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Research Article

Robust Filtering for State and Fault Estimation of Linear Stochastic Systems with Unknown Disturbance

Fayçal Ben Hmida,[1] Karim Khémiri,[1] José Ragot,[2] and Moncef Gossa[1]

[1] *Electrical Engineering Department of ESSTT, Research Unit C3S, Tunis University, 5 Avenue Taha Hussein, BP 56, 1008 Tunis, Tunisia*
[2] *Electrical Engineering Department, Research Laboratory CRAN (CNRS UMR 7039), Nancy University, 2 Avenue de la forêt de Haye, 54516 Vandoeuvre-les-Nancy Cedex, France*

Correspondence should be addressed to Fayçal Ben Hmida, faycal.benhmida@esstt.rnu.tn

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This paper presents a new robust filter structure to solve the simultaneous state and fault estimation problem of linear stochastic discrete-time systems with unknown disturbance. The method is based on the assumption that the fault and the unknown disturbance affect both the system state and the output, and no prior knowledge about their dynamical evolution is available. By making use of an optimal three-stage Kalman filtering method, an augmented fault and unknown disturbance models, an augmented robust three-stage Kalman filter (ARThSKF) is developed. The unbiasedness conditions and minimum-variance property of the proposed filter are provided. An illustrative example is given to apply this filter and to compare it with the existing literature results.

1. Introduction

The joint fault and state estimation for linear stochastic systems with unknown disturbance is concerned in this paper. The most important aim is to obtain an unbiased robust estimation of the fault and the state despite the presence of the unknown disturbance. This can be useful to solve a fault detection and isolation (FDI) problem [1–4] or a fault tolerant control (FTC) problem [5].

State estimation for stochastic linear systems with unknown inputs has gained the interest of many researchers during the last decades. In this context, this problem has been extensively studied using the Kalman filtering approach, see, for example, [1, 6–23]. When the model of the unknown input is available, it is possible to obtain an optimal estimation by using the Augmented State Kalman Filter (ASKF). To reduce computation costs of the ASKF, Friedland [7] has introduced the Two-Stage Kalman Filter (TSKF). His approach consists of decoupling the ASKF into the state subfilter and unknown-input subfilter. Friedland's filter is only optimal for constant bias. Many authors have extended the Friedland's idea to treat the stochastic bias, for example, [6, 10, 13–19]. In the same context, Hsieh and Chen [14] have generalized Friedland's filter by destroying the bias noise effect to obtain the Optimal Two-Stage Kalman Filter (OTSKF). Chen and Hsieh [15] proposed a generalization of the OTSKF to get the Optimal MultiStage Kalman Filter (OMSKF). Recently, Kim et al. [17] have developed an adaptive version of TSKF noted ATSKF (Adaptive Two-Stage Kalman Filter) and they have analysed the stability of this filter in [18].

On the other hand, when the unknown input model is not available, the unbiased minimum variance (UMV) state estimations are insensitive with the unknown inputs. Kitanidis [20] has developed a Kalman filter with unknown inputs by minimizing the trace of the state error covariance matrix under an algebraic constraint. Darouach and Zasadzinski [21] have used a parameterizing technique as an extension of the Kitanidis's results to derive an UMV estimator. Hsieh [13] has developed a robust filter in two-stage noted RTSKF (Robust Two-Stage Kalman Filter) equivalent to Kitanidis's filter. Next, the same author [11] has proposed an extension of the RTSKF (named ERTSKF) to solve the addressed general unknown-input filtering problem. To obtain ERTSKF, the author has introduced a new constrained relationship to have an equivalent structure to the optimal unbiased minimum-variance filter (OUMVF) presented in [12]. Gillijns and Moor [22] have treated the problem of estimating the state in the presence of unknown inputs which affect the system model. They developed a recursive filter which is optimal in the sense of minimum-variance. This filter has been extended by the same authors [23] for joint input and state estimation to linear discrete-time systems with direct feedthrough where the state and the unknown input estimation are interconnected. This filter is called recursive three-step filter (RTSF) and is limited to direct feedthrough matrix with full rank. Recently, Hsieh [8] has extended the RTSF [23] noted ERTSF, where he solved a general case when the direct feedthrough matrix has an arbitrary rank.

Model-based fault detection and isolation (FDI) problem for linear stochastic discrete-time systems with unknown disturbance is several studied. In [2, 3], the optimal filtering and robust fault diagnosis problem has been studied for stochastic systems with unknown disturbance. An optimal observer is proposed, which can produce disturbances decoupled state estimation with minimum-variance for linear time-varying systems with both noise and unknown disturbance. Recently, the unknown input filtering idea [8] is extended by [1] to solve the previously problem. Indeed, Ben Hmida et al. [1] present a new recursive filter to joint fault and state estimation of linear time-varying stochastic discrete-time systems in the presence of unknown disturbance. The method is based on the assumption that no prior knowledge about the dynamical evolution of the fault and the unknown disturbance is available. Moreover, it considers an arbitrary direct feedthrough matrix of the fault. However, it may in certain cases suffer from poor quality fault estimation.

The main objective of this paper is to develop a robust filter structure, that can solve the problem of simultaneously estimating the state and the fault in presence of the unknown disturbance. If the fault and the unknown disturbance affect the system state, we develop the robust three-stage Kalman filter (RThSKF) on two steps. Firstly, we make three-stage *U*-*V* transformations in order to decouple the covariance matrix on the augmented state Kalman Filter (ASKF) thus, we obtain an optimal structure named optimal three-stage Kalman filter

(OThSKF). Then, we use a modification in measurement update equations of the OThSKF in order to obtain an unbiased fault and state estimation. On the other hand, when the fault and the unknown disturbance affect both the state and the measurement equations, we propose an augmented robust three-stage Kalman filter (ARThSKF) to overcome this problem. This latter is obtained by a direct application of the RThSKF on the augmented fault and unknown disturbance models. The performances of the resulting filter are established in the sense of the unbiased minimum-variance estimation.

This paper is organized as follows. Section 2 states the problem of interest. In Section 3 we design the OThSKF. A robust three-stage Kalman filter (RThSKF) is developed in Section 4. In Section 5, the augmented robust three-stage Kalman filter is derived. Finally, an illustrative example of the proposed filter is presented.

2. Statement of the Problem

The problem consists of designing a filter that gives a robust state and fault estimation for linear stochastic systems in the presence of unknown disturbance. This problem is described by the bloc diagram in Figure 1.

The plant P represents the linear time-varying discrete stochastic system

$$x_{k+1} = A_k x_k + B_k u_k + F_k^x f_k + E_k^x d_k + w_k^x, \tag{2.1}$$

$$y_k = H_k x_k + F_k^y f_k + E_k^y d_k + v_k, \tag{2.2}$$

where $x_k \in R^n$ is the state vector, $u_k \in R^r$ is the known control input, $f_k \in R^p$ is the additive fault vector, $d_k \in R^q$ is the unknown disturbance and $y_k \in R^m$ is the observation vector. w_k^x and v_k are uncorrelated white noises sequences of zero-mean and the covariance matrices are $Q_k^x = \mathcal{E}[w_k^x w_k^T] \geq 0$, and $R_k = \mathcal{E}[v_k v_k^T] > 0$, respectively, where $\mathcal{E}[\cdot]$ denotes the expectation operator. The matrices A_k, B_k, F_k^x, H_k and F_k^y are known and have appropriate dimensions. We assume that (A_k, H_k) is observable, $m \geq p + q$, $\operatorname{rank}(H_{k+1}F_k^x) = \operatorname{rank}(F_k^x)$ and $\operatorname{rank}(H_{k+1}E_k^x) = \operatorname{rank}(E_k^x)$. The initial state is uncorrelated with the white noises processes w_k^x and v_k. The initial state x_0 is a gaussian random variable with $\mathcal{E}[x_0] = \widehat{x}_0$ and $\mathcal{E}[(x_0 - \widehat{x}_0)(x_0 - \widehat{x}_0)^T] = P_0^x$.

Under system equations (2.1)-(2.2), we consider that the proposed filter has the following form:

$$\begin{aligned}
\widehat{x}_{k+1/k+1} &= \overline{x}_{k+1/k+1} + \overline{V}_{k+1}^{12} f_{k+1/k+1} + \overline{V}_{k+1}^{13} d_{k+1/k+1}, \\
\widehat{f}_{k+1/k+1} &= f_{k+1/k+1} + \overline{V}_{k+1}^{23} d_{k+1/k+1}, \\
\overline{x}_{k+1/k+1} &= \overline{x}_{k+1/k} + \overline{K}_{k+1}^{x} (y_{k+1} - H_{k+1}\overline{x}_{k+1/k}), \\
f_{k+1/k+1} &= K_{k+1}^{f} (y_{k+1} - H_{k+1}\overline{x}_{k+1/k}), \\
d_{k+1/k+1} &= K_{k+1}^{d} (y_{k+1} - H_{k+1}\overline{x}_{k+1/k}), \\
\overline{x}_{k+1/k} &= A_k \widehat{x}_{k/k} + B_k u_k.
\end{aligned} \tag{2.3}$$

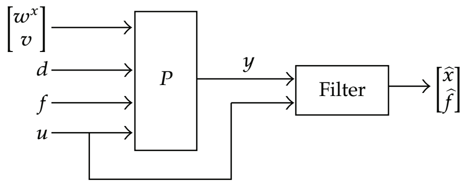


Figure 1: State and fault estimator filter.

The gain matrices $\overline{K}_{k+1}^{x}$, K_{k+1}^{f}, and K_{k+1}^{d} and the correction parameters $\overline{V}_{k+1}^{12}$, $\overline{V}_{k+1}^{13}$, and $\overline{V}_{k+1}^{23}$ will be determined later.

3. Optimal Three-Stage Kalman Filter (OThSKF)

This section is devoted to the optimal three-stage Kalman filter design. We first recall the structure of the augmented state filter, then the UV transformations are defined which will be used later to decouple the augmented state Kalman filter equations into three subfilters.

We should treat d_k and f_k as a random-walk processes with given wide-sense representation.

Thus, the dynamics of d_k may be assumed as follows:

$$d_{k+1} = d_k + w_k^d. \tag{3.1}$$

The additive faults f_k are generated by

$$f_{k+1} = f_k + w_k^f, \tag{3.2}$$

where, the noises w_k^f and w_k^d are zero-mean white noise sequences with the following covariances:

$$\begin{aligned}
\mathcal{E}\left(w_k^f w_\ell^{fT}\right) &= Q_k^f \delta_{k\ell}, \\
\mathcal{E}\left(w_k^d w_\ell^{dT}\right) &= Q_k^d \delta_{k\ell}, \\
\mathcal{E}\left(w_k^x w_\ell^{fT}\right) &= Q_k^{xf} \delta_{k\ell}, \\
\mathcal{E}\left(w_k^x w_\ell^{dT}\right) &= Q_k^{xd} \delta_{k\ell}, \\
\mathcal{E}\left(w_k^f w_\ell^{dT}\right) &= Q_k^{fd} \delta_{k\ell}.
\end{aligned} \tag{3.3}$$

The initial fault and unknown input satisfy the followings:

$$\begin{aligned}
\mathcal{E}(f_0) &= \overline{f}_0,\\
\mathcal{E}(d_0) &= \overline{d}_0,\\
\mathcal{E}\left(\left(f_0-\overline{f}_0\right)\left(f_0-\overline{f}_0\right)^T\right) &= P_0^f,\\
\mathcal{E}\left(\left(d_0-\overline{d}_0\right)\left(d_0-\overline{d}_0\right)^T\right) &= P_0^d,\\
\mathcal{E}\left(\left(x_0-\overline{f}_0\right)\left(x_0-\overline{f}_0\right)^T\right) &= P_0^{xf},\\
\mathcal{E}\left(\left(x_0-\overline{d}_0\right)\left(x_0-\overline{d}_0\right)^T\right) &= P_0^{xd},\\
\mathcal{E}\left(\left(f_0-\overline{d}_0\right)\left(f_0-\overline{d}_0\right)^T\right) &= P_0^{fd}.
\end{aligned} \tag{3.4}$$

3.1. Augmented State Kalman Filter (ASKF)

Treating x_k, f_k and d_k as the augmented system state, the ASKF is described by

$$x^a_{k+1/k} = A^a_k x^a_{k/k} + B^a_k u_k, \tag{3.5}$$

$$P^a_{k+1/k} = A^a_k P^a_{k/k} A^{aT}_k + Q^a_k, \tag{3.6}$$

$$x^a_{k+1/k+1} = x^a_{k+1/k} + K^a_{k+1}\left(y_{k+1} - H^a_{k+1} x^a_{k+1/k}\right), \tag{3.7}$$

$$K^a_{k+1} = P^a_{k+1/k} H^{aT}_{k+1}\left(H^a_{k+1} P^a_{k+1/k} H^{aT}_{k+1} + R_{k+1}\right)^{-1}, \tag{3.8}$$

$$P^a_{k+1/k+1} = \left(I - K^a_{k+1} H^a_{k+1}\right) P^a_{k+1/k}, \tag{3.9}$$

where

$$\begin{gathered}
x^a_{(\cdot)} = \begin{bmatrix} x_{(\cdot)} \\ f_{(\cdot)} \\ d_{(\cdot)} \end{bmatrix}, \qquad A^a_k = \begin{bmatrix} A_k & F^x_k & E^x_k \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \qquad B^a_k = \begin{bmatrix} B_k \\ 0 \\ 0 \end{bmatrix}, \qquad H^a_k = \begin{bmatrix} H_k & F^y_k & E^y_k \end{bmatrix},\\
P^a_{(\cdot)} = \begin{bmatrix} P^x_{(\cdot)} & P^{xf}_{(\cdot)} & P^{xd}_{(\cdot)} \\ P^{fx}_{(\cdot)} & P^f_{(\cdot)} & P^{fd}_{(\cdot)} \\ P^{dx}_{(\cdot)} & P^{df}_{(\cdot)} & P^d_{(\cdot)} \end{bmatrix}, \qquad Q^a_k = \begin{bmatrix} Q^x_k & Q^{xf}_k & Q^{xd}_k \\ Q^{fx}_k & Q^f_k & Q^{fd}_k \\ Q^{dx}_k & Q^{df}_k & Q^d_k \end{bmatrix}.
\end{gathered} \tag{3.10}$$

The filter model (3.5)–(3.9) may be used to produce the optimal state estimate. But, this filter has two main disadvantages: the increase of the computational cost with the augmentation of the state dimension and the rise of numerical problems during the implementation [13]. So, to solve these problems, we should use the three-stage Kalman filtering (ThSKF) technique.

3.2. U-V *Transformations*

According to [13, 14], the ThSKF is obtained by the application of a three-stage U-V transformations in order to decouple the ASKF covariance matrices, that is, $P^a_{k+1/k}$ and $P^a_{k+1/k+1}$. The aim is to find matrices U_{k+1} and V_{k+1} such that

$$\begin{aligned} P^a_{k+1/k} &= U_{k+1}\overline{P}^a_{k+1/k}U^T_{k+1}, \\ P^a_{k+1/k+1} &= V_{k+1}\overline{P}^a_{k+1/k+1}V^T_{k+1}, \end{aligned} \tag{3.11}$$

with $\overline{P}^a_{(\cdot)} = \mathrm{diag}\{\overline{P}^x_{(\cdot)}, \overline{P}^f_{(\cdot)}, \overline{P}^d_{(\cdot)}\}$, where $\overline{P}^x_{(\cdot)}, \overline{P}^f_{(\cdot)}$ and $\overline{P}^d_{(\cdot)}$ denote the transformed covariance matrices.

We define the structures of the U_{k+1} and V_{k+1} matrices as follows:

$$\begin{aligned} U_{k+1} &= \begin{bmatrix} I & U^{12}_{k+1} & U^{13}_{k+1} \\ 0 & I & U^{23}_{k+1} \\ 0 & 0 & I \end{bmatrix}, \\ V_{k+1} &= \begin{bmatrix} I & V^{12}_{k+1} & V^{13}_{k+1} \\ 0 & I & V^{23}_{k+1} \\ 0 & 0 & I \end{bmatrix}, \end{aligned} \tag{3.12}$$

U^{ij}_{k+1} and V^{ij}_{k+1} for $i = 1$ *or* 2 and $j = 2$ *or* 3 are to be determined later.

Using these transformations (3.12), the equations (3.5), (3.7), and (3.8) are transformed into

$$\begin{aligned} x^a_{k+1/k} &= U_{k+1}\overline{x}^a_{k+1/k}, \\ x^a_{k+1/k+1} &= V_{k+1}\overline{x}^a_{k+1/k+1}, \\ K^a_{k+1} &= V_{k+1}\overline{K}^a_{k+1}. \end{aligned} \tag{3.13}$$

The inverse transformations of U_{k+1} and V_{k+1} (3.12) will have this form

$$
\begin{aligned}
U_{k+1}^{-1} = \tilde{U}_{k+1} &= \begin{bmatrix} I & \tilde{U}_{k+1}^{12} & \tilde{U}_{k+1}^{13} \\ 0 & I & \tilde{U}_{k+1}^{23} \\ 0 & 0 & I \end{bmatrix}, \\
V_{k+1}^{-1} = \tilde{V}_{k+1} &= \begin{bmatrix} I & \tilde{V}_{k+1}^{12} & \tilde{V}_{k+1}^{13} \\ 0 & I & \tilde{V}_{k+1}^{23} \\ 0 & 0 & I \end{bmatrix}.
\end{aligned}
\tag{3.14}
$$

Using these inverse transformations (3.14), we have

$$
\begin{aligned}
\overline{x}_{k+1/k}^{a} &= \tilde{U}_{k+1} x_{k+1/k}^{a}, \\
\overline{P}_{k+1/k}^{a} &= \tilde{U}_{k+1} P_{k+1/k}^{a} \tilde{U}_{k+1}^{T}, \\
\overline{x}_{k+1/k+1}^{a} &= \tilde{V}_{k+1} x_{k+1/k+1}^{a}, \\
\overline{K}_{k+1}^{a} &= \tilde{V}_{k+1} K_{k+1}^{a}, \\
\overline{P}_{k+1/k+1}^{a} &= \tilde{V}_{k+1} P_{k+1/k+1}^{a} \tilde{V}_{k+1}^{T},
\end{aligned}
\tag{3.15}
$$

where

$$
\overline{x}_{(\cdot)}^{a} = \begin{bmatrix} \overline{x}_{(\cdot)} \\ \overline{f}_{(\cdot)} \\ \overline{d}_{(\cdot)} \end{bmatrix}, \qquad \overline{P}_{(\cdot)}^{a} = \begin{bmatrix} \overline{P}_{(\cdot)}^{x} & 0 & 0 \\ 0 & \overline{P}_{(\cdot)}^{f} & 0 \\ 0 & 0 & \overline{P}_{(\cdot)}^{d} \end{bmatrix}, \qquad \overline{K}_{(\cdot)}^{a} = \begin{bmatrix} \overline{K}_{(\cdot)}^{x} \\ \overline{K}_{(\cdot)}^{f} \\ \overline{K}_{(\cdot)}^{d} \end{bmatrix}.
\tag{3.16}
$$

3.3. Decoupling

If we use the two-step substitution method, the filter model (3.5)–(3.9) is transformed into

$$
\overline{x}_{k+1/k}^{a} = \tilde{U}_{k+1} \Pi_{k+1} \overline{x}_{k/k}^{a} + \tilde{U}_{k+1} B_{k}^{a} u_{k},
\tag{3.17}
$$

$$
\overline{P}_{k+1/k}^{a} = \tilde{U}_{k+1} \left(\Pi_{k+1} P_{k/k}^{a} \Pi_{k+1}^{T} + Q_{k}^{a} \right) \tilde{U}_{k+1},
\tag{3.18}
$$

$$
\overline{x}_{k+1/k+1}^{a} = \tilde{V}_{k+1} \Pi_{k+1} \overline{x}_{k+1/k}^{a} + \overline{K}_{k+1}^{a} \left(y_{k+1} - S_{k+1} \overline{x}_{k+1/k}^{a} \right),
\tag{3.19}
$$

$$
\overline{K}_{k+1}^{a} = \tilde{V}_{k+1} \Pi_{k+1} \overline{P}_{k+1/k}^{a} S_{k+1}^{T} \left(S_{k+1} \overline{P}_{k+1/k}^{a} S_{k+1}^{T} + R_{k+1} \right)^{-1},
\tag{3.20}
$$

$$
\overline{P}_{k+1/k+1}^{a} = \left(\tilde{V}_{k+1} \Pi_{k+1} - \overline{K}_{k+1}^{a} S_{k+1} \right) \overline{P}_{k+1/k}^{a} \left(\tilde{V}_{k+1} \Pi_{k+1} \right)^{-1},
\tag{3.21}
$$

where

$$\Pi_{k+1} = A_k^a V_k = \begin{bmatrix} A_k & \Pi_{k+1}^{12} & \Pi_{k+1}^{13} \\ 0 & I & \Pi_{k+1}^{23} \\ 0 & 0 & I \end{bmatrix}, \qquad S_{k+1} = \begin{bmatrix} S_{k+1}^1 & S_{k+1}^2 & S_{k+1}^3 \end{bmatrix}, \tag{3.22}$$

with

$$\begin{aligned} \Pi_{k+1}^{12} &= A_k V_k^{12} + F_k^x, \\ \Pi_{k+1}^{13} &= A_k V_k^{13} + F_k^x V_k^{23} + E_k^x, \\ \Pi_{k+1}^{23} &= V_k^{23}, \end{aligned} \tag{3.23}$$

$$S_{k+1}^1 = H_{k+1}, \tag{3.24a}$$

$$S_{k+1}^2 = H_{k+1} U_{k+1}^{12} + F_{k+1}^y, \tag{3.24b}$$

$$S_{k+1}^3 = H_{k+1} U_{k+1}^{13} + F_{k+1}^y U_{k+1}^{23} + E_{k+1}^y. \tag{3.24c}$$

Now, we start by developing the equations (3.18) and (3.21), respectively.

From (3.18), we obtain

$$\begin{aligned} \overline{P}_{k+1/k}^x &= A_k \overline{P}_{k/k}^x A_k^T + \overline{Q}_k^1, \\ \overline{P}_{k+1/k}^f &= \overline{P}_{k/k}^f + \overline{Q}_k^2, \\ \overline{P}_{k+1/k}^d &= \overline{P}_{k/k}^d + Q_k^d, \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} \overline{Q}_k^1 &= Q_k^x + \Pi_{k+1}^{12} \overline{P}_{k/k}^f \Pi_{k+1}^{12T} - U_{k+1}^{12} \overline{P}_{k+1/k}^f U_{k+1}^{12T} + \Pi_{k+1}^{13} \overline{P}_{k/k}^d \Pi_{k+1}^{13T} - U_{k+1}^{13} \overline{P}_{k+1/k}^d U_{k+1}^{13T}, \\ \overline{Q}_k^2 &= Q_k^f + \Pi_{k+1}^{23} \overline{P}_{k/k}^d \Pi_{k+1}^{23T} - U_{k+1}^{23} \overline{P}_{k+1/k}^d U_{k+1}^{23T}, \\ U_{k+1}^{13} &= \left(\Pi_{k+1}^{13} \overline{P}_{k/k}^d + Q_k^{xd} \right) \left(\overline{P}_{k+1/k}^d \right)^{-1}, \\ U_{k+1}^{23} &= \left(\Pi_{k+1}^{23} \overline{P}_{k/k}^d + Q_k^{fd} \right) \left(\overline{P}_{k+1/k}^d \right)^{-1}, \\ U_{k+1}^{12} &= \left(\Pi_{k+1}^{12} \overline{P}_{k/k}^f + \Pi_{k+1}^{13} \overline{P}_{k/k}^d \Pi_{k+1}^{23T} - U_{k+1}^{13} \overline{P}_{k/k}^f U_{k+1}^{23T} + Q_k^{xf} \right) \left(\overline{P}_{k+1/k}^f \right)^{-1}. \end{aligned} \tag{3.26}$$

The development of (3.21), leads to

$$\overline{P}^{x}_{k+1/k+1} = \left(I - \overline{K}^{x}_{k+1}S^{1}_{k+1}\right)\overline{P}^{x}_{k+1/k}, \tag{3.27}$$

$$\overline{P}^{f}_{k+1/k+1} = \left(I - \overline{K}^{f}_{k+1}S^{2}_{k+1}\right)\overline{P}^{f}_{k+1/k}, \tag{3.28}$$

$$\overline{P}^{d}_{k+1/k+1} = \left(I - \overline{K}^{d}_{k+1}S^{3}_{k+1}\right)\overline{P}^{d}_{k+1/k}, \tag{3.29}$$

$$V^{12}_{k+1} = U^{12}_{k+1} - \overline{K}^{x}_{k+1}S^{2}_{k+1}, \tag{3.30}$$

$$V^{13}_{k+1} = U^{13}_{k+1} - V^{12}_{k+1}\overline{K}^{f}_{k+1}S^{3}_{k+1} - \overline{K}^{x}_{k+1}S^{3}_{k+1}, \tag{3.31}$$

$$V^{23}_{k+1} = U^{23}_{k+1} - \overline{K}^{f}_{k+1}S^{3}_{k+1}. \tag{3.32}$$

With reference to (3.17), (3.19) and (3.20), we obtain, respectively,

$$\overline{x}_{k+1/k} = A_k\overline{x}_{k/k} + B_k u_k + \overline{u}^{1}_{k}, \tag{3.33}$$

$$\overline{f}_{k+1/k} = \overline{f}_{k/k} + \overline{u}^{2}_{k}, \tag{3.34}$$

$$\overline{d}_{k+1/k} = \overline{d}_{k/k}, \tag{3.35}$$

$$\overline{x}_{k+1/k+1} = \overline{x}_{k+1/k} + \overline{K}^{x}_{k+1}\left(y_{k+1} - S^{1}_{k+1}\overline{x}_{k+1/k}\right), \tag{3.36}$$

$$\overline{f}_{k+1/k+1} = \overline{f}_{k+1/k} + \overline{K}^{f}_{k+1}\left(y_{k+1} - S^{1}_{k+1}\overline{x}_{k+1/k} - S^{2}_{k+1}\overline{f}_{k+1/k}\right), \tag{3.37}$$

$$\overline{d}_{k+1/k+1} = \overline{d}_{k+1/k} + \overline{K}^{d}_{k+1}\left(y_{k+1} - S^{1}_{k+1}\overline{x}_{k+1/k} - S^{2}_{k+1}\overline{f}_{k+1/k} - S^{3}_{k+1}\overline{d}_{k+1/k}\right), \tag{3.38}$$

$$\overline{K}^{x}_{k+1} = \overline{P}^{x}_{k+1/k}S^{1T}_{k+1}\left(S^{1}_{k+1}\overline{P}^{x}_{k+1/k}S^{1T}_{k+1} + R_{k+1}\right)^{-1}, \tag{3.39}$$

$$\overline{K}^{f}_{k+1} = \overline{P}^{f}_{k+1/k}S^{2T}_{k+1}\left(S^{2}_{k+1}\overline{P}^{f}_{k+1/k}S^{2T}_{k+1} + S^{1}_{k+1}\overline{P}^{x}_{k+1/k}S^{1T}_{k+1} + R_{k+1}\right)^{-1}, \tag{3.40}$$

$$\overline{K}^{d}_{k+1} = \overline{P}^{d}_{k+1/k}S^{3T}_{k+1}\left(S^{3}_{k+1}\overline{P}^{d}_{k+1/k}S^{3T}_{k+1} + S^{2}_{k+1}\overline{P}^{f}_{k+1/k}S^{2T}_{k+1} + S^{1}_{k+1}\overline{P}^{x}_{k+1/k}S^{1T}_{k+1} + R_{k+1}\right)^{-1}, \tag{3.41}$$

where

$$\begin{aligned} \overline{u}^{1}_{k} &= \left(\Pi^{12}_{k+1} - U^{12}_{k+1}\right)\overline{f}_{k/k} + \left(\Pi^{13}_{k+1} - U^{13}_{k+1} - U^{12}_{k+1}\left(\Pi^{23}_{k+1} - U^{23}_{k+1}\right)\right)\overline{d}_{k/k}, \\ \overline{u}^{2}_{k} &= \left(\Pi^{23}_{k+1} - U^{23}_{k+1}\right)\overline{d}_{k/k}. \end{aligned} \tag{3.42}$$

Finally, to correct the estimation of the state and the fault, we should follow these equations

$$
\begin{aligned}
\hat{x}_{k+1/k+1} &= \overline{x}_{k+1/k+1} + V_{k+1}^{12}\overline{f}_{k+1/k+1} + V_{k+1}^{13}\overline{d}_{k+1/k+1},\\
\hat{P}_{k+1/k+1}^{x} &= \overline{P}_{k+1/k+1}^{x} + V_{k+1}^{12}\overline{P}_{k+1/k+1}^{f}V_{k+1}^{12T} + V_{k+1}^{13}\overline{P}_{k+1/k+1}^{d}V_{k+1}^{13T},\\
\hat{f}_{k+1/k+1} &= \overline{f}_{k+1/k+1} + V_{k+1}^{23}\overline{d}_{k+1/k+1},\\
\hat{P}_{k+1/k+1}^{f} &= \overline{P}_{k+1/k+1}^{f} + V_{k+1}^{23}\overline{P}_{k+1/k+1}^{d}V_{k+1}^{23T}.
\end{aligned}
\tag{3.43}
$$

The OThSKF is optimal in the minimum mean square error (MMSE) sense. However, this filter loses its optimality, when the statistical properties of the models (3.1) and (3.2) are unknown or not perfectly known. So, it would be better to use a robust three-stage Kalman filter (RThKF) to get a good estimation of state and fault in the presence of unknown disturbance.

4. Robust Three-Stage Kalman Filter (RThSKF)

In this section, we present a robust version of a filter to solve the joint fault and state estimation problem for linear stochastic system with unknown disturbance. We consider that the fault and the unknown disturbance only affect the state equation, that is, ($F_k^y = E_k^y = 0$). This filter is obtained by modifying the measurement update equations of unknown disturbance subfilter and fault subfilter of the OThSKF.

Equations (3.24b)-(3.24c) will be rewritten as follow:

$$
\begin{aligned}
S_{k+1}^{2} &= H_{k+1}U_{k+1}^{12},\\
S_{k+1}^{3} &= H_{k+1}U_{k+1}^{13}.
\end{aligned}
\tag{4.1}
$$

By substituting (3.34) and (3.35) into (3.37) and (3.38) and combining (3.28) and (3.29) into (3.40) and (3.41), respectively, the measurement update equations of the unknown input subfilter and the fault subfilter are given as follow:

$$\overline{f}_{k+1/k+1} = \left(I - \overline{K}_{k+1}^{f}S_{k+1}^{2}\right)\overline{f}_{k+1/k} + \overline{K}_{k+1}^{f}\left(y_{k+1} - S_{k+1}^{1}\overline{x}_{k+1/k}\right), \tag{4.2}$$

$$\overline{K}_{k+1}^{f} = \overline{P}_{k+1/k+1}^{f}S_{k+1}^{2T}C_{k+1}^{-1}, \tag{4.3}$$

$$\overline{d}_{k+1/k+1} = \left(I - \overline{K}_{k+1}^{d}S_{k+1}^{3}\right)\overline{d}_{k/k} + \overline{K}_{k+1}^{d}\left(y_{k+1} - S_{k+1}^{1}\overline{x}_{k+1/k} - S_{k+1}^{2}\overline{f}_{k+1/k}\right), \tag{4.4}$$

$$\overline{K}_{k+1}^{d} = \overline{P}_{k+1/k+1}^{d}S_{k+1}^{3T}\left(S_{k+1}^{2}\overline{P}_{k+1/k}^{f}S_{k+1}^{2T} + C_{k+1}\right)^{-1}, \tag{4.5}$$

where $C_{k+1} = H_{k+1}\overline{P}_{k+1/k}^{x}H_{k+1}^{T} + R_{k+1}$.

Firstly, to eliminate the two terms $\overline{f}_{k+1/k}$ and $\overline{d}_{k/k}$, we will choose the matrix gains $\overline{K}^f_{k+1}$ and $\overline{K}^d_{k+1}$ that can satisfy the following algebraic constraints:

$$\left(I - \overline{K}^f_{k+1} S^2_{k+1}\right) = 0, \tag{4.6}$$

$$\left(I - \overline{K}^d_{k+1} S^3_{k+1}\right) = 0, \tag{4.7}$$

$$\overline{K}^d_{k+1} S^2_{k+1} = 0. \tag{4.8}$$

In this case, (4.2) and (4.4) become

$$\begin{aligned} \overline{f}_{k+1/k+1} &= \overline{K}^f_{k+1}\left(y_{k+1} - S^1_{k+1}\overline{x}_{k+1/k}\right), \\ \overline{d}_{k+1/k+1} &= \overline{K}^d_{k+1}\left(y_{k+1} - S^1_{k+1}\overline{x}_{k+1/k}\right). \end{aligned} \tag{4.9}$$

With substituting (4.3) and (4.5) into (4.6) and (4.7) and using (4.8), $\overline{P}^f_{k+1/k+1}$ and $\overline{P}^d_{k+1/k+1}$ can be rewritten as follows

$$\begin{aligned} \overline{P}^f_{k+1/k+1} &= \left(S^{2T}_{k+1} C^{-1}_{k+1} S^2_{k+1}\right)^+, \\ \overline{P}^d_{k+1/k+1} &= \left(S^{3T}_{k+1} C^{-1}_{k+1} S^3_{k+1}\right)^+, \end{aligned} \tag{4.10}$$

where M^+ denotes any one-condition generalized inverse of M, that is, $MM^+M = M$.

The gain matrix $\overline{K}^d_{k+1}$ is calculated in the assumption that the constraints (4.7) and (4.8) are satisfied:

$$\overline{K}^d_{k+1} = \overline{P}^d_{k+1/k+1} S^{3T}_{k+1} C^{-1}_{k+1} \alpha_{k+1}\left(I - S^2_{k+1} S^{2+}_{k+1}\right), \tag{4.11}$$

where α_{k+1} is an arbitrary matrix.

Equations (3.33) and (3.34) are presented, respectively, as follows:

$$\begin{aligned} \overline{x}_{k+1/k} &= A_k \hat{x}_{k/k} + B_k u_k + \tilde{u}^1_k, \\ \overline{f}_{k+1/k} &= \overline{f}_{k/k} + \tilde{u}^2_k, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} \tilde{u}^1_k &= \left(F^x_k - U^{12}_{k+1}\right)\overline{f}_{k/k} + \left(E^x_k - U^{13}_{k+1} + U^{12}_{k+1} U^{23}_{k+1}\right)\overline{d}_{k/k}, \\ \tilde{u}^2_k &= \left(\Pi^{23}_{k+1} - U^{23}_{k+1}\right)\overline{d}_{k/k}. \end{aligned} \tag{4.13}$$

In order to return (4.12) robust against the fault and the unknown disturbance, we can choose $\tilde{u}_k^1 = 0$ and $\tilde{u}_k^2 = 0$.

In this case, the new matrices U_{k+1}^{12}, U_{k+1}^{13}, and U_{k+1}^{23} can be written as follow:

$$\begin{aligned} U_{k+1}^{23} &= \Pi_{k+1}^{23} = V_k^{23}, \\ U_{k+1}^{12} &= F_k^x, \\ U_{k+1}^{13} &= E_k^x + F_k^x V_k^{23}. \end{aligned} \tag{4.14}$$

Equation (4.12) becomes

$$\begin{aligned} \overline{x}_{k+1/k} &= A_k \widehat{x}_{k/k} + B_k u_k, \\ \overline{f}_{k+1/k} &= \overline{f}_{k/k}. \end{aligned} \tag{4.15}$$

Finally, an unbiased estimation of the state and the fault is obtained if and only if the constraints (4.6)–(4.8) are satisfied and

$$\begin{aligned} V_{k+1}^{12} &= F_k^x - \overline{K}_{k+1}^x S_{k+1}^2, \\ V_{k+1}^{23} &= -\overline{K}_{k+1}^f H_{k+1} E_k^x, \\ V_{k+1}^{13} &= E_k^x - \overline{K}_{k+1}^x H_{k+1} E_k^x + V_{k+1}^{12} V_{k+1}^{23}. \end{aligned} \tag{4.16}$$

Now, we summarize the robust three-stage Kalman filter (RThSKF) equations as follow:

$$\begin{aligned} \widehat{x}_{k+1/k+1} &= \overline{x}_{k+1/k+1} + V_{k+1}^{12} \overline{f}_{k+1/k+1} + V_{k+1}^{13} \overline{d}_{k+1/k+1}, \\ \widehat{P}_{k+1/k+1}^x &= \overline{P}_{k+1/k+1}^x + V_{k+1}^{12} \overline{P}_{k+1/k+1}^f V_{k+1}^{12T} + V_{k+1}^{13} \overline{P}_{k+1/k+1}^d V_{k+1}^{13T}, \\ \widehat{f}_{k+1/k+1} &= \overline{f}_{k+1/k+1} + V_{k+1}^{23} \overline{d}_{k+1/k+1}, \\ \widehat{P}_{k+1/k+1}^f &= \overline{P}_{k+1/k+1}^f + V_{k+1}^{23} \overline{P}_{k+1/k+1}^d V_{k+1}^{23T}, \end{aligned} \tag{4.17}$$

where $\overline{x}_{k+1/k+1}$ is given by

$$\begin{aligned} \overline{x}_{k+1/k} &= A_k \widehat{x}_{k/k} + B_k u_k, \\ \overline{P}_{k+1/k}^x &= A_k \widehat{P}_{k/k}^x A_k^T + Q_k^x, \\ \overline{K}_{k+1}^x &= \overline{P}_{k+1/k}^x H_{k+1}^T C_{k+1}^{-1}, \\ \overline{x}_{k+1/k+1} &= \overline{x}_{k+1/k} + \overline{K}_{k+1}^x \left(y_{k+1} - H_{k+1} \overline{x}_{k+1/k} \right), \\ \overline{P}_{k+1/k+1}^x &= \left(I - \overline{K}_{k+1}^x H_{k+1} \right) \overline{P}_{k+1/k}^x, \end{aligned} \tag{4.18}$$

the fault $\overline{f}_{k+1/k+1}$ is given by

$$\begin{aligned}\overline{P}^{f}_{k+1/k+1} &= \left(S^{2T}_{k+1}C^{-1}_{k+1}S^{2}_{k+1}\right)^{+},\\ \overline{K}^{f}_{k+1} &= \overline{P}^{f}_{k+1/k+1}S^{2T}_{k+1}C^{-1}_{k+1},\\ \overline{f}_{k+1/k+1} &= \overline{K}^{f}_{k+1}\left(y_{k+1}-H_{k+1}\overline{x}_{k+1/k}\right),\end{aligned}\tag{4.19}$$

$\overline{d}_{k+1/k+1}$ is given by

$$\begin{aligned}\overline{P}^{d}_{k+1/k+1} &= \left(S^{3T}_{k+1}C^{-1}_{k+1}S^{3}_{k+1}\right)^{+},\\ \overline{K}^{d}_{k+1} &= \overline{P}^{d}_{k+1/k+1}S^{3T}_{k+1}C^{-1}_{k+1}\alpha_{k+1}\left(I-S^{2}_{k+1}\left(S^{2}_{k+1}\right)^{+}\right),\\ \overline{d}_{k+1/k+1} &= \overline{K}^{d}_{k+1}\left(y_{k+1}-H_{k+1}\overline{x}_{k+1/k}\right).\end{aligned}\tag{4.20}$$

The initial conditions are given as follow: $\hat{x}_{0/0}=\overline{x}_{0}$, $\hat{P}^{x}_{0/0}=P^{x}_{0}$ and V^{23}_{0}.

5. Augmented Robust Three-Stage Kalman Filter (ARThSKF)

In this section, we consider that the fault and the unknown disturbance affect both the state and the measurement equations, that is, ($F^{y}_{k}\neq 0$ and $E^{y}_{k}\neq 0$). Using the same technique proposed by [9, 10], the measurement equation (2.2) can be rewritten as follows:

$$y_{k}=H_{k}x_{k}+F^{y}_{k}\widetilde{f}_{k}+E^{y}_{k}\widetilde{d}_{k}+v_{k}.\tag{5.1}$$

Hence, we augment the fault and the unknown disturbance as follow:

$$\begin{aligned}d_{k}&\longrightarrow d^{a}_{k}=\begin{bmatrix}\widetilde{d}^{T}_{k} & d^{T}_{k}\end{bmatrix}^{T},\\ f_{k}&\longrightarrow f^{a}_{k}=\begin{bmatrix}\widetilde{f}^{T}_{k} & f^{T}_{k}\end{bmatrix}^{T}.\end{aligned}\tag{5.2}$$

Thus, the system model (2.1) and (5.1) can be represented, respectively, by:

$$\begin{aligned}x_{k+1}&=A_{k}x_{k}+B_{k}u_{k}+\overline{F}^{x}_{k}f^{a}_{k}+\overline{E}^{x}_{k}d^{a}_{k}+w^{x}_{k},\\ y_{k}&=H_{k}x_{k}+\overline{F}^{y}_{k}f^{a}_{k}+\overline{E}^{y}_{k}d^{a}_{k}+v_{k},\end{aligned}\tag{5.3}$$

where

$$\begin{aligned}\overline{F}^{x}_{k}&=\begin{bmatrix}0 & F^{x}_{k}\end{bmatrix}, & \overline{F}^{y}_{k}&=\begin{bmatrix}F^{y}_{k} & 0\end{bmatrix},\\ \overline{E}^{x}_{k}&=\begin{bmatrix}0 & E^{x}_{k}\end{bmatrix}, & \overline{E}^{y}_{k}&=\begin{bmatrix}E^{y}_{k} & 0\end{bmatrix}.\end{aligned}\tag{5.4}$$

Referring to (4.14) and (5.4), we define

$$\begin{aligned} \overline{U}_{k+1}^{12} &= \overline{F}_{k}^{x}, \\ \overline{U}_{k+1}^{13} &= \overline{E}_{k}^{x} + \overline{F}_{k}^{x}\overline{V}_{k}^{23}, \\ \overline{U}_{k+1}^{23} &= \overline{V}_{k}^{23}. \end{aligned} \tag{5.5}$$

Using (5.5), equations (3.24b)-(3.24c) can be written as follow:

$$\begin{aligned} \overline{S}_{k+1}^{2} &= H_{k+1}\overline{F}_{k}^{x} + \overline{F}_{k+1}^{y} = \left[F_{k}^{y} \quad S_{k+1}^{2} \right], \\ \overline{S}_{k+1}^{3} &= H_{k+1}\overline{E}_{k}^{x} + \overline{E}_{k+1}^{y} + \overline{S}_{k+1}^{2}\overline{V}_{k}^{23} \\ &= \left[E_{k+1}^{y} + F_{k+1}^{y}V_{k}^{23} \quad H_{k+1}E_{k+1}^{x} + S_{k+1}^{2}V_{k}^{23} \right] \end{aligned} \tag{5.6}$$

The augmented fault and unknown disturbance RThKF (ARThSKF) is obtained by a direct application of the RThSKF with a minor modification

$$\begin{aligned} \widehat{x}_{k+1/k+1} &= \overline{x}_{k+1/k+1} + \overline{V}_{k+1}^{12} f_{k+1/k+1} + \overline{V}_{k}^{13} d_{k+1/k+1}, \\ \widehat{P}_{k+1/k+1}^{x} &= \overline{P}_{k+1/k+1}^{x} + \overline{V}_{k+1}^{12} P_{k+1/k+1}^{f} \overline{V}_{k+1}^{12T} + \overline{V}_{k+1}^{13} P_{k+1/k+1}^{d} \overline{V}_{k+1}^{13T}, \\ \widehat{f}_{k+1/k+1} &= f_{k+1/k+1} + \overline{V}_{k+1}^{23} d_{k+1/k+1}, \\ \widehat{P}_{k+1/k+1}^{f} &= P_{k+1/k+1}^{f} + \overline{V}_{k+1}^{23} P_{k+1/k+1}^{d} \overline{V}_{k+1}^{23T}, \end{aligned} \tag{5.7}$$

where $\overline{x}_{k+1/k+1}$ is given by (4.18), $f_{k+1/k+1}$ and $d_{k+1/k+1}$ are given by

$$\begin{aligned} f_{k+1/k+1} &= K_{k+1}^{f}\left(y_{k+1} - H_{k+1}\overline{x}_{k+1/k}\right), \\ K_{k+1}^{f} &= P_{k+1/k+1}^{f} \overline{S}_{k+1}^{2T} C_{k+1}^{-1}, \\ P_{k+1/k+1}^{f} &= \left\{ \overline{S}_{k+1}^{2T} C_{k+1}^{-1} \overline{S}_{k+1}^{2} \right\}^{+}, \\ d_{k+1/k+1} &= K_{k+1}^{d}\left(y_{k+1} - H_{k+1}\overline{x}_{k+1/k}\right), \\ K_{k+1}^{d} &= P_{k+1/k+1}^{d} \overline{S}_{k+1}^{3T} C_{k+1}^{-1} \alpha_{k+1}\left(I - \overline{S}_{k+1}^{2}\overline{S}_{k+1}^{2+}\right), \\ P_{k+1/k+1}^{d} &= \left\{ \overline{S}_{k+1}^{3T} C_{k+1}^{-1} \overline{S}_{k+1}^{3} \right\}^{+}, \end{aligned} \tag{5.8}$$

where α_{k+1} is an arbitrary matrix.

The unknown disturbance estimation error $\tilde{d}_{k+1}$ is given by

$$\begin{aligned}\tilde{d}_{k+1} &= d^a_{k+1} - d_{k+1/k+1} \\ &= \left(I - K^d_{k+1}\overline{S}^3_{k+1} - K^d_{k+1}\overline{S}^2_{k+1}V^{23}_{k+1}\right)d^a_{k+1} - K^d_{k+1}\overline{S}^2_{k+1}f^a_{k+1} - K^d_{k+1}e_{k+1},\end{aligned} \tag{5.9}$$

where $e_{k+1} = H_{k+1}(A_k\tilde{x}_k + w^x_k) + v_{k+1}$.

The unbiasedness constraints to have an unbiased estimation of the unknown disturbance are given as follow:

$$\begin{aligned}I - K^d_{k+1}\overline{S}^3_{k+1} = 0, \\ K^d_{k+1}\overline{S}^2_{k+1} = 0.\end{aligned} \tag{5.10}$$

The fault estimation error $\tilde{f}_{k+1}$ is presented as follows

$$\begin{aligned}\tilde{f}_{k+1} &= f^a_{k+1} - \hat{f}_{k+1/k+1} \\ &= f^a_{k+1} - f_{k+1/k+1} - V^{23}_{k+1}d_{k+1/k+1} \\ &= \left(I - K^f_{k+1}\overline{S}^2_{k+1}\right)f^a_{k+1} - K^f_{k+1}\left(\overline{S}^3_{k+1} - \overline{S}^2_{k+1}\overline{V}^{23}_k\right)d^a_{k+1} - \overline{V}^{23}_{k+1}d_{k+1/k+1} - K^f_{k+1}e_{k+1}.\end{aligned} \tag{5.11}$$

The estimator $\hat{f}_{k+1}$ is unbiased if and only if

$$\begin{aligned}\overline{V}^{23}_{k+1} &= -K^f_{k+1}\left(\overline{S}^3_{k+1} - \overline{S}^2_{k+1}\overline{V}^{23}_k\right), \\ 0 &= I - K^f_{k+1}\overline{S}^2_{k+1}.\end{aligned} \tag{5.12}$$

The state estimation error $\tilde{x}_{k+1}$ has the following form:

$$\begin{aligned}\tilde{x}_{k+1} &= x_{k+1} - \hat{x}_{k+1/k+1} \\ &= \left(I - \overline{K}^x_{k+1}\right)(A_k\tilde{x}_k + w_k) - \overline{K}^x_{k+1}v_{k+1} + \left(\overline{E}^x_k - K^x_{k+1}\overline{S}^2_{k+1}\right)f^a_{k+1} - \overline{V}^{12}_{k+1}f_{k+1/k+1} \\ &\quad + \left(\overline{E}^x_k - \overline{K}^x_{k+1}\overline{S}^3_{k+1} + \overline{K}^x_{k+1}\overline{S}^2_{k+1}V^{23}_k\right)d^a_{k+1} - \overline{V}^{13}_{k+1}d_{k+1/k+1}.\end{aligned} \tag{5.13}$$

The parameters $\overline{V}^{12}_{k+1}$ and $\overline{V}^{13}_{k+1}$ have the following relations:

$$\begin{aligned}\overline{V}^{12}_{k+1} &= \overline{F}^x_k - \overline{K}^x_{k+1}\overline{S}^2_k, \\ \overline{V}^{13}_{k+1} &= \overline{E}^x_k - \overline{K}^x_{k+1}\left(\overline{S}^3_{k+1} - \overline{S}^2_{k+1}\overline{V}^{23}_k\right) + \overline{V}^{12}_{k+1}\overline{V}^{23}_{k+1}.\end{aligned} \tag{5.14}$$

If the gain matrices $\overline{K}^{x}_{k+1}$, K^{f}_{k+1} and K^{d}_{k+1} are determined to satisfy the unbiasedness conditions (5.10) and (5.12), then the proposed filter ensure an unbiased minimum-variance estimation of state, fault and unknown disturbance. Indeed, the unknown disturbance, fault and state estimation errors are given, respectively, as follows:

$$\begin{aligned}\tilde{d}_{k+1} &= -K^{d}_{k+1}e_{k+1},\\ \tilde{f}_{k+1} &= -K^{f}_{k+1}e_{k+1} - \overline{V}^{23}_{k+1}K^{d}_{k+1}e_{k+1},\\ \tilde{x}_{k+1} &= \left(I - \overline{K}^{x}_{k+1}\right)(A_k\tilde{x}_k + w_k) - \overline{K}^{x}_{k+1}v_{k+1} - \overline{V}^{12}_{k+1}K^{f}_{k+1}e_{k+1} - \overline{V}^{13}_{k+1}K^{d}_{k+1}e_{k+1}.\end{aligned} \tag{5.15}$$

6. Illustrative Example

We consider the linearized model of a simplified longitudinal flight control system as follows:

$$\begin{aligned}x_{k+1} &= (A_k + \Delta A_k)x_k + B_k u_k + F^{a}_{k}f^{a}_{k} + w^{x}_{k},\\ y_k &= (H_k + \Delta H_k)x_k + F^{s}_{k}f^{s}_{k} + v_k,\end{aligned} \tag{6.1}$$

where the state variables are: pitch angle δ_z, pitch rate ω_z and normal velocity η_y, the control input u_k is the elevator control signal. F^{a}_{k} and F^{s}_{k} are the matrices distribution of the actuator fault f^{a}_{k} and sensor fault f^{s}_{k}.

The presented system equations can be rewritten as follow:

$$\begin{aligned}x_{k+1} &= A_k x_k + B_k u_k + F^{x}_{k}f_k + E^{x}_{k}d_k + w^{x}_{k},\\ y_k &= H_k x_k + F^{y}_{k}f_k + E^{y}_{k}d_k + v_k,\end{aligned} \tag{6.2}$$

where F^{x}_{k} and F^{y}_{k} are the distribution matrices of the fault vector in the state and measurement equations

$$\begin{aligned}F^{x}_{k} &= \begin{bmatrix}F^{a}_{k} & 0\end{bmatrix},\\ F^{y}_{k} &= \begin{bmatrix}0 & F^{s}_{k}\end{bmatrix}\end{aligned} \tag{6.3}$$

The terms $E^{x}_{k}d_k$ and $E^{y}_{k}d_k$ represent the parameter perturbations in matrices A_k and H_k:

$$\begin{aligned}\begin{bmatrix}\Delta A_k\\ \Delta H_k\end{bmatrix} &= \begin{bmatrix}E^{x}_{k}\\ E^{y}_{k}\end{bmatrix}\Delta_k M,\\ d_k &= \Delta_k M x_k,\end{aligned} \tag{6.4}$$

where $\Delta_k\Delta^{T}_{k} \leq I$.

The numerical example given in [2, 3] is considered and slightly modified, where the parameters system are given as follows:

$$A_k = \begin{bmatrix} 0.9944 & -0.1203 & -0.4302 \\ 0.0017 & 0.9902 & -0.0747 \\ 0 & 0.8187 & 0 \end{bmatrix}, \qquad B_k = \begin{bmatrix} 0.4252 \\ -0.0082 \\ 0.1813 \end{bmatrix},$$
$$H_k = I_{3\times3}, \qquad x_k = \begin{bmatrix} \eta_y & \omega_z & \delta_z \end{bmatrix}^T,$$
$$Q_k^x = \text{diag}\left\{0.1^2, 0.1^2, 0.01^2\right\}, \qquad R_k = 0.1^2 I_{3\times3}. \tag{6.5}$$

We inject simultaneously two faults in the system,

$$\begin{bmatrix} f_k^a \\ f_k^s \end{bmatrix} = \begin{bmatrix} 2u_s(k-20) - 2u_s(k-60) \\ -u_s(k-30) + u_s(k-70) \end{bmatrix}, \tag{6.6}$$

where $u_s(k)$ is the unit-step function.

In the simulation, the aerodynamic coefficients are perturbed; $M = [-0.02 \quad 0 \quad 0]$ and $\Delta_k = 0.2\sin(0.1k)$. In addition, we set $u_k = 10$, $x_0 = [0 \quad -1 \quad 2]^T$, $P_0 = 0.1I_{3\times3}$.

We propose to apply the proposed filter (ARThSKF) to obtain a robust estimation of simultaneous actuator and sensor faults. The obtained results will be compared with existing filters in literature, in particular the ARTSKF [10].

6.1. Case 1

The matrices distribution of the fault and the unknown disturbance are taken as follow:

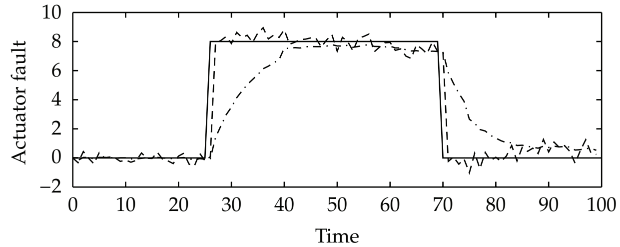
$$E_k^x = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, \qquad E_k^y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T,$$
$$F_k^a = \begin{bmatrix} 0.4252 & -0.0082 & 0.1813 \end{bmatrix}^T, \qquad F_k^s = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T. \tag{6.7}$$

In Figures 2 and 3, we focused on the presentation of the fault (f_k) and the unknown disturbance (d_k), respectively. We have plotted the actual value, the estimated values obtained by ARThSKF and ARTSKF.

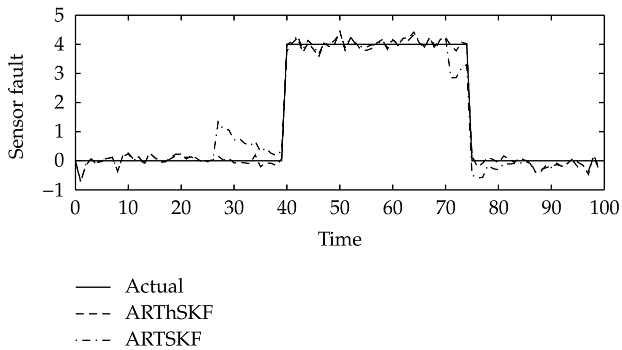
The simulation results in Table 1, show the average root mean square errors (RMSE) in the estimated states, fault and unknown disturbance. For example, the RMSE of the first component of state vector is calculated by

$$\text{RMSE}(x_{1,k}) = \sqrt{\frac{1}{N}\sum_{k=1}^{N}\left(x_{1,k} - \hat{x}_{1,k/k}\right)^2}. \tag{6.8}$$

From Figures 2, 3, and Table 1, the filtering performances of the ARThSKF appears to be much better that those of ARTSKF.



(a)



(b)

Figure 2: Actual fault f_k and estimated fault $\widehat{f}_{k/k}$ (Case 1).

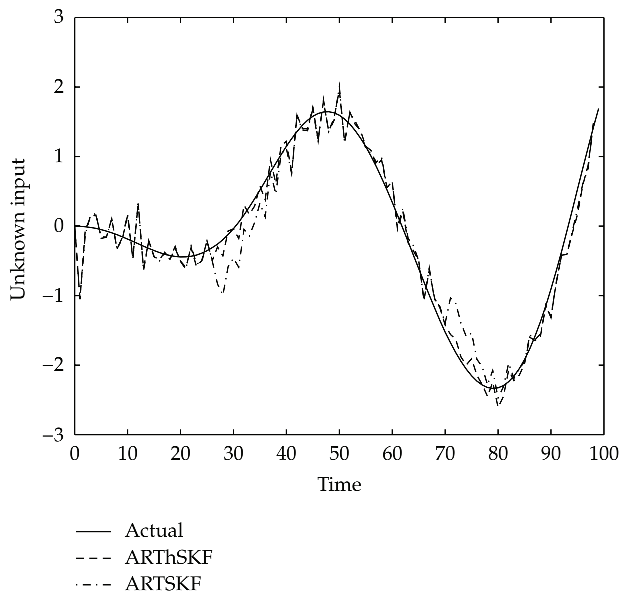


Figure 3: Actual unknown disturbance d_k and its estimated $\widehat{d}_{k/k}$ (Case 1).

Table 1: Performance of the ARThSKF and ARTSKF (Case 1).

RMSE	ARThSKF	ARTSKF
$x_{1,k}$	0.9731	10.0330
$x_{2,k}$	0.1418	0.1418
$x_{3,k}$	0.1827	0.3576
f_k^a	1.1796	2.7494
f_k^s	0.1887	0.3795
d_k	0.2225	0.3035

Table 2: Performance of the ARThSKF and ARTSKF (Case 2).

RMSE	ARThSKF	ARTSKF
$x_{1,k}$	1.2893	5.3844
$x_{2,k}$	0.1422	0.1418
$x_{3,k}$	0.5028	0.5920
f_k^a	2.5954	4.6809
f_k^s	0.5087	0.5944
d_k	1.2805	9.6895

6.2. Case 2

Here, we assume that the distribution matrices of the fault and the unknown disturbance are the followings:

$$\begin{aligned} E_k^x &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \qquad E_k^y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \\ F_k^a &= \begin{bmatrix} 0.4252 & -0.0082 & 0.1813 \end{bmatrix}^T, \qquad F_k^s = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T. \end{aligned} \tag{6.9}$$

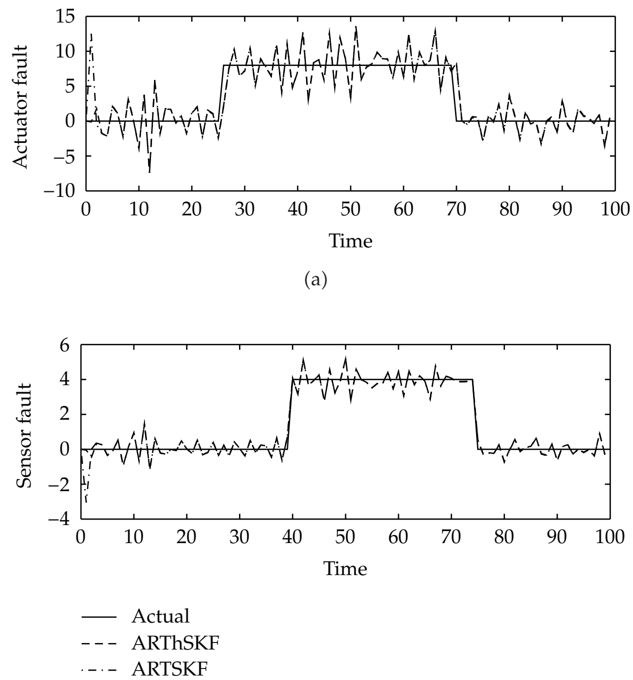
According to Table 2 and Figures 4 and 5, we note that the ARThSKF gives the best estimation of the first component of the state vector, the actuator fault and the unknown disturbance in comparison with the ARTSKF.

6.3. Case 3

In this case, the distribution matrices of the fault and the unknown disturbance are given by:

$$\begin{aligned} E_k^x &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \qquad E_k^y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \\ F_k^a &= \begin{bmatrix} 0.4252 & -0.0082 & 0.1813 \end{bmatrix}^T, \qquad F_k^s = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T. \end{aligned} \tag{6.10}$$

In Figures 6 and 7, we deduce that the ARThSKF and ARTSKF have similar filtering performances. Indeed, the evaluation of the RMSE values presented in the Table 3 confirms this deduction.



(a)

(b)

Figure 4: Actual fault f_k and estimated fault $\widehat{f}_{k/k}$ (Case 2).

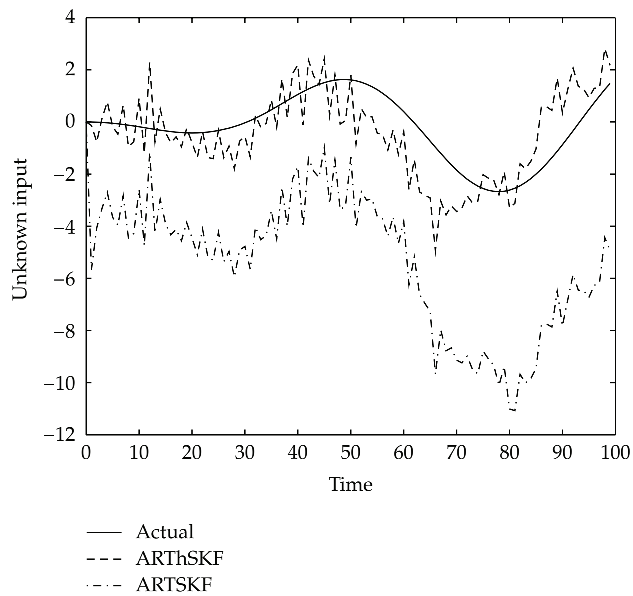
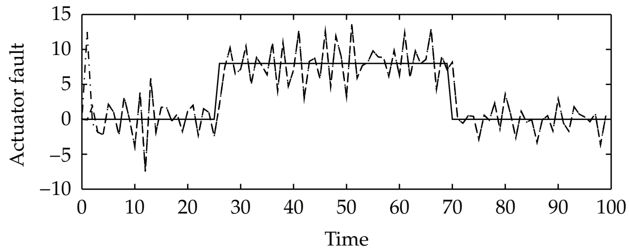
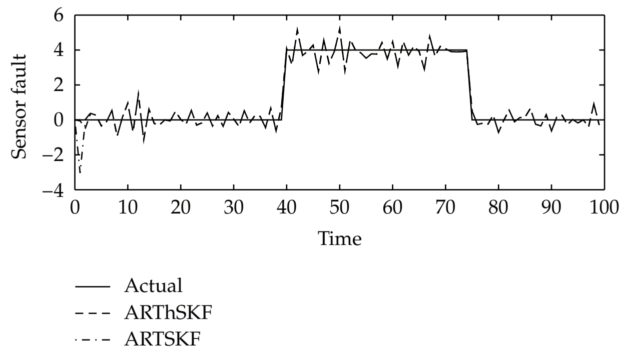


Figure 5: Actual unknown disturbance d_k and its estimated $\widehat{d}_{k/k}$ (Case 2).



(a)



(b)

Figure 6: Actual fault f_k and estimated fault $\widehat{f}_{k/k}$ (Case 3).

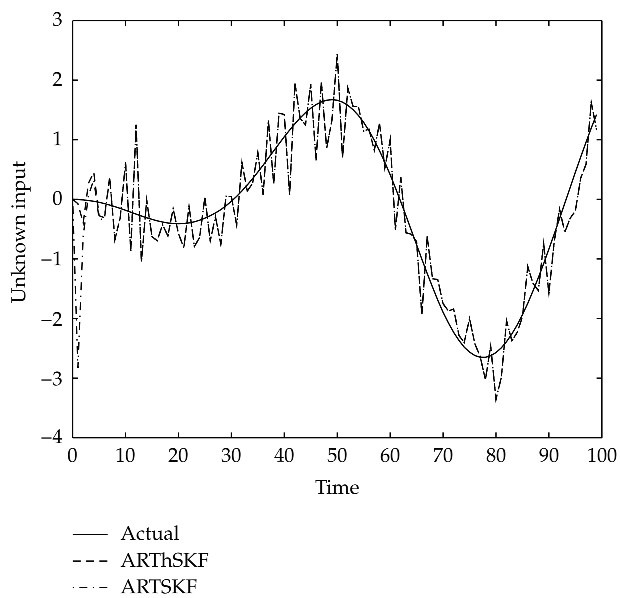


Figure 7: Actual unknown disturbance d_k and its estimated $\widehat{d}_{k/k}$ (Case 3).

Table 3: Performance of the ARThSKF and ARTSKF (Case 3).

RMSE	ARThSKF	ARTSKF
$x_{1,k}$	0.4566	0.5315
$x_{2,k}$	0.1424	0.1418
$x_{3,k}$	0.5072	0.5924
f_k^a	2.6148	4.6444
f_k^s	0.5130	0.5946
d_k	0.4569	0.5379

Table 4: Performance of the ARThSKF and ARTSKF (Case 4).

RMSE	ARThSKF	ARTSKF
$x_{1,k}$	2.7895	8.6757
$x_{2,k}$	0.1416	0.1418
$x_{3,k}$	0.1516	0.1516
f_k^a	2.3823	4.6188
f_k^s	3.7220	5.3151
d_k	0.4706	0.5691

6.4. Case 4

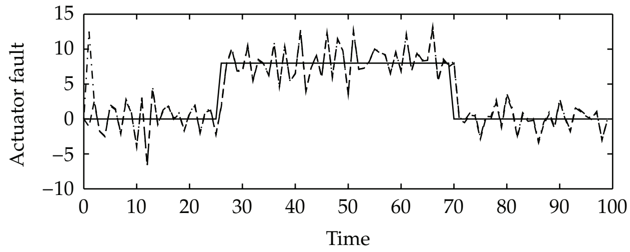
Now, we take the following values of the distrubition matrices of the fault and the unknown disturbance:

$$\begin{gathered} E_k^x = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \qquad E_k^y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \\ F_k^a = \begin{bmatrix} 0.4252 & -0.0082 & 0.1813 \end{bmatrix}^T, \qquad F_k^s = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T. \end{gathered} \tag{6.11}$$

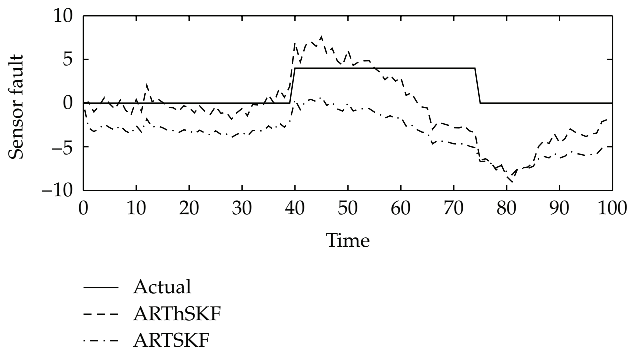
The simulation results presented in Figures 8 and 9 and Table 4 show that the ARThSKF is a little better than the ARTSKF. In this case, we notes that the two filters ARThSKF and ARTSKF do not estimate suitably the sensor fault, since $E_k^y = F_k^s$.

7. Conclusion

In this paper, the problem of joint state and fault estimation for linear discrete-time stochastic systems with unknown disturbance is solved by using the robust three-stage Kalman filtering technique. We assume that the fault and the unknown disturbance affect both the system state and the output. To achieve this aim, a new robust filter named ARThSKF has been proposed by using an optimal three-stage Kalman filtering method and an augmented fault and unknown disturbance models. The unbiasedness conditions and minimum-variance property of this filter are also established. The proposed filter is applied efficiently to solve two problems: Firstly, it estimates the actuator and sensor faults simultaneously; Secondly, it establishes a comparative study with the existing literature results.



(a)



(b)

Figure 8: Actual fault f_k and estimated fault $\widehat{f}_{k/k}$ (Case 4).

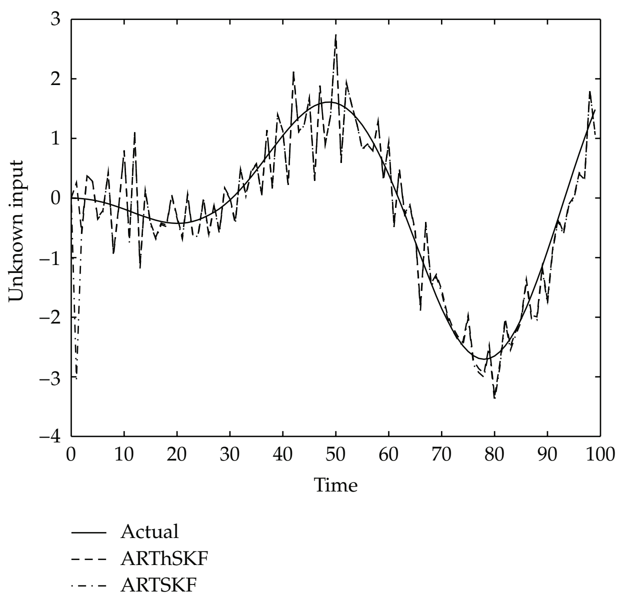


Figure 9: Actual unknown disturbance d_k and its estimated $\widehat{d}_{k/k}$ (Case 4).

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