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Research Article

Existence of Three Positive Solutions to Nonlinear Boundary Value Problems

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Criteria are established for existence of three solutions to the boundary value problems $Lx = f(t,x)$, $w_1x(0) - w_2x'(0) = 0 = w_3x(1) + w_4x'(1)$, where $Lx := -(px')' + qx$. Here, $p \in C^1[0,1]$, $p > 0$, $q \in C[0,1]$, $q \geq 0$.

1. Introduction

In this paper, we are concerned with the existence of three positive solutions for the boundary value problem (BVP)

$$Lx = f(t,x), \quad 0 < t < 1, \tag{1.1}$$

$$\begin{gathered} w_1x(0) - w_2x'(0) = 0, \\ w_3x(1) + w_4x'(1) = 0, \end{gathered} \tag{1.2}$$

where $f \in C([0,1] \times [0,+\infty)), [0,+\infty)$, $w_i \geq 0$ $(i = 1,\ldots,4)$ with $\rho := w_2w_3 + w_1w_3 + w_1w_4 > 0$ and $Lx := -(p(t)x')' + q(t)x$. Here $p \in C^1([0,1],(0,\infty)), q \in C([0,1],[0,\infty))$. We shall also assume that $\lambda = 0$ is not an eigenvalue of $Lx = \lambda x$ subject to conditions (1.2). As a consequence, it follows that the the smallest eigenvalue λ_1 of the problem $Lx = \lambda x$ subject to (1.2) satisfies $\lambda_1 > 0$ and the corresponding eigenfunction $\varphi_1(t)$ does not vanish on $(0,1)$. Without loss of generality, we may assume $\varphi_1(t) > 0$ on $(0,1)$ and $\|\varphi_1\| = \max_{0\leq t\leq 1}|\varphi_1(t)| = 1$.

Let $G(t,s)$ denote Green's function for the problem $Lx = 0$ subject to condition (1.2). It is well known that $G(t,s)$ may be written as

$$G(t,s) = \frac{1}{d}\begin{cases} \phi(t)\psi(s), & 0 \le t \le s \le 1, \\ \phi(s)\psi(t), & 0 \le s \le t \le 1, \end{cases} \tag{1.3}$$

where $\phi(t)$ and $\psi(t)$ satisfy

$$\begin{aligned} L\phi &= 0, \qquad \phi(0) = w_2, \qquad \phi'(0) = w_1, \\ L\psi &= 0, \qquad \psi(1) = w_4, \qquad \psi'(1) = -w_3, \end{aligned} \tag{1.4}$$

and where

$$p(t)\left(\phi(t)\psi'(t) - \phi'(t)\psi(t)\right) \equiv -d. \tag{1.5}$$

It may be shown that $\phi(t) \ge 0$ and is increasing on $[0,1]$ while $\psi(t) \ge 0$ and is decreasing on $[0,1]$. As a consequence, it follows that $d > 0$ and, furthermore, we have

$$0 \le G(t,s) \le G(s,s), \quad 0 \le t,s \le 1. \tag{1.6}$$

We define the positive number $\eta,\ \mu$ by

$$\eta^{-1} := \max_{0\le t\le 1}\left(\int_0^1 G(t,s)ds\right), \qquad \mu^{-1} := \int_{1/4}^{3/4} G(s,s)ds. \tag{1.7}$$

For the case $Lx = -x''$ (i.e, $p(t) \equiv 1, q(t) \equiv 0$), the corresponding BVP

$$-x'' = f(t,x), \quad 0 < t < 1 \tag{1.8}$$

subject to (1.2) has attracted considerable attention over the last number of years. Under certain condition, positive solutions of (1.8) and (1.2) are obtained in [1, 2]. In a recent paper, Erbe [3] investigated the existence of multiple positive solutions to (1.1)-(1.2) by applying the fixed point index.

The aim of this paper is to establish criteria for the existence of three positive solutions to (1.1) and (1.2), which improve the corresponding result of [3]. Our tool in this paper will be well-known Five Functionals Fixed Point Theorem [4–7].

2. Preliminaries

Definition 2.1. Suppose P is a cone in a Banach. The map α is a nonnegative continuous concave functional on P provided $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y) \tag{2.1}$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly, the map β is a nonnegative continuous convex functional on P provided $\beta : P \to [0,\infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y) \tag{2.2}$$

for all $x, y \in P$ and $t \in [0,1]$.

Let γ, β, θ be nonnegative, continuous, convex functionals on P and α, ψ be nonnegative, continuous, concave functionals on P. Then, for nonnegative real numbers h, a, b, d and c, we define the convex sets

$$\begin{aligned}
P(\gamma, c) &= \{x \in P : \gamma(x) < c\}, \\
P(\gamma, \alpha, a, c) &= \{x \in P : a \leq \alpha(x), \gamma(x) \leq c\}, \\
Q(\gamma, \beta, d, c) &= \{x \in P : \beta(x) \leq d, \gamma(x) \leq c\}, \\
P(\gamma, \theta, \alpha, a, b, c) &= \{x \in P : a \leq \alpha(x), \theta(x) \leq b, \gamma(x) \leq c\}, \\
Q(\gamma, \beta, \psi, h, d, c) &= \{x \in P : h \leq \psi(x), \beta(x) \leq d, \gamma(x) \leq c\}.
\end{aligned} \tag{2.3}$$

To prove our main results, we need the following theorem, which is the Five Functionals Fixed Point Theorem [4].

Theorem 2.2. *Let P be a cone in a real Banach space E. Suppose there exist positive numbers c and M, nonnegative, continuous, concave functionals α and ψ on P, and nonnegative, continuous, convex functionals γ, β and θ on P, with*

$$\alpha(x) \leq \beta(x), \qquad \|x\| \leq M\gamma(x) \tag{2.4}$$

for all $x \in \overline{P(\gamma, c)}$. Suppose

$$\Phi : \overline{P(\gamma, c)} \longrightarrow \overline{P(\gamma, c)} \tag{2.5}$$

is completely continuous and there exist nonnegative numbers h, a, k, b, with $0 < a < b$ such that

(i) *$\{x \in P(\gamma, \theta, \alpha, b, k, c) : \alpha(x) > b\} \neq \emptyset$ and $\alpha(\Phi x) > b$ for $x \in P(\gamma, \theta, \alpha, b, k, c)$;*

(ii) *$\{x \in Q(\gamma, \beta, \psi, h, a, c) : \beta(x) < a\} \neq \emptyset$ and $\beta(\Phi x) < a$ for $x \in Q(\gamma, \beta, \psi, h, a, c)$;*

(iii) *$\alpha(\Phi x) > b$ for $x \in P(\gamma, \alpha, b, c)$ with $\theta(\Phi x) > k$;*

(iv) *$\beta(\Phi x) < a$ for $x \in Q(\gamma, \beta, a, c)$ with $\psi(\Phi x) < h$.*

Then Φ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$\begin{gathered}
\beta(x_1) < a, \\
b < \alpha(x_2), \\
a < \beta(x_3) \quad \text{with } \alpha(x_3) < b.
\end{gathered} \tag{2.6}$$

3. Main Result

In this section, we shall obtain existence results for BVP (1.1) and (1.2) by using the Five Functional Fixed Point Theorem.

By [3], it is well known that BVP associated with (1.1), (1.2) is equivalent to the operator equation

$$x = Ax, \quad x \in C[0,1], \tag{3.1}$$

where

$$(Ax)(t) = \int_0^1 G(t,s) f(s, x(s)) ds. \tag{3.2}$$

Now with $X = C[0,1], \|x\| = \max_{0 \le t \le 1} |x(t)|$, it is easy to see that $A : X \to X$ is completely continuous. We define a cone $P \subset X$ by

$$P := \left\{ x \in X : x(t) \ge 0, \min_{1/4 \le t \le 3/4} x(t) \ge \sigma \|x\| \right\}, \tag{3.3}$$

where σ is defined by

$$\sigma := \min \left\{ \frac{G(t,s)}{G(s,s)} : \frac{1}{4} \le t \le \frac{3}{4},\ 0 \le s \le 1 \right\}. \tag{3.4}$$

By (1.3) and the properties of $\varphi(t),\ \psi(t)$, we have

$$\sigma := \min \left\{ \frac{\phi(1/4)}{\phi(1)}, \frac{\psi(3/4)}{\psi(0)} \right\}. \tag{3.5}$$

Clearly, $0 < \sigma < 1$ and $G(t,s) \ge \sigma G(s,s)$ for $1/4 \le t \le 3/4,\ 0 \le s \le 1$.

Lemma 3.1. *The operator A maps P into P.*

Proof. Let $x \in P$. From (1.6) and the condition of f, we see that $Ax \ge 0$. Next, for $x \in P$, we have

$$|(Ax)(t)| = (Ax)(t) = \int_0^1 G(t,s) f(s,x(s)) ds \le \int_0^1 G(s,s) f(s,x(s)) ds. \tag{3.6}$$

Hence,

$$\|Ax\| \le \int_0^1 G(s,s) f(s,x(s)) ds. \tag{3.7}$$

Now, from $G(t,s) \geq \sigma G(s,s)$ for $1/4 \leq t \leq 3/4,\ 0 \leq s \leq 1$, we have

$$\begin{aligned}\min_{t\in[1/4,3/4]}(Ax)(t) &= \min_{t\in[1/4,3/4]}\int_0^1 G(t,s)f(s,x(s))ds \\ &\geq \sigma\int_0^1 G(s,s)f(s,x(s))ds \geq \sigma\|Ax\|.\end{aligned} \tag{3.8}$$

This show that $Ax \in P$, which completes this proof. □

Theorem 3.2. *Let $0 < a < b$ and $\mu b < \eta\sigma c$, and suppose $f(t,x)$ satisfies the following conditions:*

(H1) $f(t,x) < \eta a$ *for* $0 \leq t \leq 1$ *and* $0 \leq x \leq a$,

(H2) $f(t,x) \geq (\mu b)/\sigma$ *for* $1/4 \leq t \leq 3/4$ *and* $b \leq x \leq b/\sigma$,

(H3) $f(t,x) \leq \eta c$ *for* $0 \leq t \leq 1$ *and* $0 \leq x \leq c$.

Then the BVP (1.1)-(1.2) *has at least three positive solutions.*

Proof. Theorem 2.2 will be applied. We begin by defining the nonnegative continuous concave functional α, ψ and the nonnegative continuous convex functional β, θ, γ on P

$$\begin{aligned}\psi(x) &= \min_{t\in[0,1]} x(t), \\ \beta(x) = \theta(x) &= \max_{t\in[0,1]} x(t), \\ \alpha(x) = \min_{t\in[1/4,3/4]} x(t), &\qquad \gamma(x) = \|x\|.\end{aligned} \tag{3.9}$$

It is clear that $\alpha(x) \leq \beta(x)$ for all $x \in P$.

First, we shall show that the operator A maps $\overline{P(\gamma,c)}$ into $\overline{P(\gamma,c)}$. Let $x \in \overline{P(\gamma,c)}$. Thus we have $0 \leq x(t) \leq c$ for $0 \leq t \leq 1$. Using (H3), we have

$$|(Ax)(t)| = (Ax)(t) = \int_0^1 G(t,s)f(s,x(s))ds \leq \eta c\int_0^1 G(t,s)ds \leq c. \tag{3.10}$$

Hence

$$\gamma(Ax) = \|Ax\| \leq c. \tag{3.11}$$

Therefore, we have shown that $A : \overline{P(\gamma,c)} \rightarrow \overline{P(\gamma,c)}$.

We next prove that Condition (i) of Theorem 2.2 holds. Let $x \equiv (1/2)(b+k), k = b/\sigma$. Then

$$\alpha(x) = \frac{1}{2}(b+k) > b, \qquad \theta(x) = \frac{1}{2}(b+k) < k, \qquad \gamma(x) = \frac{1}{2}(b+k) < c, \tag{3.12}$$

which shows that $\{x \in P(\gamma, \theta, \alpha, b, k, c), \alpha(x) > b\} \neq \emptyset$. Let $x \in P(\gamma, \theta, \alpha, b, k, c)$. Then $\alpha(x) > b$, $\theta(x) < k = b/\sigma$ imply that

$$b < x(t) < \frac{b}{\sigma}, \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{3.13}$$

By (H2) we can obtain

$$\begin{aligned}
\alpha(Ax) &= \min_{t\in[1/4,3/4]} (Ax)(t) = \min_{t\in[1/4,3/4]} \int_0^1 G(t,s) f(s,x(s))ds \\
&\geq \sigma \int_0^1 G(s,s) f(s,x(s))ds > \sigma \int_{1/4}^{3/4} G(s,s) f(s,x(s))ds \\
&\geq \mu b \int_{1/4}^{3/4} G(s,s)ds = b.
\end{aligned} \tag{3.14}$$

Hence, $\alpha(Ax) > b$ for all $x \in P(\gamma, \theta, \alpha, b, k, c)$ and so Condition (i) of Theorem 2.2 holds.

Next, we verify that Condition (ii) of Theorem 2.2 is satisfied. Take $x \equiv \sigma a, h = \sigma a$, then

$$\gamma(x) = \sigma a < c, \qquad \psi(x) = \sigma a = h, \qquad \beta(x) = \sigma a < a. \tag{3.15}$$

From this we know that $\{x \in Q(\gamma, \beta, \psi, h, a, c), \beta(x) < a\} \neq \emptyset$. Let $x \in Q(\gamma, \beta, \psi, h, a, c)$. Then we have $\beta(x) \leq a$, which lead to $0 \leq x(t) \leq a$, for $t \in [0,1]$. In view of (H1), we have

$$\begin{aligned}
\beta(Ax) &= \max_{t\in[0,1]} (Ax)(t) = \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,x(s))ds \\
&\leq \eta a \cdot \max_{t\in[0,1]} \int_0^1 G(t,s)ds = a.
\end{aligned} \tag{3.16}$$

Hence, $\beta(Ax) < a$ for all $x \in Q(\gamma, \beta, \psi, h, \mathrm{a}, c)$. Thus, Condition (ii) of Theorem 2.2 is fulfilled.

We shall next show that Condition (iii) of Theorem 2.2 is met. Observe that for $x \in P$

$$\begin{aligned}
\theta(Ax) &= \max_{t\in[0,1]} (Ax)(t) = \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,x(s))ds \\
&\leq \int_0^1 G(s,s) f(s,x(s))ds.
\end{aligned} \tag{3.17}$$

On the other hand,

$$\begin{aligned}\alpha(Ax) &= \min_{t\in[1/4,3/4]}(Ax)(t) = \min_{t\in[1/4,3/4]}\int_0^1 G(t,s)f(s,x(s))ds \\ &\geq \sigma\int_0^1 G(s,s)f(s,x(s))ds.\end{aligned} \tag{3.18}$$

(3.17) together (3.18) implies that

$$\alpha(Ax) \geq \sigma\theta(Ax), \quad x \in P. \tag{3.19}$$

Let $x \in P(\gamma,\alpha,b,c)$ with $\theta(Ax) > k = b/\sigma$. Then, it follows from (3.19) that

$$\alpha(Ax) \geq \sigma\theta(Ax) > b. \tag{3.20}$$

Thus, $\alpha(Ax) > b$ for all $x \in P(\gamma,\alpha,b,c)$ with $\theta(Ax) > b/\sigma$. Hence, Condition (iii) of Theorem 2.2 holds.

Finally, we shall prove that Condition (iv) of Theorem 2.2 is fulfilled. Let $x \in Q(\gamma,\beta,a,c)$ and $\psi(Ax) < h = \sigma a$. Then $0 \leq x(t) \leq a,\ t \in [0,1]$. By (H1), we have

$$\begin{aligned}\beta(Ax) &= \max_{t\in[0,1]}(Ax)(t) = \max_{t\in[0,1]}\int_0^1 G(t,s)f(s,x(s))ds \\ &< \eta a \cdot \max_{t\in[0,1]}\int_0^1 G(t,s)ds = a.\end{aligned} \tag{3.21}$$

Thus, Condition (iv) of Theorem 2.2 is satisfied.

Now, an application of Theorem 2.2 ensures that the BVP (1.1) and (1.2) has at least three positive solutions x_1, x_2, x_3 such that

$$\beta(x_1) < a, \quad b < \alpha(x_2), \quad a < \beta(x_3) \quad \text{with } \alpha(x_3) < b. \tag{3.22}$$

This proof is complete. □

Remark 3.3. This Theorem improves the Corollary 2.5 in [3].

Example 3.4. For simplicity, we consider the boundary value problem

$$\begin{gathered}-x'' = f(t,x), \quad 0 < t < 1, \\ x(0) = x'(1) = 0,\end{gathered} \tag{3.23}$$

where

$$f(t,x)=\begin{cases}\dfrac{1}{10}|\sin t|+20x^5, & x\le 1,\\[2mm] \dfrac{1}{10}|\sin t|+20, & x\ge 1.\end{cases} \tag{3.24}$$

By direct calculation we can obtain that $\eta=2,\ \mu=4,\ \sigma=1/4$. Set $a=1/2,\ b=1,\ c=12$, so the nonlinear term f satisfies

$$\begin{gathered} f(t,x)\le 0.1+20\times\left(\frac{1}{2}\right)^5<1=\eta a,\quad (t,x)\in[0,1]\times\left[0,\frac{1}{2}\right],\\ f(t,x)>20>16=\frac{(\mu b)}{\sigma},\quad (t,x)\in\left[\frac{1}{4},\frac{3}{4}\right]\times[1,4],\\ f(t,x)<21<24=\eta c,\quad (t,x)\in[0,1]\times[0,12]. \end{gathered} \tag{3.25}$$

Then the conditions in Theorem 3.2 are all satisfied, so the boundary value problem (3.23) has at least three positive solutions x_1, x_2, x_3 such that

$$\max_{0\le t\le 1} x_1(t)<\frac{1}{2},\quad 1<\min_{1/4\le t\le 3/4} x_2(t),\quad 1<\max_{0\le t\le 1} x_3(t)\quad \text{with}\quad \min_{1/4\le t\le 3/4} x_3(t)<1. \tag{3.26}$$

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