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*Research Article*

# A Note on Periodic Solutions of Second Order Nonautonomous Singular Coupled Systems

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We establish the existence of periodic solutions of the second order nonautonomous singular coupled systems $x'' + a_1(t)x = f_1(t, y(t)) + e_1(t)$ for a.e. $t \in [0, T]$, $y'' + a_2(t)y = f_2(t, x(t)) + e_2(t)$ for a.e. $t \in [0, T]$. The proof relies on Schauder's fixed point theorem.

## 1. Introduction

Some classical tools have been used in the literature to study the positive solutions for two-point boundary value problems of a coupled system of differential equations. These classical tools include some fixed point theorems in cones for completely continuous operators and Leray-Schauder fixed point theorem; for examples, see [1–3] and literatures therein.

Recently, Schauder's fixed point theorem has been used to study the existence of positive solutions of periodic boundary value problems in several papers; see, for example, Torres [4], Chu et al. [5, 6], Cao and Jiang [7], and the references contained therein. However, there are few works on periodic solutions of second-order nonautonomous singular coupled systems. In these papers above, there are the major assumption that their associated Green's functions are positive. Since Green's functions are positive, in the paper, we continue to study the existence of periodic solutions to second-order nonautonomous singular coupled systems in the following form:

$$\begin{aligned} x'' + a_1(t)x &= f_1\big(t, y(t)\big) + e_1(t) \quad \text{for a.e. } t \in [0, T], \\ y'' + a_2(t)y &= f_2(t, x(t)) + e_2(t) \quad \text{for a.e. } t \in [0, T], \end{aligned} \tag{1.1}$$

with $a_1, a_2, e_1, e_2 \in L^1[0,T]$, $f_1, f_2 \in \mathrm{Car}([0,T] \times (0,+\infty), (0,+\infty))$. Here we write $f \in \mathrm{Car}([0,T] \times (0,+\infty), (0,+\infty))$ if $f : [0,T] \times (0,+\infty) \to (0,+\infty)$ is an $L^1$-caratheodory function, that is, the map $x \mapsto f(t,x)$ is continuous for a.e. $t \in (0,1)$ and the map $t \mapsto f(t,x)$ is measurable for all $x \in (0,+\infty)$, and for every $0 < r < s$ there exists $h_{r,s} \in L^1(0,T)$ such that $|f(t,x)| \le h_{r,s}(t)$ for all $x \in [r,s]$ and a.e. $t \in [0,T]$; here "for a.e." means "for almost every".

This paper is mainly motivated by the recent papers [4–6, 8, 9], in which the periodic singular problems have been studied. Some results in [4–6, 9] prove that in some situations weak singularities may help create periodic solutions. In [6], the authors consider the periodic solutions of second-order nonautonomous singular dynamical systems, in which the scalar periodic singular problems have been studied by Leray-Schauder alternative principle, a well-known fixed point theorem in cones, and Schauder's fixed point theorem, respectively.

The remaining part of the paper is organized as follows. In Section 2, some preliminary results will be given. In Sections 3–5, by employing a basic application of Schauder's fixed point theorem, we state and prove the existence results for (1.1) under the nonnegative of the Green's function associated with (2.1)-(2.2). Our view point sheds some new light on problems with weak force potentials and proves that in some situations weak singularities may stimulate the existence of periodic solutions, just as pointed out in [9] for the scalar case.

To illustrate our results, for example, we can select the system

$$\begin{aligned} x'' + a_1(t)x &= y^{-\alpha_1} + e_1(t), \\ y'' + a_2(t)y &= x^{-\alpha_2} + e_2(t), \end{aligned} \tag{1.2}$$

with $a_1, a_2, e_1, e_2 \in \mathbb{C}[0,T]$, $0 < \alpha_i < 1$, $i = 1,2$. Here we emphasize that in the new results $e_1, e_2$ do not need to be positive.

Let us fix some notation to be used in the following: given $a \in L^1(0,1)$, we write $a \succ 0$ if $a \ge 0$ for a.e. $t \in [0,1]$ and it is positive in a set of positive measures. For a given function $p \in L^1[0,T]$, we denote the essential supremum and infimum by $p^*$ and $p_*$, if they exist. The usual $L^p$-norm is denoted by $\|\cdot\|_p$. The conjugate exponent of $p$ is denoted by $\tilde{p} : 1/p + 1/\tilde{p} = 1$.

## 2. Preliminaries

We consider the scalar equation

$$x'' + a_i(t)x = e_i(t), \quad i = 1,2, \tag{2.1}$$

with periodic boundary conditions

$$x(0) = x(T), \qquad x'(0) = x'(T). \tag{2.2}$$

In this paper, we assume that the following standing hypothesis is satisfied.

$(H_1)$ The Green function $G_i(t,s)$, associated with (2.1)-(2.2), is nonnegative for all $(t,s) \in [0,T] \times [0,T]$, $i = 1,2$.

In other words, the (strict) antimaximum principle holds for (2.1)-(2.2). Under the conditions $(H_1)$, the solution of (2.1)-(2.2) is given by

$$x(t) = \int_0^T G_i(t,s)e_i(s)ds. \tag{2.3}$$

For a nonconstant function $a(t)$, there is an $L^p$-criterion proved in [9], which is given in the following lemma for the sake of completeness. Let $\mathbf{K}(q)$ denote the best Sobolev constant in the following inequality:

$$C\|u\|_q^2 \le \|u'\|_2^2, \quad \forall u \in H_0^1(0,T). \tag{2.4}$$

The explicit formula for $\mathbf{K}(q)$ is

$$\mathbf{K}(q) = \begin{cases} \dfrac{2\pi}{qT^{1+2/q}}\left(\dfrac{2}{2+q}\right)^{1-2/q}\left(\dfrac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\right)^2 & \text{if } 1 \le q < \infty, \\ \dfrac{4}{T} & \text{if } q = \infty, \end{cases} \tag{2.5}$$

where $\Gamma$ is the Gamma function. See [10].

**Lemma 2.1.** *For each $i = 1,2$, assume that $a_i(t) \succ 0$ and $a_i \in L^p[0,T]$ for some $1 \le p \le \infty$. If*

$$\|a_i\|_p \le \mathbf{K}(2\tilde{p}), \tag{2.6}$$

*then the standing hypothesis $(H_1)$ holds.*

We define the function $\gamma_i : \mathbb{R} \to \mathbb{R}$ by

$$\gamma_i(t) = \int_0^T G_i(t,s)e_i(s)ds, \quad i = 1,2, \tag{2.7}$$

which is the unique $T$-periodic solution of

$$x'' + a_i(t)x = e_i(t). \tag{2.8}$$

Throughout this paper, we use the following notations:

$$\gamma_{i*} = \min_{i,t} \gamma_i(t), \qquad \gamma_i^* = \max_{i,t} \gamma_i(t). \tag{2.9}$$

## 3. The Case $\gamma_{1*} \geq 0,\ \gamma_{2*} \geq 0$

**Theorem 3.1.** *Assume that* $(H_1)$ *is satisfied; furthermore, we assume that there exist* $b_i \succ 0$, $\widehat{b}_i \succ 0$, *and* $0 < \alpha_i < 1$ *such that*

$(H2)$

$$0 \leq \frac{\widehat{b}_i(t)}{x^{\alpha_i}} \leq f_i(t,x) \leq \frac{b_i(t)}{x^{\alpha_i}}, \quad \forall x > 0,\ a.e.\ t \in (0,T),\ i = 1,2. \tag{3.1}$$

*If* $\gamma_{1*} \geq 0$, $\gamma_{2*} \geq 0$, *then there exists a positive T-periodic solution of* (1.1).

*Proof.* A $T$-periodic solution of (1.1) is just a fixed point of the completely continuous map $A(x,y) = (Ax, Ay) : C_T \times C_T \to C_T \times C_T$ defined as

$$\begin{aligned}(Ax)(t) &:= \int_0^T G_1(t,s)\left[f_1(s,y(s)) + e_1(s)\right]ds \\ &= \int_0^T G_1(t,s) f_1(s,y(s))ds + \gamma_1(t), \\ (Ay)(t) &:= \int_0^T G_2(t,s)\left[f_2(s,x(s)) + e_2(s)\right]ds \\ &= \int_0^T G_2(t,s) f_2(s,x(s))ds + \gamma_2(t).\end{aligned} \tag{3.2}$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove that $A$ maps the closed convex set defined as

$$K = \left\{(x,y) \in C_T \times C_T : r_1 \leq x(t) \leq R_1, r_2 \leq y(t) \leq R_2,\ \forall t \in [0,T]\right\}, \tag{3.3}$$

into itself, where $R_1 > r_1 > 0$, $R_2 > r_2 > 0$ are positive constants to be fixed properly. For convenience, we introduce the following notations:

$$\beta_i(t) = \int_0^T G_i(t,s) b_i(s)ds, \qquad \widehat{\beta}_i(t) = \int_0^T G_i(t,s)\widehat{b}_i(s)ds, \quad i = 1,2. \tag{3.4}$$

Given $(x,y) \in K$, by the nonnegative sign of $G_i$ and $f_i$, $i = 1,2$, we have

$$\begin{aligned}(Ax)(t) &= \int_0^T G_1(t,s) f_1(s,y(s))ds + \gamma_1(t) \\ &\geq \int_0^T G_1(t,s)\frac{\widehat{b}_1(s)}{y^{\alpha_1}(s)}ds \geq \int_0^T G_1(t,s)\frac{\widehat{b}_1(s)}{R_2^{\alpha_1}}ds \geq \widehat{\beta}_{1*} \cdot \frac{1}{R_2^{\alpha_1}},\end{aligned} \tag{3.5}$$

and note for every $(x,y) \in K$ that

$$\begin{aligned}(Ax)(t) &= \int_0^T G_1(t,s) f_1(s, y(s))\,ds + \gamma_1(t) \\ &\le \int_0^T G_1(t,s)\frac{b_1(s)}{y^{\alpha_1}(s)}ds + \gamma_1^* \le \int_0^T G_1(t,s)\frac{b_1(s)}{r_2^{\alpha_1}}ds + \gamma_1^* \le \beta_1^* \cdot \frac{1}{r_2^{\alpha_1}} + \gamma_1^*.\end{aligned} \tag{3.6}$$

Also, follow the same strategy,

$$\begin{aligned}(Ay)(t) &= \int_0^T G_2(t,s) f_2(s, x(s))\,ds + \gamma_2(t) \\ &\ge \int_0^T G_2(t,s)\frac{\widehat{b}_2(s)}{x^{\alpha_2}(s)}ds \ge \int_0^T G_2(t,s)\frac{\widehat{b}_2(s)}{R_1^{\alpha_2}}ds \ge \widehat{\beta}_{2*} \cdot \frac{1}{R_1^{\alpha_2}}, \\ (Ay)(t) &= \int_0^T G_2(t,s) f_2(s, x(s))\,ds + \gamma_2(t) \\ &\le \int_0^T G_2(t,s)\frac{b_2(s)}{x^{\alpha_2}(s)}ds + \gamma_2^* \le \int_0^T G_2(t,s)\frac{b_2(s)}{r_1^{\alpha_2}}ds + \gamma_2^* \le \beta_2^* \cdot \frac{1}{r_1^{\alpha_2}} + \gamma_2^*.\end{aligned} \tag{3.7}$$

Thus $(Ax, Ay) \in K$ if $r_1, r_2, R_1$, and $R_2$ are chosen so that

$$\begin{aligned}&\widehat{\beta}_{1*} \cdot \frac{1}{R_2^{\alpha_1}} \ge r_1, \qquad \beta_1^* \cdot \frac{1}{r_2^{\alpha_1}} + \gamma_1^* \le R_1, \\ &\widehat{\beta}_{2*} \cdot \frac{1}{R_1^{\alpha_2}} \ge r_2, \qquad \beta_2^* \cdot \frac{1}{r_1^{\alpha_2}} + \gamma_2^* \le R_2.\end{aligned} \tag{3.8}$$

Note that $\widehat{\beta}_{i*}, \beta_{i*} > 0$ and taking $R = R_1 = R_2$, $r = r_1 = r_2$, $r = 1/R$, it is sufficient to find $R > 1$ such that

$$\begin{aligned}&\widehat{\beta}_{1*} \cdot R^{1-\alpha_1} \ge 1, \qquad \beta_1^* \cdot R^{\alpha_1} + \gamma_1^* \le R, \\ &\widehat{\beta}_{2*} \cdot R^{1-\alpha_2} \ge 1, \qquad \beta_2^* \cdot R^{\alpha_2} + \gamma_2^* \le R,\end{aligned} \tag{3.9}$$

and these inequalities hold for $R$ being big enough because $\alpha_i < 1$. □

## 4. The Case $\gamma_{1*} < 0 < \gamma_1^*,\ \gamma_{2*} < 0 < \gamma_2^*$

**Theorem 4.1.** *Assume* $(H_1)$ *and* $(H_2)$ *are satisfied. If* $\gamma_{1*} < 0 < \gamma_1^*$, $\gamma_{2*} < 0 < \gamma_2^*$, *and*

$$\begin{aligned} \gamma_{1*} &\geq r_{10} - \widehat{\beta}_{1*} \cdot \frac{r_{10}^{\alpha_1\alpha_2}}{\left(\beta_2^* + \gamma_2^* r_{10}^{\alpha_2}\right)^{\alpha_1}}, \\ \gamma_{2*} &\geq r_{20} - \widehat{\beta}_{2*} \cdot \frac{r_{20}^{\alpha_1\alpha_2}}{\left(\beta_1^* + \gamma_1^* r_{20}^{\alpha_1}\right)^{\alpha_2}}, \end{aligned} \tag{4.1}$$

*where* $0 < r_{10} < +\infty$ *is a unique positive solution of the equation*

$$r_1^{1-\alpha_1\alpha_2}\left(\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2}\right)^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*}, \tag{4.2}$$

*and* $0 < r_{20} < +\infty$ *is a unique positive solution of the equation*

$$r_2^{1-\alpha_1\alpha_2}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\widehat{\beta}_{2*}, \tag{4.3}$$

*then there exists a positive T-periodic solution of* (1.1).

*Proof.* We follow the same strategy and notation as in the proof of ahead theorem. In this case, to prove that $A : K \to K$, it is sufficient to find $r_1 < R_1$, $r_2 < R_2$ such that

$$\frac{\widehat{\beta}_{1*}}{R_2^{\alpha_1}} + \gamma_{1*} \geq r_1, \qquad \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^* \leq R_1, \tag{4.4}$$

$$\frac{\widehat{\beta}_{2*}}{R_1^{\alpha_2}} + \gamma_{2*} \geq r_2, \qquad \frac{\beta_2^*}{r_1^{\alpha_2}} + \gamma_2^* \leq R_2. \tag{4.5}$$

If we fix $R_1 = \beta_1^*/r_2^{\alpha_1} + \gamma_1^*$, $R_2 = \beta_2^*/r_1^{\alpha_2} + \gamma_2^*$, then the first inequality of (4.5) holds if $r_2$ satisfies

$$\gamma_{2*} \geq g(r_2) := r_2 - \widehat{\beta}_{2*} \cdot \frac{r_2^{\alpha_1\alpha_2}}{\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{\alpha_2}}. \tag{4.6}$$

According to

$$\begin{aligned} g'(r_2) &= 1 - \widehat{\beta}_{2*} \cdot \frac{1}{\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{2\alpha_2}} \\ &\quad \cdot \left[\alpha_1\alpha_2 r_2^{\alpha_1\alpha_2-1}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{\alpha_2} - r_2^{\alpha_1\alpha_2}\alpha_2\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{\alpha_2-1}\alpha_1\gamma_1^* r_2^{\alpha_1-1}\right] \\ &= 1 - \frac{\widehat{\beta}_{2*}\alpha_1\alpha_2 r_2^{\alpha_1\alpha_2-1}}{\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{\alpha_2}}\left[1 - \frac{r_2^{\alpha_1}\gamma_1^*}{\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}}\right] \\ &= 1 - \alpha_1\alpha_2\beta_1^*\widehat{\beta}_{2*} r_2^{\alpha_1\alpha_2-1}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{-1-\alpha_2}, \end{aligned} \tag{4.7}$$

we have $g'(0) = -\infty$, $g'(+\infty) = 1$; then there exists $r_{20}$ such that $g'(r_{20}) = 0$, and

$$\begin{aligned} g''(r_2) = -\Big[&\alpha_1\alpha_2\beta_1^*\widehat{\beta}_{2*}(\alpha_1\alpha_2 - 1)r_2^{\alpha_1\alpha_2-2}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{-1-\alpha_2} \\ &+\alpha_1\alpha_2\beta_1^*\widehat{\beta}_{2*}r_2^{\alpha_1\alpha_2-1}(-1-\alpha_2)\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{-2-\alpha_2}\gamma_1^*\alpha_1 r_2^{\alpha_1-1}\Big] > 0. \end{aligned} \tag{4.8}$$

Then the function $g(r_2)$ possesses a minimum at $r_{20}$, that is, $g(r_{20}) = \min_{r_2\epsilon(0,+\infty)} g(r_2)$. Note $g'(r_{20}) = 0$; then we have

$$1 - \alpha_1\alpha_2\beta_1^*\widehat{\beta}_{2*}r_{21}^{\alpha_1\alpha_2-1}\left(\beta_1^* + \gamma_1^* \cdot r_{21}^{\alpha_1}\right)^{-1-\alpha_2} = 0, \tag{4.9}$$

or equivalently,

$$r_{20}^{1-\alpha_1\alpha_2}\left(\beta_1^* + \gamma_1^* \cdot r_{20}^{\alpha_1}\right)^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\widehat{\beta}_{2*}. \tag{4.10}$$

Similarly,

$$\gamma_{1*} \geq g(r_1) := r_1 - \widehat{\beta}_{1*} \cdot \frac{r_1^{\alpha_1\alpha_2}}{\left(\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2}\right)^{\alpha_1}}. \tag{4.11}$$

$g(r_{10}) = \min_{r_1\epsilon(0,+\infty)} g(r_1)$, and

$$r_{10}^{1-\alpha_1\alpha_2}\left(\beta_2^* + \gamma_2^* \cdot r_{10}^{\alpha_2}\right)^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*}. \tag{4.12}$$

Taking $r_1 = r_{10}$ and $r_2 = r_{20}$, then the first inequality in (4.4) and (4.5) holds if $\gamma_{1*} \geq g(r_{10})$, $\gamma_{2*} \geq g(r_{20})$, which are just condition (4.1). The second inequalities hold directly by the choice of $R_1$ and $R_2$, and it would remain to prove that $r_{10} < R_1$ and $r_{20} < R_2$. This is easily verified through elementary computations

$$\begin{aligned} R_1 &= \frac{\beta_1^*}{r_{20}^{\alpha_1}} + \gamma_1^* = \frac{\beta_1^* + \gamma_1^* \cdot r_{20}^{\alpha_1}}{r_{20}^{\alpha_1}} \\ &= \frac{\left(\alpha_1\alpha_2\beta_1^*\widehat{\beta}_{2*}\right)^{1/(1+\alpha_2)} \cdot r_{20}^{(\alpha_1\alpha_2-1)/(1+\alpha_2)}}{r_{20}^{\alpha_1}} \\ &= \left(\alpha_1\alpha_2\beta_1^*\widehat{\beta}_{2*}\right)^{1/(1+\alpha_2)} \cdot r_{20}^{-(1+\alpha_1)/(1+\alpha_2)}. \end{aligned} \tag{4.13}$$

The proof is the same as that in $R_1$,$R_2 = (\alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*})^{1/(1+\alpha_1)} \cdot r_{10}^{-(1+\alpha_2)/(1+\alpha_1)}$.

Next,we will prove $r_{10} < R_1$, $r_{20} < R_2$, or equivalently,

$$r_{10} r_{20}^{(1+\alpha_1)/(1+\alpha_2)} < \left(\alpha_1 \alpha_2 \beta_1^* \widehat{\beta}_{2*}\right)^{1/(1+\alpha_2)},$$
$$r_{20} r_{10}^{(1+\alpha_2)/(1+\alpha_1)} < \left(\alpha_1 \alpha_2 \beta_2^* \widehat{\beta}_{1*}\right)^{1/(1+\alpha_1)}. \tag{4.14}$$

Namely,

$$r_{10}^{1+\alpha_2} r_{20}^{1+\alpha_1} < \alpha_1 \alpha_2 \beta_1^* \widehat{\beta}_{2*}, \qquad r_{20}^{1+\alpha_1} r_{10}^{1+\alpha_2} < \alpha_1 \alpha_2 \beta_2^* \widehat{\beta}_{1*}. \tag{4.15}$$

On the other hand,

$$r_{20}^{1-\alpha_1\alpha_2} \left(\beta_1^*\right)^{1+\alpha_2} \le \alpha_1 \alpha_2 \beta_1^* \widehat{\beta}_{2*}. \tag{4.16}$$

Then

$$r_{20} \le \left(\alpha_1 \alpha_2 \left(\beta_1^*\right)^{-\alpha_2} \widehat{\beta}_{2*}\right)^{1/(1-\alpha_1\alpha_2)}. \tag{4.17}$$

Similarly,

$$r_{10} \le \left(\alpha_1 \alpha_2 \left(\beta_2^*\right)^{-\alpha_1} \widehat{\beta}_{1*}\right)^{1/(1-\alpha_1\alpha_2)}. \tag{4.18}$$

By (4.17) and (4.18),

$$r_{10}^{1+\alpha_2} r_{20}^{1+\alpha_1} \le \left(\alpha_1 \alpha_2 \left(\beta_2^*\right)^{-\alpha_1} \widehat{\beta}_{1*}\right)^{(1+\alpha_2)/(1-\alpha_1\alpha_2)} \left(\alpha_1 \alpha_2 \left(\beta_1^*\right)^{-\alpha_2} \widehat{\beta}_{2*}\right)^{(1+\alpha_1)/(1-\alpha_1\alpha_2)}. \tag{4.19}$$

Now if we can prove

$$\left(\alpha_1 \alpha_2 \left(\beta_2^*\right)^{-\alpha_1} \widehat{\beta}_{1*}\right)^{(1+\alpha_2)/(1-\alpha_1\alpha_2)} \left(\alpha_1 \alpha_2 \left(\beta_1^*\right)^{-\alpha_2} \widehat{\beta}_{2*}\right)^{(1+\alpha_1)/(1-\alpha_1\alpha_2)} < \alpha_1 \alpha_2 \beta_1^* \widehat{\beta}_{2*}, \tag{4.20}$$

then

$$r_{10}^{1+\alpha_2} r_{20}^{1+\alpha_1} < \alpha_1 \alpha_2 \beta_1^* \widehat{\beta}_{2*}. \tag{4.21}$$

In fact,

$$\left(\alpha_1 \alpha_2\right)^{(2+\alpha_2+\alpha_1-1)/(1-\alpha_1\alpha_2)} \cdot \left(\frac{\widehat{\beta}_{1*}}{\beta_1^*}\right)^{(1+\alpha_2)/(1-\alpha_1\alpha_2)} \cdot \left(\frac{\widehat{\beta}_{2*}}{\beta_2^*}\right)^{\alpha_1(1+\alpha_2)/(1-\alpha_1\alpha_2)} < 1, \tag{4.22}$$

since $\widehat{\beta} i* \le \beta_i^*$, $i = 1, 2$. Similarly, we have $r_{20}^{1+\alpha_1} r_{10}^{1+\alpha_2} < \alpha_1 \alpha_2 \beta_2^* \widehat{\beta}_{1*}$; we omit the details. Now we can obtain $r_{10} < R_1$, $r_{20} < R_2$. The proof is complete. □

## 5. The Case $\gamma_1^* \le 0,\ \gamma_{2*} < 0 < \gamma_2^*\ (\gamma_2^* \le 0,\ \gamma_{1*} < 0 < \gamma_1^*)$

**Theorem 5.1.** *Assume* $(H_1)$ *and* $(H_2)$ *are satisfied. If* $\gamma_1^* \le 0$, $\gamma_{2*} < 0 < \gamma_2^*$, *and*

$$\gamma_{2*} \ge \left(1 - \frac{1}{\alpha_1\alpha_2}\right)\left[\alpha_1\alpha_2 \frac{\widehat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}\right]^{1/(1-\alpha_1\alpha_2)},$$
$$\gamma_{1*} \ge r_{11} - \widehat{\beta}_{1*} \cdot \frac{r_{11}^{\alpha_1\alpha_2}}{(\beta_2^* + \gamma_2^* r_{11}^{\alpha_2})^{\alpha_1}}, \tag{5.1}$$

*where* $0 < r_{11} < +\infty$ *is a unique positive solution of the equation*

$$r_1^{1-\alpha_1\alpha_2}\left(\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2}\right)^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*}, \tag{5.2}$$

*then there exists a positive T-periodic solution of* (1.1).

*Proof.* In this case, to prove that $A : K \to K$, it is sufficient to find $r_1 < R_1, r_2 < R_2$ such that

$$\frac{\widehat{\beta}_{1*}}{R_2^{\alpha_1}} + \gamma_{1*} \ge r_1, \qquad \frac{\beta_1^*}{r_2^{\alpha_1}} \le R_1,$$
$$\frac{\widehat{\beta}_{2*}}{R_1^{\alpha_2}} + \gamma_{2*} \ge r_2, \qquad \frac{\beta_2^*}{r_1^{\alpha_2}} + \gamma_2^* \le R_2. \tag{5.3}$$

If we fix $R_1 = \beta_1^*/r_2^{\alpha_1}$, $R_2 = \beta_2^*/r_1^{\alpha_2} + \gamma_2^*$, then the first inequality of (6.4) holds if $r_2$ satisfies

$$\gamma_{2*} \ge r_2 - \frac{\widehat{\beta}_{2*}}{R_1^{\alpha_2}} = r_2 - \frac{\widehat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}} \cdot r_2^{\alpha_1\alpha_2}, \tag{5.4}$$

or equivalently

$$\gamma_{2*} \ge f(r_2) := r_2 - \frac{\widehat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}} \cdot r_2^{\alpha_1\alpha_2}. \tag{5.5}$$

Then the function $f(r_2)$ possesses a minimum at

$$r_{21} = \left[\alpha_1\alpha_2 \cdot \frac{\widehat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}\right]^{1/(1-\alpha_1\alpha_2)}, \tag{5.6}$$

that is, $f(r_{21}) = \min_{r_2 \in (0,+\infty)} f(r_2)$.

On the analogy of (5.4), we obtain

$$\gamma_{1*} \geq r_1 - \widehat{\beta}_{1*} \cdot \frac{r_1^{\alpha_1\alpha_2}}{\left(\beta_2^* + \gamma_2^* r_1^{\alpha_2}\right)^{\alpha_1}}, \tag{5.7}$$

or equivalently,

$$\gamma_{1*} \geq h(r_1) := r_1 - \widehat{\beta}_{1*} \cdot \frac{r_1^{\alpha_1\alpha_2}}{\left(\beta_2^* + \gamma_2^* r_1^{\alpha_2}\right)^{\alpha_1}}. \tag{5.8}$$

According to

$$h'(r_1) := 1 - \alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*} r_1^{\alpha_1\alpha_2-1}\left(\beta_2^* + \gamma_2^* r_1^{\alpha_2}\right)^{-1-\alpha_1}, \tag{5.9}$$

we have $h'(0) = -\infty$, $h'(+\infty) = 1$; then there exists $r_{11}$ such that $h'(r_{11}) = 0$, and

$$\begin{aligned} h''(r_1) = -\Big[&\alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*}(\alpha_1\alpha_2 - 1) r_1^{\alpha_1\alpha_2-2}\left(\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2}\right)^{-1-\alpha_1} \\ &+\alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*} r_1^{\alpha_1\alpha_2-1}(-1-\alpha_1)\left(\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2}\right)^{-2-\alpha_1}\gamma_2^*\alpha_2 r_1^{\alpha_2-1}\Big] > 0. \end{aligned} \tag{5.10}$$

Then the function $h(r_1)$ possesses a minimum at $r_{11}$, that is, $h(r_{11}) = \min_{r_1\epsilon(0,+\infty)} f(r_1)$.

Note $h'(r_{11}) = 0$; then we have

$$1 - \alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*} r_{11}^{\alpha_1\alpha_2-1}\left(\beta_2^* + \gamma_2^* \cdot r_{11}^{\alpha_2}\right)^{-1-\alpha_1} = 0. \tag{5.11}$$

Namely,

$$r_{11}^{1-\alpha_1\alpha_2}\left(\beta_2^* + \gamma_2^* \cdot r_{11}^{\alpha_2}\right)^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*}. \tag{5.12}$$

Taking $r_2 = r_{21}$ and $r_1 = r_{11}$, then the first inequality in (5.3) hold if $\gamma_{2*} \geq h(r_{21})$ and $\gamma_{1*} \geq h(r_{11})$ which are just condition (5.1). The second inequalities hold directly by the choice of $R_2$ and $R_1$, so it would remain to prove that $R_1 = \beta_1^*/r_{21}^{\alpha_1} > r_{11}$, $R_2 = \beta_2^*/r_{11}^{\alpha_2} + \gamma_2^* > r_{21}$. Now we turn to prove that $R_1 > r_{11}$, $R_2 > r_{21}$.

First,

$$
\begin{aligned}
R_1 &= \frac{\beta_1^*}{r_{21}^{\alpha_1}} = \frac{\beta_1^*}{\left\{\left[\alpha_1\alpha_2 \cdot \widehat{\beta}_{2*}/(\beta_1^*)^{\alpha_2}\right]^{1/(1-\alpha_1\alpha_2)}\right\}^{\alpha_1}} \\
&= \frac{\beta_1^*}{\left[\alpha_1\alpha_2 \cdot \widehat{\beta}_{2*}/(\beta_1^*)^{\alpha_2}\right]^{\alpha_1/(1-\alpha_1\alpha_2)}} = \frac{(\beta_1^*)^{1+(\alpha_1\alpha_2)/(1-\alpha_1\alpha_2)}}{\left(\alpha_1\alpha_2 \cdot \widehat{\beta}_{2*}\right)^{\alpha_1/(1-\alpha_1\alpha_2)}} \\
&= \frac{(\beta_1^*)^{1/(1-\alpha_1\alpha_2)}}{\left[\left(\alpha_1\alpha_2 \cdot \widehat{\beta}_{2*}\right)^{\alpha_1}\right]^{1/(1-\alpha_1\alpha_2)}} = \left[\frac{\beta_1^*}{\left(\alpha_1\alpha_2 \cdot \widehat{\beta}_{2*}\right)^{\alpha_1}}\right]^{1/(1-\alpha_1\alpha_2)} \\
&= \left[\frac{1}{(\alpha_1\alpha_2)^{\alpha_1}} \cdot \frac{\beta_1^*}{\left(\widehat{\beta}_{2*}\right)^{\alpha_1}}\right]^{1/(1-\alpha_1\alpha_2)} > \left[\alpha_1\alpha_2 \cdot \frac{\widehat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\right]^{1/(1-\alpha_1\alpha_2)} = r_{11},
\end{aligned}
\tag{5.13}
$$

since $\widehat{\beta}_{i*} \le \beta_i^*$, $i = 1, 2$.

On the other hand,

$$
R_2 = \frac{\beta_2^*}{r_{11}^{\alpha_2}} + \gamma_2^* = \frac{\beta_2^* + \gamma_2^* \cdot r_{11}^{\alpha_2}}{r_{11}^{\alpha_2}}. \tag{5.14}
$$

By (5.2), we have

$$
\beta_2^* + \gamma_2^* \cdot r_{11}^{\alpha_2} = \left(\alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*}\right)^{1/(1+\alpha_1)} r_{11}^{(\alpha_1\alpha_2-1)/(1+\alpha_1)}. \tag{5.15}
$$

Combing (5.14) and (5.15),

$$
R_2 = \left(\alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*}\right)^{1/(1+\alpha_1)} r_{11}^{-(1+\alpha_2)/(1+\alpha_1)}. \tag{5.16}
$$

In what follows, we will verify that $R_2 > r_{21}$. In fact,

$$
(\alpha_1\alpha_2)^{(2+\alpha_2+\alpha_1)/(1-\alpha_1\alpha_2)-1} \cdot \left(\frac{\widehat{\beta}_{2*}}{\beta_2^*}\right)^{(1+\alpha_1)/(1-\alpha_1\alpha_2)} \cdot \left(\frac{\widehat{\beta}_{1*}}{\beta_1^*}\right)^{\alpha_2(1+\alpha_1)/(1-\alpha_1\alpha_2)} < 1, \tag{5.17}
$$

since $\widehat{\beta}_{i*} \le \beta_i^*$, $i = 1, 2$. Thus

$$
\left(\alpha_1\alpha_2\beta_1^{*(-\alpha_2)}\widehat{\beta}_{2*}\right)^{(1+\alpha_1)/(1-\alpha_1\alpha_2)} \cdot \left(\alpha_1\alpha_2\beta_2^{*(-\alpha_1)}\widehat{\beta}_{1*}\right)^{(1+\alpha_2)/(1-\alpha_1\alpha_2)} < \alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*}. \tag{5.18}
$$

On the other hand,

$$r_{21}^{1-\alpha_1\alpha_2}\beta_1^{*(1+\alpha_2)} \le \alpha_1\alpha_2\beta_1^*\widehat{\beta}_{2*},$$
$$r_{11}^{1-\alpha_1\alpha_2}\beta_2^{*(1+\alpha_1)} \le \alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*}. \tag{5.19}$$

Thus one can see easily that

$$r_{21} \le \left(\alpha_1\alpha_2\beta_1^{*(-\alpha_2)}\widehat{\beta}_{2*}\right)^{1/(1-\alpha_1\alpha_2)},$$
$$r_{11} \le \left(\alpha_1\alpha_2\beta_2^{*(-\alpha_1)}\widehat{\beta}_{1*}\right)^{1/(1-\alpha_1\alpha_2)}. \tag{5.20}$$

From (5.20),

$$r_{11}^{1+\alpha_2}r_{21}^{1+\alpha_1} \le \left(\alpha_1\alpha_2\beta_2^{*(-\alpha_1)}\widehat{\beta}_{1*}\right)^{(1+\alpha_2)/(1-\alpha_1\alpha_2)}\left(\alpha_1\alpha_2\beta_1^{*(-\alpha_2)}\widehat{\beta}_{2*}\right)^{(1+\alpha_1)/(1-\alpha_1\alpha_2)}. \tag{5.21}$$

Combing (5.18) and (5.21),

$$r_{11}^{1+\alpha_2}r_{21}^{1+\alpha_1} < \alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*}. \tag{5.22}$$

Therefore,

$$r_{21}r_{11}^{(1+\alpha_2)/(1+\alpha_1)} < \left(\alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*}\right)^{1/(1+\alpha_1)}. \tag{5.23}$$

Recall (5.16), we obtain $r_{21} < R_2$ immediately. The proof is complete. □

Similarly, we have the following theorem.

**Theorem 5.2.** *Assume* $(H_1)$ *and* $(H_2)$ *are satisfied. If* $\gamma_2^* \le 0$, $\gamma_{1*} < 0 < \gamma_1^*$, *and*

$$\gamma_{1*} \ge \left(1 - \frac{1}{\alpha_1\alpha_2}\right)\cdot\left[\alpha_1\alpha_2\frac{\widehat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\right]^{1/(1-\alpha_1\alpha_2)},$$
$$\gamma_{2*} \ge r_{21} - \widehat{\beta}_{2*}\cdot\frac{r_{21}^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* r_{21}^{\alpha_1})^{\alpha_2}}, \tag{5.24}$$

*where* $0 < r_{21} < +\infty$ *is a unique positive solution of the equation*

$$r_2^{1-\alpha_1\alpha_2}(\beta_1^* + \gamma_1^*\cdot r_2^{\alpha_1})^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\widehat{\beta}_{2*}, \tag{5.25}$$

*then there exists a positive T-periodic solution of* (1.1).

## 6. The Case $\gamma_{1*} \geq 0,\ \gamma_{2*} < 0 < \gamma_2^*$ $(\gamma_{2*} \geq 0,\ \gamma_{1*} < 0 < \gamma_1^*)$

**Theorem 6.1.** *Assume* $(H_1)$ *and* $(H_2)$ *are satisfied. If* $\gamma_{1*} \geq 0$, $\gamma_{2*} < 0 < \gamma_2^*$, *and*

$$\gamma_{2*} \geq r_{22} - \widehat{\beta}_{2*} \cdot \frac{r_{22}^{\alpha_1\alpha_2}}{\left(\beta_1^* + \gamma_1^* r_{22}^{\alpha_1}\right)^{\alpha_2}}, \tag{6.1}$$

*where* $0 < r_{22} < +\infty$ *is a unique positive solution of the equation*

$$r_2^{1-\alpha_1\alpha_2}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\widehat{\beta}_{2*}, \tag{6.2}$$

*then there exists a positive $T$-periodic solution of* (1.1).

*Proof.* The following proof is the same as the proof of ahead theorem. In this case, to prove that $A : K \to K$, it is sufficient to find $r_1 < R_1$, $r_2 < R_2$ such that

$$\frac{\widehat{\beta}_{1*}}{R_2^{\alpha_1}} \geq r_1, \qquad \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^* \leq R_1, \tag{6.3}$$

$$\frac{\widehat{\beta}_{2*}}{R_1^{\alpha_2}} + \gamma_{2*} \geq r_2, \qquad \frac{\beta_2^*}{r_1^{\alpha_2}} + \gamma_2^* \leq R_2. \tag{6.4}$$

If we fix $R_1 = \beta_1^*/r_2^{\alpha_1} + \gamma_1^*$, $R_2 = \beta_2^*/r_1^{\alpha_2} + \gamma_2^*$, then the first inequality of (6.4) satisfies

$$\widehat{\beta}_{2*} \cdot \left(\frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^*\right)^{-\alpha_2} + \gamma_{2*} \geq r_2, \tag{6.5}$$

or equivalently

$$\gamma_{2*} \geq l(r_2) := r_2 - \frac{\widehat{\beta}_{2*}}{\left(\beta_1^* + \gamma_1^* r_2^{\alpha_1}\right)^{\alpha_2}} \cdot r_2^{\alpha_1\alpha_2}. \tag{6.6}$$

Then the function $l(r_2)$ possesses a minimum at $r_{22}$, that is, $l(r_{22}) = \min_{r_2 \in (0,+\infty)} l(r_2)$.

Note $l'(r_{22}) = 0$; then we have

$$1 - \alpha_1\alpha_2\beta_1^*\widehat{\beta}_{2*}r_{22}^{\alpha_1\alpha_2-1}\left(\beta_1^* + \gamma_1^* \cdot r_{22}^{\alpha_1}\right)^{-1-\alpha_2} = 0. \tag{6.7}$$

Therefore,

$$r_{22}^{1-\alpha_1\alpha_2}\left(\beta_1^* + \gamma_1^* \cdot r_{22}^{\alpha_1}\right)^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\widehat{\beta}_{2*}. \tag{6.8}$$

Note that $\widehat{\beta}i*, \beta_{i*} > 0$, $i = 1, 2$. And taking $r_2 = r_{22}$, $R_1 = \beta_1^*/r_{22}^{\alpha_1} + \gamma_1^*$, $r_1 = 1/R_2$, it is sufficient to find $r_1 < R_1$, $r_2 < R_2$ such that

$$R_2^{\alpha_1-1} \leq \beta_1^*, \qquad R_2^{\alpha_2}\beta_2^* + \gamma_2^* \leq R_2, \tag{6.9}$$

and these inequalities hold for $R_2$ being big enough because $\alpha_i < 1$. The proof is completed. □

Similarly, we have the following theorem.

**Theorem 6.2.** *Assume* $(H_1)$ *and* $(H_2)$ *are satisfied. If* $\gamma_{2*} \geq 0$, $\gamma_{1*} < 0 < \gamma_1^*$, *and*

$$\gamma_{1*} \geq r_{12} - \widehat{\beta}_{1*} \cdot \frac{r_{12}^{\alpha_1\alpha_2}}{\left(\beta_2^* + \gamma_2^* r_{12}^{\alpha_2}\right)^{\alpha_1}}, \tag{6.10}$$

*where* $0 < r_{12} < +\infty$ *is a unique positive solution of the equation*

$$r_1^{1-\alpha_1\alpha_2}\left(\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2}\right)^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\widehat{\beta}_{1*}, \tag{6.11}$$

*then there exists a positive T-periodic solution of* (1.1).

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