to that of Reynolds number R_e presents the effects of nonNewtonian parameter W_{e2} on the squeezing flow of the second grade fluid. It is noted that effect of W_{e2} on the velocity components is opposite to that of the nonNewtonian parameter W_{e1}.

The HAM was applied successfully to find an explicit, totally analytic, uniformly valid solution for unsteady squeezing flow between circular parallel plates. The validity of our analytic solutions is verified by numerical results. The results show that HAM is very accurate (Figures 3, 4, 6, 7, 9, and 10). Also, the HAM gives rapidly convergent series with specific significant features for each scheme (Figures 2, 5, and 8). The accuracy of the method is very good.
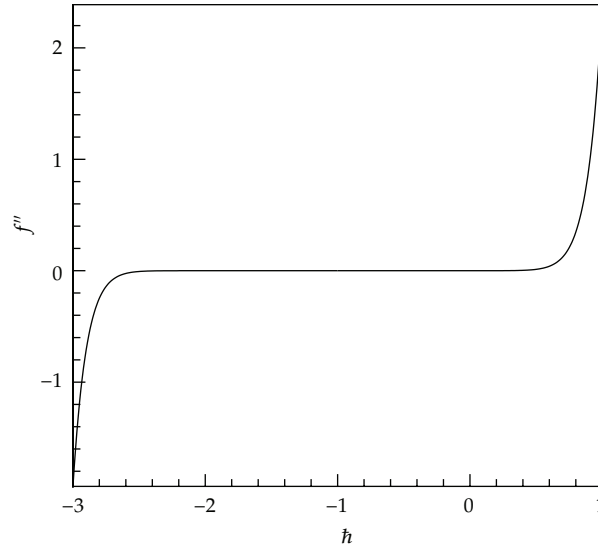
Figure 2: $\hbar$ curve by the HAM with $x = 0.00001$, $K = 1.732$, $L = 1$, $M = 0$, and $N = 0.05$, by 20th-order approximation solution.

7. Conclusions

A similarity solution of the problem for the unsteady flow of nonconducting, incompressible second grade fluid between two circular plates approaching or receding from each other has been investigated. It is noted that a similarity solution exists only when the distance between the plates varies as $(Kt + a_0)^{1/2}$, and squeezing flow takes place for $K < 0$ and $a_0 > 0$ as for as $Kt + a_0 > 0$, Approximate solutions for the fluid velocity have been found for the flow of unsteady 2-dimensional second grade fluid between two circular plates by the homotopy analysis method and numerical method. The major finding of the present paper can be summarized as follows:

> The Reynolds number R_e and nonNewtonian parameter W_{e1} have similar effect on the normal and longitudinal velocity components.

> The nonNewtonian parameter W_{e2} has opposite trend on the velocity profiles to that of Reynolds number and W_{e1}.

> It is found that at a given time and for a fixed positive value of Reynolds number, nonNewtonian parameters W_{e1}, W_{e2}, the normal velocity increases monotonically from $\eta = 0$ to $\eta = 1$.

> The longitudinal component of velocity increases near the central axis of the channel but deceases near the walls when R_e increases from 1.0 to 2.0. Similar effect of W_{e2} on this component is observed when W_{e2} varies from 0.0 to 0.1.
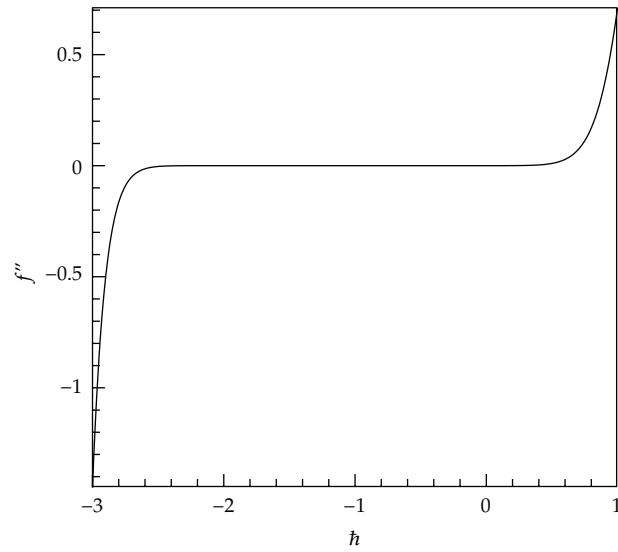
> The results are recovered for $W_e \rightarrow 0$.

> The obtained results are valid for all values of Reynolds number and nonNewtonian parameters.
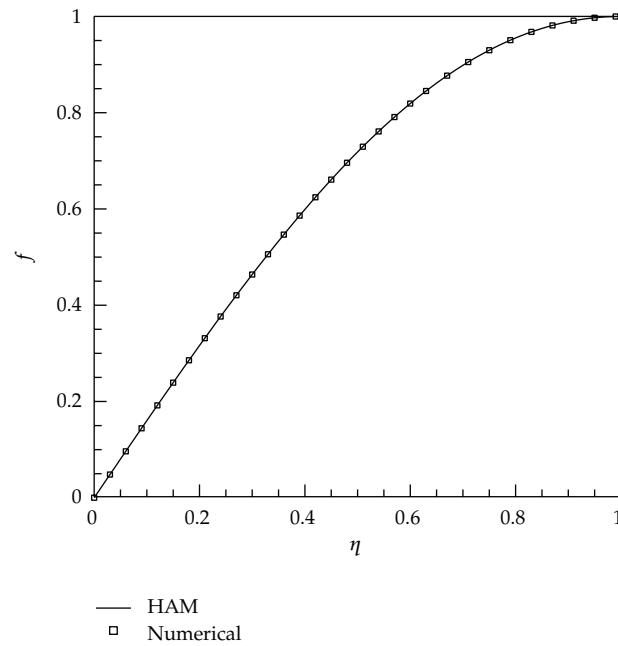
— HAM
□ Numerical
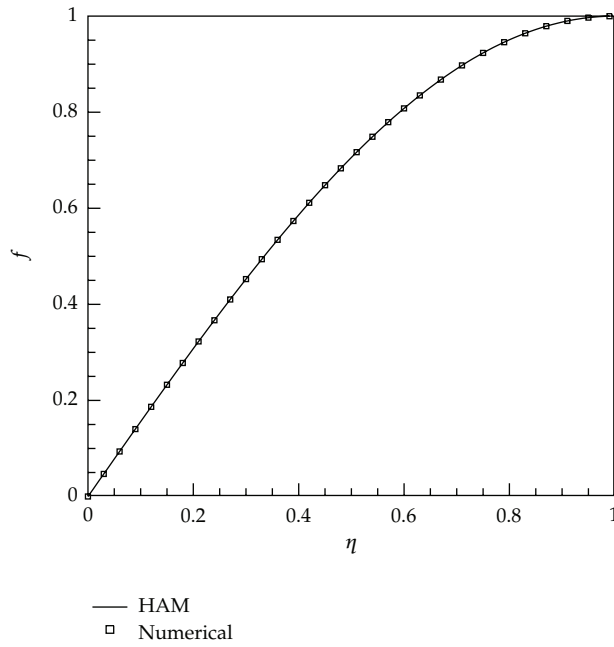
Figure 3: Obtained solution by the HAM with $\hbar = -1$, $K = 1.732$, $L = 1$, $M = 0$, and $N = 0.05$, by 20th-order approximation solution.
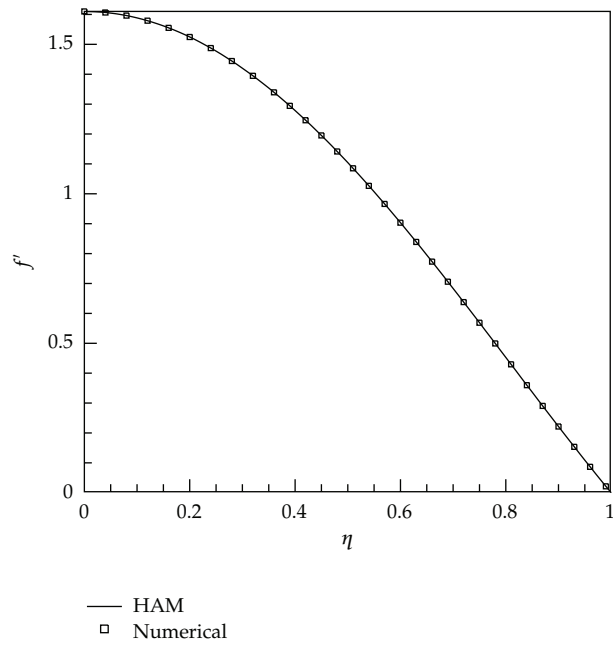

— HAM
□ Numerical
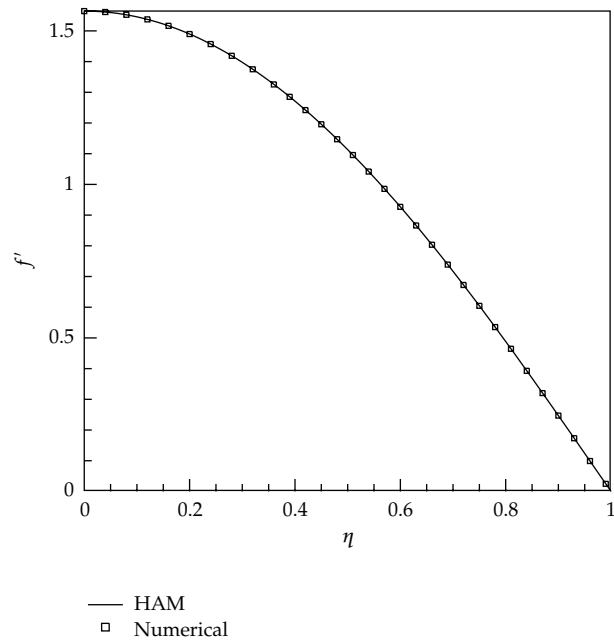
Figure 4: Obtained solution by the HAM with $\hbar = -1$, $K = 1.732$, $L = 1$, $M = 0$, and $N = 0.05$, by 20th-order approximation solution.
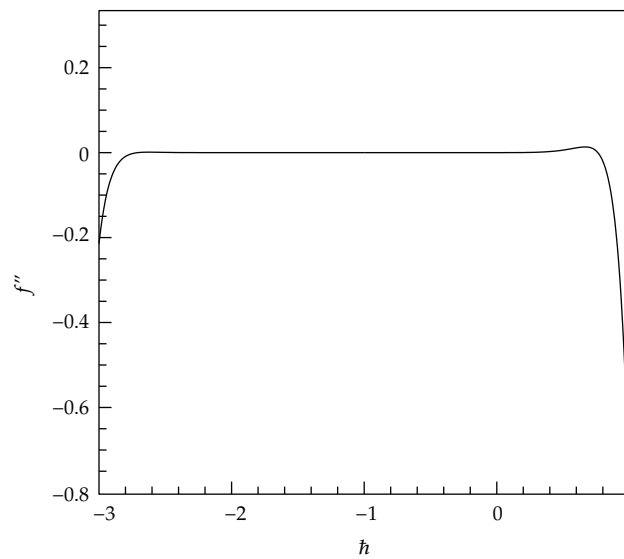
Figure 5: $\hbar$ curve by the HAM with $x = 0.00001$, $K = 2.121$, $L = 1.5$, $M = 0$, and $N = 0.05$, by 20th-order approximation solution.
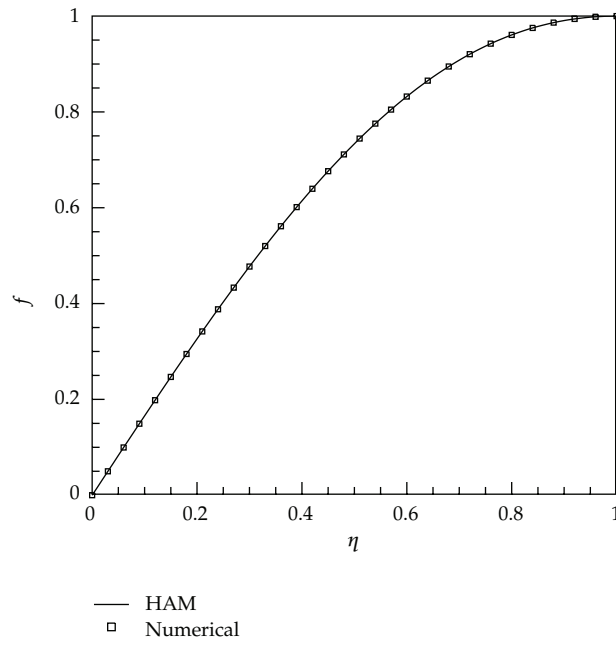

Figure 6: Obtained solution by the HAM with $\hbar = -1$, $K = 2.121$, $L = 1.5$, $M = 0$, and $N = 0.05$, by 20th-order approximation solution.

HAM
□ Numerical
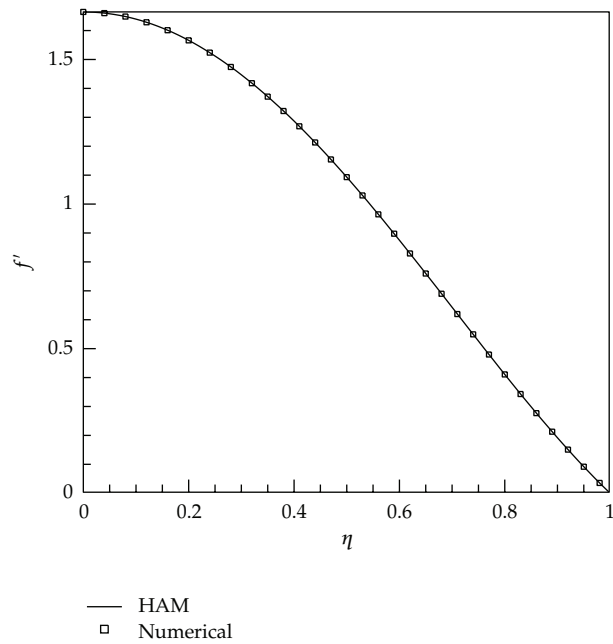
Figure 7: Obtained solution by the HAM with $\hbar = -1$, $K = 2.121$, $L = 1.5$, $M = 0$, and $N = 0.05$, by 20th-order approximation solution.

Figure 8: $\hbar$ curve by the HAM with $x = 0.00001$, $K = 2.449$, $L = 2$, $M = 0$, and $N = 0.05$, by 10th-order approximation solution.

Figure 9: Obtained solution by the HAM with $\hbar = -1$, $K = 2.449$, $L = 2$, $M = 0$, and $N = 0.05$, by 20th-order approximation solution.

Figure 10: Obtained solution by the HAM with $\hbar = -1$, $K = 2.449$, $L = 2$, $M = 0$, and $N = 0.05$, by 20th-order approximation solution.
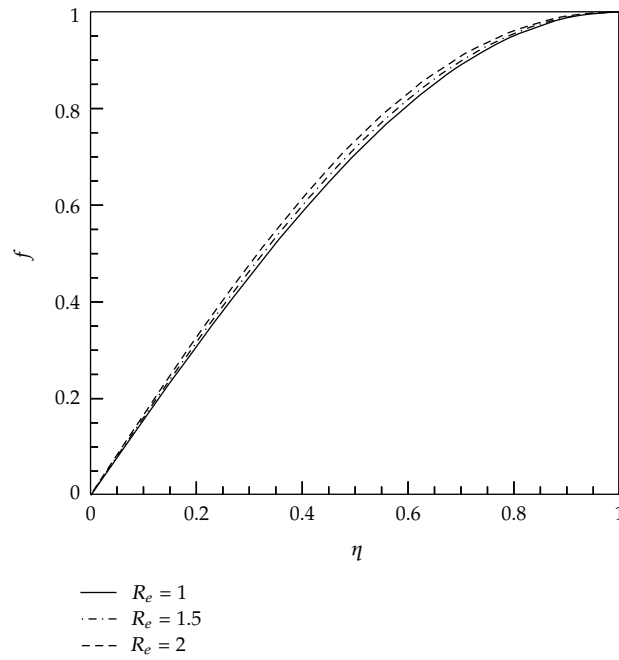
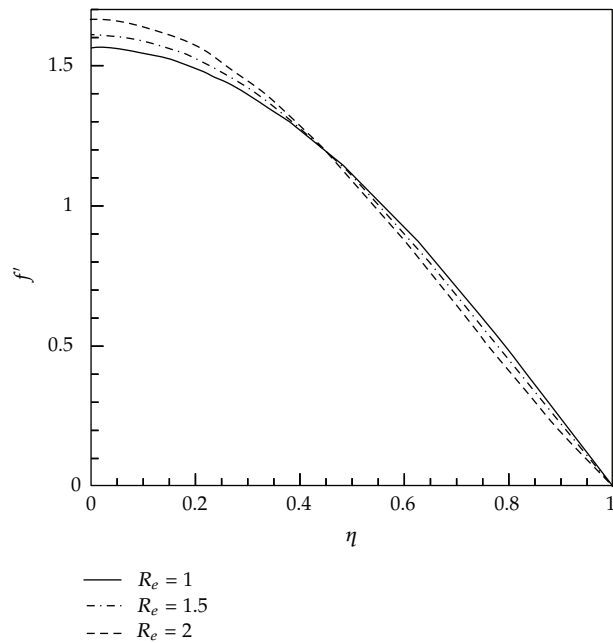**Figure 11:** The effect of R_e on the axial velocity profiles.

Figure 12: The effect of R_e on the radial velocity profiles.

Nomenclature

ρ: Density
v: Kinematic viscosity
V: Velocity vector
P: Pressure
T: Stress tensor
μ: Viscosity coefficient
α_1:   Material constant
α_2:   Material constant
Ω: Vorticity function
h: Pressure gradient
∇^2:   Laplacian operator
η: Dimensionless variable
$\hbar$:   Auxiliary linear parameter
A_i: Rivlin-Ericksen tensor
u: Velocity component along the radial
w: Velocity component along the axial
$2a(t)$:   Distance between the plates at any time t
W_{e1}:   Non-Newton parameter
W_{e2}:   Non-Newton parameter
R_e: Reynolds number.

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