

Research Article

Order Level Inventory Models for Deteriorating Seasonable/Fashionable Products with Time Dependent Demand and Shortages

K. Skouri¹ and I. Konstantaras^{1,2}

¹ *Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece*

² *Hellenic Army Academy, Vari 16673, Attica, Greece*

Correspondence should be addressed to I. Konstantaras, ikonst@cc.uoi.gr

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An order level inventory model for seasonable/fashionable products subject to a period of increasing demand followed by a period of level demand and then by a period of decreasing demand rate (three branches ramp type demand rate) is considered. The unsatisfied demand is partially backlogged with a time dependent backlogging rate. In addition, the product deteriorates with a time dependent, namely, Weibull, deterioration rate. The model is studied under the following different replenishment policies: (a) starting with no shortages and (b) starting with shortages. The optimal replenishment policy for the model is derived for both the above mentioned policies.

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1. Introduction

It is observed that the life cycle of many seasonal products, over the entire time horizon, can be portrayed as a period of growth, followed by a period of relatively level demand and finishing with a period of decline. So researchers commonly use a time-varying demand pattern to reflect sales in different phases of product life cycle. Resh et al. [1] and Donaldson [2] are the first researchers who considered an inventory model with a linear trend in demand. Thereafter, numerous research works have been carried out incorporating time-varying demand patterns into inventory models. The time dependent demand patterns, mainly, used in literature are, (i) linearly time dependent and, (ii) exponentially time dependent (Dave and Patel [3], Goyal [4], Hariga [5], Hariga and Benkherouf [6], Yang et al. [7]). The time dependent demand patterns reported above are unidirectional, that is, increase continuously or decrease continuously. Hill [8] proposed a time dependent demand pattern by considering it as the combination of two different types of demand in two successive

time periods over the entire time horizon and termed it as “ramp-type” time dependent demand pattern. Then, inventory models with ramp type demand rate also are studied by Mandal and Pal [9], Wu et al. [10] and Wu and Ouyang [11], Wu [12], Giri et al. [13], and Manna and Chaudhuri [14]. In these papers, the determination of the optimal replenishment policy requires the determination of the time point, when the inventory level falls to zero. So the following two cases should be examined: (1) this time point occurs before the point, where the demand is stabilized, and (2) this time point occurs after the point, where the demand is stabilized. Almost all of the researchers examine only the first case. Deng et al. [15] reconsidered the inventory model of Mandal and Pal [9] and Wu and Ouyang [11] and studied it exploring these two cases. Skouri et al. [16] extend the work of Deng et al. [15] by introducing a general ramp type demand rate and considering Weibull distributed deterioration rate.

The assumption that the goods in inventory always preserve their physical characteristics is not true in general. There are some items, which are subject to risks of breakage, deterioration, evaporation, obsolescence, and so forth. Food items, pharmaceuticals, photographic film, chemicals, and radioactive substances are few items in which appreciable deterioration can take place during the normal storage of the units. A model with exponentially decaying inventory was initially proposed by Ghare and Schrader [17]. Covert and Phillip [18] and Tadikamalla [19] developed an economic order quantity model with Weibull and Gamma distributed deterioration rates, respectively. Thereafter, a great deal of research efforts have been devoted to inventory models of deteriorating items, the details can be found in the review articles by Raafat [20], and Goyal and Giri [21].

In most of the above-mentioned papers, the demand during stockout period is totally backlogged. In practice, there are customers who are willing to wait and receive their orders at the end of stockout period, while others are not. In the last few years, considerable attention has been paid to inventory models with partial backlogging. The backlogging rate can be modelled taking into account the customers’ behavior. The first paper in which customers’ impatience functions are proposed seems to be that by Abad [22]. Chang and Dye [23] developed a finite horizon inventory model using Abad’s reciprocal backlogging rate. Skouri and Papachristos [24] studied a multiperiod inventory model using the negative exponential backlogging rate proposed by Abad [22]. Teng et al. [25] extended Chang and Dye’s [23] and Skouri and Papachristos’ [24] models, assuming as backlogging rate any decreasing function of the waiting time up to the next replenishment. Research on models with partial backlogging continues with Wang [26] and San Jose et al. [27] and [28].

Manna and Chaudhuri [14] noted that ramp type demand pattern is generally followed by new brand of consumer goods coming to the market. But for fashionable products as well as for seasonal products, the steady demand will never be continued indefinitely. Rather it would be followed by decrement with respect to time after a period of time and becomes asymptotic in nature. Thus the demand may be illustrated by three successive time periods that classified time dependent ramp-type function, in which in the first phase the demand increases with time and after that it becomes steady, and towards the end in the final phase it decreases and becomes asymptotic. Chen et al. [29] proposed a search procedure based on Nelder-Mead algorithm to find a solution for the case of inventory systems with shortage allowance and nonlinear demand pattern. Also, Chen et al. [30] proposed a net present value approach for the previous inventory system without shortages. For both models, the demand rate is a revised version of the Beta distribution function and so is a differentiable with respect to time.

The purpose of the present paper is to study an order level inventory model when the demand is described by a three successive time periods that classified time dependent ramp-type function. Any such function has points, at least one, where differentiation is not possible, and this introduces extra complexity in the analysis of the relevant models. The unsatisfied demand is partially backlogged with time dependent backlogging rate, and units in inventory are subject to deterioration with Weibull-distributed deterioration rate.

The rest of the paper is organized as follows. In the next section the assumptions and notations for the development of the model are provided. The model starting with no shortages is studied in Section 3, and the corresponding one starting with shortages is studied in Section 4. For each model the optimal policy is obtained. Numerical examples highlighting the results obtained are given in Section 5. The paper closes with concluding remarks in Section 6.

2. Notation and Assumptions

The following notations and assumptions are used in developing the model:

Notations

- T The constant scheduling period (cycle)
- t_1 The time when the inventory level reaches zero
- S The maximum inventory level at each scheduling period (cycle)
- c_1 The inventory holding cost per unit per unit time
- c_2 The shortage cost per unit per unit time
- c_3 The cost incurred from the deterioration of one unit
- c_4 The per unit opportunity cost due to the lost sales
- μ The time point that increasing demand becomes steady
- γ The time point, after μ , until the demand is steady and then decreases
- $I(t)$ The inventory level at time $t \in [0, T]$.

Assumptions

- (1) The ordering quantity brings the inventory level up to the order level S . Replenishment rate is infinite.
- (2) Shortages are backlogged at a rate $\beta(x)$, which is a nonincreasing function of x ($\beta'(x) \leq 0$) with $0 \leq \beta(x) \leq 1$, $\beta(0) = 1$, and x is the waiting time up to the next replenishment. Moreover it is assumed that $\beta(x)$ satisfies the relation $\beta(x) + T\beta'(x) \geq 0$, where $\beta'(x)$ is the derivate of $\beta(x)$. The cases with $\beta(x) = 1$ (or 0) correspond to complete backlogging (or complete lost sales) models.
- (3) The time to deterioration of the item is distributed as Weibull (a, b) ; that is, the deterioration rate is $\theta(t) = abt^{b-1}$ ($a > 0, b > 0, t > 0$). There is no replacement or repair of deteriorated units during the period T . For $b = 1$, $\theta(t)$ becomes constant, which corresponds to exponentially decaying case.

- (4) The demand rate $D(t)$ is a time dependent ramp-type function and is of the following form:

$$D(t) = \begin{cases} f(t), & 0 < t < \mu, \\ f(\mu) = g(\gamma), & \mu \leq t \leq \gamma, \\ g(t), & \gamma < t, \end{cases} \quad (2.1)$$

where $f(t)$ is a positive, continuous, and increasing function of t , and $g(t)$ is a positive, continuous and decreasing function of t .

3. The Mathematical Formulation of the Model Starting with No Shortages

The replenishment at the beginning of the cycle brings the inventory level up to S . Due to demand and deterioration, the inventory level gradually depletes during the period $(0, t_1)$ and falls to zero at $t = t_1$. Thereafter shortages occur during the period (t_1, T) , which are partially backlogged. Consequently, the inventory level, $I(t)$, during the time interval $0 \leq t \leq T$, satisfies the following differential equations:

$$\frac{dI(t)}{dt} + \theta(t)I(t) = -D(t), \quad 0 \leq t \leq t_1, \quad I(t_1) = 0, \quad (3.1)$$

$$\frac{dI(t)}{dt} = -D(t)\beta(T-t), \quad t_1 \leq t \leq T, \quad I(t_1) = 0. \quad (3.2)$$

The solutions of these differential equations are affected from the relation between t_1 , μ , and γ through the demand rate function. Since the demand has three components in three successive time periods, the following cases: (i) $t_1 < \mu < \gamma < T$, (ii) $\mu < t_1 < \gamma < T$, and (iii) $\mu < \gamma < t_1 < T$ must be considered to determine the total cost and then the optimal replenishment policy.

Case 1 ($t_1 < \mu < \gamma < T$). In this case, (3.1) becomes

$$\frac{dI(t)}{dt} + abt^{b-1}I(t) = -f(t), \quad 0 \leq t \leq t_1, \quad I(t_1) = 0. \quad (3.3)$$

Equation (3.2) leads to the following three:

$$\frac{dI(t)}{dt} = -f(t)\beta(T-t), \quad t_1 \leq t \leq \mu, \quad I(t_1) = 0, \quad (3.4)$$

$$\frac{dI(t)}{dt} = -f(\mu)\beta(T-t), \quad \mu \leq t \leq \gamma, \quad I(\mu_-) = I(\mu_+), \quad (3.5)$$

$$\frac{dI(t)}{dt} = -g(t)\beta(T-t), \quad \gamma \leq t \leq T. \quad (3.6)$$

The solutions of (3.3), (3.4), (3.5), and (3.6), are, respectively,

$$I(t) = e^{-at^b} \int_t^{t_1} f(x) e^{ax^b} dx, \quad 0 \leq t \leq t_1, \quad (3.7)$$

$$I(t) = - \int_{t_1}^t f(x) \beta(T-x) dx, \quad t_1 \leq t \leq \mu, \quad (3.8)$$

$$I(t) = -f(\mu) \int_{\mu}^t \beta(T-x) dx - \int_{t_1}^{\mu} f(x) \beta(T-x) dx, \quad \mu \leq t \leq \gamma, \quad (3.9)$$

$$I(t) = - \int_{\gamma}^t g(x) \beta(T-x) dx - \int_{t_1}^{\mu} f(x) \beta(T-x) dx - f(\mu) \int_{\mu}^{\gamma} \beta(T-x) dx, \quad \gamma \leq t \leq T. \quad (3.10)$$

The total amount of deteriorated items during $[0, t_1]$ is

$$D = \int_0^{t_1} f(t) e^{at^b} dt - \int_0^{t_1} f(t) dt. \quad (3.11)$$

The cumulative inventory carried in the interval $[0, t_1]$ is found from (3.7) and is

$$I_1 = \int_0^{t_1} I(t) dt = \int_0^{t_1} e^{-at^b} \left[\int_t^{t_1} f(x) e^{ax^b} dx \right] dt. \quad (3.12)$$

Due to (3.8), (3.9), and (3.10), the time-weighted backorders during the interval $[t_1, T]$ are

$$\begin{aligned} I_2 &= \int_{t_1}^T [-I(t)] dt \\ &= \int_{t_1}^{\mu} [-I(t)] dt + \int_{\mu}^{\gamma} [-I(t)] dt + \int_{\gamma}^T [-I(t)] dt \\ &= \int_{t_1}^{\mu} (\mu - t) f(t) \beta(T-t) dt + f(\mu) \int_{\mu}^{\gamma} \left[\int_{\mu}^t \beta(T-x) dx \right] dt + \int_{\mu}^{\gamma} \left[\int_{t_1}^{\mu} f(x) \beta(T-x) dx \right] dt \\ &\quad + \int_{\gamma}^T \left[\int_{\gamma}^t g(x) \beta(T-x) dx \right] dt + \int_{\gamma}^T \left[\int_{t_1}^{\mu} f(x) \beta(T-x) dx \right] dt + f(\mu) \int_{\gamma}^T \left[\int_{\mu}^{\gamma} \beta(T-x) dx \right] dt. \end{aligned} \quad (3.13)$$

The amount of lost sales during $[t_1, T]$ is

$$L = \int_{t_1}^{\mu} (1 - \beta(T-t)) f(t) dt + f(\mu) \int_{\mu}^{\gamma} (1 - \beta(T-t)) dt + \int_{\gamma}^T (1 - \beta(T-t)) g(t) dt. \quad (3.14)$$

The total cost in the time interval $[0, T]$ is the sum of holding, shortage, deterioration, and opportunity costs and is given by

$$\begin{aligned}
TC_1(t_1) &= c_1 I_1 + c_2 I_2 + c_3 D + c_4 L \\
&= c_1 \left\{ \int_0^{t_1} e^{-at^b} \left[\int_t^{t_1} f(x) e^{ax^b} dx \right] dt \right\} + c_3 \left\{ \int_0^{t_1} f(t) e^{at^b} dt - \int_0^{t_1} f(t) dt \right\} \\
&\quad + c_2 \left\{ \int_{t_1}^{\mu} (\mu - t) f(t) \beta(T - t) dt + f(\mu) \int_{\mu}^{\gamma} \left[\int_{\mu}^t \beta(T - x) dx \right] dt \right. \\
&\quad + \int_{\mu}^{\gamma} \left[\int_{t_1}^{\mu} f(x) \beta(T - x) dx \right] dt + \int_{\gamma}^T \left[\int_{\gamma}^t g(x) \beta(T - x) dx \right] dt \\
&\quad \left. + \int_{\gamma}^T \left[\int_{t_1}^{\mu} f(x) \beta(T - x) dx \right] dt + f(\mu) \int_{\gamma}^T \left[\int_{\mu}^{\gamma} \beta(T - x) dx \right] dt \right\} \\
&\quad + c_4 \left\{ \int_{t_1}^{\mu} (1 - \beta(T - t)) f(t) dt + f(\mu) \int_{\mu}^{\gamma} (1 - \beta(T - t)) dt + \int_{\gamma}^T (1 - \beta(T - t)) g(t) dt \right\}.
\end{aligned} \tag{3.15}$$

Case 2 ($\mu < t_1 < \gamma < T$). In this case, (3.1) reduces to the following two:

$$\begin{aligned}
\frac{dI(t)}{dt} + abt^{b-1}I(t) &= -f(t), \quad 0 \leq t \leq \mu, \quad I(\mu^-) = I(\mu^+), \\
\frac{dI(t)}{dt} + abt^{b-1}I(t) &= -f(\mu), \quad \mu \leq t \leq t_1, \quad I(t_1) = 0.
\end{aligned} \tag{3.16}$$

Equation (3.2) leads to the following two:

$$\begin{aligned}
\frac{dI(t)}{dt} &= -f(\mu)\beta(T - t), \quad t_1 \leq t \leq \gamma, \quad I(t_1) = 0, \\
\frac{dI(t)}{dt} &= -g(t)\beta(T - t), \quad \gamma \leq t \leq T, \quad I(\gamma^-) = I(\gamma^+).
\end{aligned} \tag{3.17}$$

Their solutions are, respectively,

$$\begin{aligned}
I(t) &= e^{-at^b} \left[\int_t^{\mu} f(x) e^{ax^b} dx + f(\mu) \int_{\mu}^{t_1} e^{ax^b} dx \right], \quad 0 \leq t \leq \mu, \\
I(t) &= e^{-at^b} f(\mu) \int_t^{t_1} e^{ax^b} dx, \quad \mu \leq t \leq t_1, \\
I(t) &= -f(\mu) \int_{t_1}^t \beta(T - x) dx, \quad t_1 \leq t \leq \gamma, \\
I(t) &= -\int_{\gamma}^t g(x) \beta(T - x) dx - f(\mu) \int_{t_1}^{\gamma} \beta(T - x) dx, \quad \gamma \leq t \leq T.
\end{aligned} \tag{3.18}$$

The total amount of deteriorated items during $[0, t_1]$ is

$$D = I(0) - \int_0^{t_1} D(t)dt = \int_0^\mu f(t)e^{at^b} dt + f(\mu) \int_\mu^{t_1} e^{at^b} dt - \int_0^\mu f(t)dt - f(\mu)(t_1 - \mu). \quad (3.19)$$

The total inventory carried during the interval $[0, t_1]$ is

$$\begin{aligned} I_1 &= \int_0^{t_1} I(t)dt = \int_0^\mu I(t)dt + \int_\mu^{t_1} I(t)dt \\ &= \int_0^\mu e^{-at^b} \left[\int_t^\mu f(x)e^{ax^b} dx + f(\mu) \int_\mu^{t_1} e^{ax^b} dx \right] dt + f(\mu) \int_\mu^{t_1} e^{-at^b} \left[\int_t^{t_1} e^{ax^b} dx \right] dt. \end{aligned} \quad (3.20)$$

The time-weighted backorders during the interval $[t_1, T]$ are

$$\begin{aligned} I_2 &= \int_{t_1}^T [-I(t)]dt = \int_{t_1}^Y [-I(t)]dt + \int_Y^T [-I(t)]dt \\ &= \int_{t_1}^Y f(\mu) \int_{t_1}^t \beta(T-x)dx dt + \int_Y^T \int_Y^t g(x)\beta(T-x)dx dt + f(\mu) \int_Y^T \int_{t_1}^Y \beta(T-x)dx dt. \end{aligned} \quad (3.21)$$

The lost sales in the interval $[t_1, T]$ are

$$L = f(\mu) \int_{t_1}^Y [1 - \beta(T-t)]dt + \int_Y^T [1 - \beta(T-t)]g(t)dt. \quad (3.22)$$

The inventory cost for this case is

$$\begin{aligned} TC_2(t_1) &= c_1 I_1 + c_2 I_2 + c_3 D + c_4 L \\ &= c_1 \left\{ \int_0^\mu e^{-at^b} \left[\int_t^\mu f(x)e^{ax^b} dx + f(\mu) \int_\mu^{t_1} e^{ax^b} dx \right] dt + f(\mu) \int_\mu^{t_1} e^{-at^b} \left[\int_t^{t_1} e^{ax^b} dx \right] dt \right\} \\ &\quad + c_2 \left\{ \int_{t_1}^Y f(\mu) \int_{t_1}^t \beta(T-x)dx dt + \int_Y^T \int_Y^t g(x)\beta(T-x)dx dt \right. \\ &\quad \left. + f(\mu) \int_Y^T \int_{t_1}^Y \beta(T-x)dx dt \right\} \\ &\quad + c_3 \left\{ \int_0^\mu f(t)e^{at^b} dt + f(\mu) \int_\mu^{t_1} e^{at^b} dt - \int_0^\mu f(t)dt - f(\mu)(t_1 - \mu) \right\} \\ &\quad + c_4 \left\{ f(\mu) \int_{t_1}^Y [1 - \beta(T-t)]dt + \int_Y^T [1 - \beta(T-t)]g(t)dt \right\}. \end{aligned} \quad (3.23)$$

Case 3 ($\mu < \gamma < t_1 < T$). In this case, (3.1) reduces to the following three:

$$\begin{aligned} \frac{dI(t)}{dt} + abt^{b-1}I(t) &= -f(t), \quad 0 \leq t \leq \mu, \quad I(\mu^-) = I(\mu^+), \\ \frac{dI(t)}{dt} + abt^{b-1}I(t) &= -f(\mu), \quad \mu \leq t \leq \gamma, \quad I(\gamma^-) = I(\gamma^+), \\ \frac{dI(t)}{dt} + abt^{b-1}I(t) &= -g(t), \quad \gamma \leq t \leq t_1, \quad I(t_1) = 0. \end{aligned} \quad (3.24)$$

Equation (3.2) leads to the following:

$$\frac{dI(t)}{dt} = -g(t)\beta(T-t), \quad t_1 \leq t \leq T, \quad I(t_1) = 0. \quad (3.25)$$

Their solutions are, respectively,

$$I(t) = e^{-at^b} \left[\int_t^\mu f(x)e^{ax^b} dx + f(\mu) \int_\mu^\gamma e^{ax^b} dx + \int_\gamma^{t_1} g(x)e^{ax^b} dx \right], \quad 0 \leq t \leq \mu, \quad (3.26)$$

$$I(t) = e^{-at^b} \left[f(\mu) \int_t^\gamma e^{ax^b} dx + \int_\gamma^{t_1} g(x)e^{ax^b} dx \right], \quad \mu \leq t \leq \gamma, \quad (3.27)$$

$$I(t) = e^{-at^b} \int_t^{t_1} g(x)e^{ax^b} dx, \quad \gamma \leq t \leq t_1, \quad (3.28)$$

$$I(t) = - \int_{t_1}^t g(x)\beta(T-x)dx, \quad t_1 \leq t \leq T. \quad (3.29)$$

The total amount of deteriorated items during $[0, t_1]$ is

$$\begin{aligned} D &= I(0) - \int_0^{t_1} D(t)dt \\ &= \int_0^\mu f(x)e^{ax^b} dx + f(\mu) \int_\mu^\gamma e^{ax^b} dx + \int_\gamma^{t_1} g(x)e^{ax^b} dx \\ &\quad - \int_0^\mu f(t)dt - f(\mu)(\gamma - \mu) - \int_\gamma^{t_1} g(t)dt. \end{aligned} \quad (3.30)$$

The total inventory carried during the interval $[0, t_1]$, using (3.26), (3.27), and (3.28) is

$$\begin{aligned}
I_1 &= \int_0^{t_1} I(t) dt = \int_0^\mu I(t) dt + \int_\mu^\gamma I(t) dt + \int_\gamma^{t_1} I(t) dt \\
&= \int_0^\mu e^{-at^b} \left[\int_t^\mu f(x) e^{ax^b} dx + f(\mu) \int_\mu^\gamma e^{ax^b} dx + \int_\gamma^{t_1} g(x) e^{ax^b} dx \right] dt \\
&\quad + \int_\mu^\gamma e^{-at^b} \left[f(\mu) \int_t^\gamma e^{ax^b} dx + \int_\gamma^{t_1} g(x) e^{ax^b} dx \right] dt + \int_\gamma^{t_1} e^{-at^b} \left[\int_t^{t_1} g(x) e^{ax^b} dx \right] dt.
\end{aligned} \tag{3.31}$$

The time-weighted backorders during the interval $[t_1, T]$ are

$$I_2 = \int_{t_1}^T [-I(t)] dt = \int_{t_1}^T \int_{t_1}^t g(x) \beta(T-x) dx. \tag{3.32}$$

The lost sales in the interval $[t_1, T]$ are

$$L = \int_{t_1}^T [1 - \beta(T-t)] g(t) dt. \tag{3.33}$$

The inventory cost for this case is

$$\begin{aligned}
TC_3(t_1) &= c_1 I_1 + c_2 I_2 + c_3 D + c_4 L \\
&= c_1 \left\{ \int_0^\mu e^{-at^b} \left[\int_t^\mu f(x) e^{ax^b} dx + f(\mu) \int_\mu^\gamma e^{ax^b} dx + \int_\gamma^{t_1} g(x) e^{ax^b} dx \right] dt \right. \\
&\quad \left. + \int_\mu^\gamma e^{-at^b} \left[f(\mu) \int_t^\gamma e^{ax^b} dx + \int_\gamma^{t_1} g(x) e^{ax^b} dx \right] dt + \int_\gamma^{t_1} e^{-at^b} \left[\int_t^{t_1} g(x) e^{ax^b} dx \right] dt \right\} \\
&\quad + c_2 \left\{ \int_{t_1}^T \int_{t_1}^t g(x) \beta(T-t) dx \right\} + c_4 \left\{ \int_{t_1}^T [1 - \beta(T-t)] g(t) dt \right\} \\
&\quad + c_3 \left\{ \int_0^\mu f(x) e^{ax^b} dx + f(\mu) \int_\mu^\gamma e^{ax^b} dx + \int_\gamma^{t_1} g(x) e^{ax^b} dx \right. \\
&\quad \left. - \int_0^\mu f(t) dt - f(\mu)(\gamma - \mu) - \int_\gamma^{t_1} g(t) dt \right\}.
\end{aligned} \tag{3.34}$$

Finally the total cost function of the system over $[0, T]$ takes the following form:

$$TC(t_1) = \begin{cases} TC_1(t_1), & \text{if } t_1 \leq \mu, \\ TC_2(t_1), & \text{if } \mu < t_1 < \gamma, \\ TC_3(t_1), & \text{if } \gamma \leq t_1. \end{cases} \tag{3.35}$$

It is easy to check that this function is continuous at μ and γ . The problem now is the minimization of this, three branches, function $TC(t_1)$. This requires, separately, studying each of these branches and then combining the results to state the algorithm giving the optimal policy.

3.1. The Optimal Replenishment Policy

In this subsection we present the results, which ensure the existence of a unique optimal value for t_1 , say t_1^* , which minimizes the total cost function. Although the optimality procedure requires the constrained optimization of the functions $TC_1(t_1)$, $TC_2(t_1)$, and $TC_3(t_1)$, we will, firstly, search for their unconstrained minimum. The first- and second-order derivatives of $TC_1(t_1)$, $TC_2(t_1)$, and $TC_3(t_1)$ are, respectively,

$$\begin{aligned}\frac{dTC_1(t_1)}{dt_1} &= f(t_1)h(t_1), \\ \frac{d^2TC_1(t_1)}{dt_1^2} &= \frac{df(t_1)}{dt_1}h(t_1) + f(t_1)\frac{dh(t_1)}{dt_1}, \\ \frac{dTC_2(t_1)}{dt_1} &= f(\mu)h(t_1), \\ \frac{d^2TC_2(t_1)}{dt_1^2} &= f(\mu)\frac{dh(t_1)}{dt_1}, \\ \frac{dTC_3(t_1)}{dt_1} &= g(t_1)h(t_1), \\ \frac{d^2TC_3(t_1)}{dt_1^2} &= \frac{dg(t_1)}{dt_1}h(t_1) + g(t_1)\frac{dh(t_1)}{dt_1},\end{aligned}\tag{3.36}$$

where

$$h(t_1) = c_1 e^{at_1^b} \int_0^{t_1} e^{-at^b} dt + c_3 (e^{at_1^b} - 1) - c_2 (T - t_1) \beta (T - t_1) - c_4 (1 - \beta (T - t_1)).\tag{3.37}$$

Equation (3.37) is the same as (16) of the paper of Skouri et al. [16]. So, following the methodology proposed by Skouri et al. [16], the algorithm, which gives the optimal replenishment policy, is as follows.

Step 1. Compute t_1^* from $h(t_1) = 0$.

Step 2. If $t_1^* \leq \mu$, then the optimal order quantity is given by

$$Q^* = \int_0^{t_1^*} f(t) e^{at^b} dt + \int_{t_1^*}^{\mu} \beta (T - t) f(t) dt + f(\mu) \int_{\mu}^{\gamma} \beta (T - t) dt + \int_{\gamma}^T \beta (T - t) g(t) dt\tag{3.38}$$

and the total cost is given by $TC_1(t_1^*)$.

If $\mu < t_1^* < \gamma$, then the optimal order quantity is given by

$$Q^* = \int_0^\mu f(t)e^{at^b} dt + f(\mu) \int_\mu^{t_1^*} e^{at^b} dt + f(\mu) \int_{t_1^*}^\gamma \beta(T-t) dt + \int_\gamma^T g(t)\beta(T-t) dt \quad (3.39)$$

and the total cost is given by $TC_2(t_1^*)$.

If $\gamma < t_1^* < T$, then the optimal order quantity is given by

$$Q^* = \int_0^\mu f(t)e^{at^b} dt + f(\mu) \int_\mu^\gamma e^{at^b} dt + \int_\gamma^{t_1^*} g(t)e^{at^b} dt + \int_{t_1^*}^T g(t)\beta(T-t) dt \quad (3.40)$$

and the total cost is given by $TC_3(t_1^*)$.

Remark 3.1. The previous analysis shows that t_1^* is independent from the demand rate $D(t)$. This very interesting result agrees with the classical result, in many order level inventory systems, that the point t_1^* is independent from the demand rate (Naddor [31, page 67]).

3.2. The Special Case $\beta(x) = 1$ and $a = 0$

If we are considering the case that there is no deterioration of the product ($a = 0$) and unsatisfied demand is complete backlogged ($\beta(x) = 1$), then the total cost function of the model starting with no shortages over $[0, T]$ takes the following form:

$$TC(t_1) = \begin{cases} \left\{ \begin{aligned} &c_1 \int_0^{t_1} \int_t^{t_1} f(x) dx dt + c_2 \int_{t_1}^\mu \int_{t_1}^\mu f(x) dx dt + c_2 f(\mu) \int_\mu^\gamma (t-\mu) dt + c_2 \int_\mu^\gamma \int_{t_1}^\mu f(x) dx dt \\ &+ c_2 \int_\gamma^T \int_\gamma^t g(x) dx dt + c_2 f(\mu) (\gamma - \mu) (T - \gamma) + c_2 \int_\gamma^\mu \int_{t_1}^\mu f(x) dx dt, \\ &t_1 < \mu < \gamma < T, \end{aligned} \right. \\ \left\{ \begin{aligned} &c_1 \int_0^\mu \int_t^\mu f(x) dx dt + c_1 \int_0^\mu \int_\mu^{t_1} f(\mu) dx dt + c_1 \int_\mu^{t_1} \int_t^{t_1} f(\mu) dx dt + c_2 \int_{t_1}^\gamma \int_{t_1}^t f(\mu) dx dt \\ &+ c_2 \int_\gamma^T \int_\gamma^t g(x) dx dt + c_2 \int_\gamma^T \int_{t_1}^\gamma f(\mu) dx dt, \quad \mu < t_1 < \gamma < T, \end{aligned} \right. \\ \left\{ \begin{aligned} &c_1 \int_0^\mu \int_t^\mu f(x) dx dt + c_1 \int_0^\mu \int_\mu^\gamma f(\mu) dx dt + c_1 \int_0^\mu \int_\gamma^{t_1} g(x) dx dt + c_1 \int_\mu^\gamma \int_t^\gamma f(\mu) dx dt \\ &+ c_1 \int_\mu^\gamma \int_\gamma^{t_1} g(x) dx dt + c_1 \int_\gamma^{t_1} \int_t^{t_1} g(x) dx dt + c_2 \int_{t_1}^T \int_{t_1}^t g(x) dx dt, \\ &\mu < \gamma < t_1 < T. \end{aligned} \right. \end{cases} \quad (3.41)$$

Following the previous procedure for the optimal replenishment policy, the optimal value of t_1 , say t_1^* , is given by the very simple and known, in classical order level inventory system (Naddor [31]), equation:

$$t_1^* = \frac{c_2 T}{c_1 + c_2}. \quad (3.42)$$

4. The Mathematical Formulation of the Model Starting with Shortages

In this section the inventory model starting with shortages is studied. The cycle now starts with shortages, which occur during the period $(0, t_1)$, and are partially backlogged. At time t_1 a replenishment brings the inventory level up to S . Demand and deterioration of the items deplete the inventory level during the period (t_1, T) until this falls to zero at $t = T$. Again the three cases $t_1 < \mu < \gamma < T$, $\mu < t_1 < \gamma < T$, and $\mu < \gamma < t_1 < T$ must be examined.

Case 4 ($t_1 < \mu < \gamma < T$). The inventory level, $I(t)$, $0 \leq t \leq T$ satisfies the following differential equations:

$$\begin{aligned} \frac{dI(t)}{dt} &= -f(t)\beta(t_1 - t), \quad 0 \leq t \leq t_1, \quad I(0) = 0, \\ \frac{dI(t)}{dt} + abt^{b-1}I(t) &= -f(t), \quad t_1 \leq t \leq \mu, \quad I(\mu_-) = I(\mu_+), \\ \frac{dI(t)}{dt} + abt^{b-1}I(t) &= -f(\mu), \quad \mu \leq t \leq \gamma, \quad I(\gamma_-) = I(\gamma_+), \\ \frac{dI(t)}{dt} + abt^{b-1}I(t) &= -g(t), \quad \gamma \leq t \leq T, \quad I(T) = 0. \end{aligned} \quad (4.1)$$

The solutions of (4.1), are, respectively,

$$I(t) = -\int_0^t f(x)\beta(t_1 - x)dx, \quad 0 \leq t \leq t_1, \quad (4.2)$$

$$I(t) = e^{-at^b} \left[\int_t^\mu e^{ax^b} f(x)dx + f(\mu) \int_\mu^\gamma e^{ax^b} dx + \int_\gamma^T e^{ax^b} g(x)dx \right], \quad t_1 \leq t \leq \mu, \quad (4.3)$$

$$I(t) = e^{-at^b} \left[f(\mu) \int_t^\gamma e^{ax^b} dx + \int_\gamma^T e^{ax^b} g(x)dx \right], \quad \mu \leq t \leq \gamma, \quad (4.4)$$

$$I(t) = e^{-at^b} \int_t^T e^{ax^b} g(x)dx, \quad \gamma \leq t \leq T. \quad (4.5)$$

The total amount of deteriorated units during $[t_1, T]$ is

$$\begin{aligned} D &= e^{-at_1^b} \left[\int_{t_1}^\mu f(x)e^{ax^b} dx + f(\mu) \int_\mu^\gamma e^{ax^b} dx + \int_\gamma^T e^{ax^b} g(x)dx \right] \\ &\quad - \int_{t_1}^\mu f(x)dx - (\gamma - \mu)f(\mu) - \int_\gamma^T g(x)dx. \end{aligned} \quad (4.6)$$

The total inventory carried during the interval $[t_1, T]$ is found using (4.3), (4.4), and (4.5) and is

$$I_1 = \int_{t_1}^{\mu} e^{-at^b} \left[\int_t^{\mu} e^{\alpha x^b} f(x) dx + f(\mu) \int_{\mu}^{\gamma} e^{\alpha x^b} dx + \int_{\gamma}^T e^{\alpha x^b} g(x) dx \right] dt \\ + \int_{\mu}^{\gamma} e^{-at^b} \left[f(\mu) \int_t^{\gamma} e^{\alpha x^b} dx + \int_{\gamma}^T e^{\alpha x^b} g(x) dx \right] dt + \int_{\gamma}^T e^{-at^b} \left[\int_t^T e^{\alpha x^b} g(x) dx \right] dt. \quad (4.7)$$

Due to (4.2) the time-weighted backorders during the time interval $[0, t_1]$ are

$$I_2 = \int_0^{t_1} \int_0^t f(x) \beta(t_1 - x) dx dt. \quad (4.8)$$

The amount of lost sales during $[0, t_1]$ is

$$L = \int_0^{t_1} [1 - \beta(t_1 - t)] f(t) dt. \quad (4.9)$$

The inventory cost during the time interval $[0, T]$ is the sum of holding, shortage, deterioration, and opportunity costs and is given by

$$TC_1(t_1) = c_1 I_1 + c_2 I_2 + c_3 D + c_4 L \\ = c_1 \left(\int_{t_1}^{\mu} e^{-at^b} \left[\int_t^{\mu} e^{\alpha x^b} f(x) dx + f(\mu) \int_{\mu}^{\gamma} e^{\alpha x^b} dx + \int_{\gamma}^T e^{\alpha x^b} g(x) dx \right] dt \right) \\ + c_1 \left(\int_{\mu}^{\gamma} e^{-at^b} \left[f(\mu) \int_t^{\gamma} e^{\alpha x^b} dx + \int_{\gamma}^T e^{\alpha x^b} g(x) dx \right] dt + \int_{\gamma}^T e^{-at^b} \left[\int_t^T e^{\alpha x^b} g(x) dx \right] dt \right) \\ + c_2 \int_0^{t_1} \int_0^t f(x) \beta(t_1 - x) dx dt + c_4 \int_0^{t_1} [1 - \beta(t_1 - t)] f(t) dt \\ + c_3 \left[e^{-at_1^b} \left[\int_{t_1}^{\mu} f(x) e^{\alpha x^b} dx + f(\mu) \int_{\mu}^{\gamma} e^{\alpha x^b} dx + \int_{\gamma}^T e^{\alpha x^b} g(x) dx \right] \right. \\ \left. - \int_{t_1}^{\mu} f(x) dx - (\gamma - \mu) f(\mu) - \int_{\gamma}^T g(x) dx \right]. \quad (4.10)$$

Case 5 ($\mu < t_1 < \gamma < T$). The inventory level, $I(t)$, $0 \leq t \leq T$ satisfies the following differential equations:

$$\begin{aligned} \frac{dI(t)}{dt} &= -f(t)\beta(t_1 - t), \quad 0 \leq t \leq \mu, \quad I(0) = 0, \\ \frac{dI(t)}{dt} &= -f(\mu)\beta(t_1 - t), \quad \mu \leq t \leq t_1, \quad I(\mu^-) = I(\mu^+), \\ \frac{dI(t)}{dt} + abt^{b-1}I(t) &= -f(\mu), \quad t_1 \leq t \leq \gamma, \quad I(\gamma^-) = I(\gamma^+), \\ \frac{dI(t)}{dt} + abt^{b-1}I(t) &= -g(t), \quad \gamma \leq t \leq T, \quad I(T) = 0. \end{aligned} \quad (4.11)$$

The solutions of (4.11), are, respectively,

$$\begin{aligned} I(t) &= -\int_0^t f(x)\beta(t_1 - x)dx, \quad 0 \leq t \leq \mu, \\ I(t) &= -\int_0^\mu f(x)\beta(t_1 - x)dx - f(\mu)\int_\mu^t \beta(t_1 - x)dx, \quad \mu \leq t \leq t_1, \\ I(t) &= e^{-at^b} \left[f(\mu)\int_t^\gamma e^{ax^b} dx + \int_\gamma^T e^{ax^b} g(x)dx \right], \quad t_1 \leq t \leq \gamma, \\ I(t) &= e^{-at^b} \int_t^T e^{ax^b} g(x)dx, \quad \gamma \leq t \leq T. \end{aligned} \quad (4.12)$$

The total cost of this case is obtained with a similar way of the previous cases and is,

$$\begin{aligned} TC_2(t_1) &= c_1 \left[\int_{t_1}^\gamma e^{-at^b} \left(f(\mu)\int_t^\gamma e^{ax^b} dx + \int_\gamma^T e^{ax^b} g(x)dx \right) dt \right. \\ &\quad \left. + \int_\gamma^T \left(e^{-at^b} \int_t^T e^{ax^b} g(x)dx \right) dt \right] \\ &\quad + c_3 \left[e^{-at_1^b} \left(\int_{t_1}^\gamma f(\mu)e^{ax^b} dx + \int_\gamma^T e^{ax^b} g(x)dx \right) - f(\mu)(\gamma - t_1) - \int_\gamma^T g(x)dx \right] \\ &\quad + c_2 \left[\int_0^\mu \left[\int_0^t f(x)\beta(t_1 - x)dx \right] dt \right. \\ &\quad \left. + \int_\mu^{t_1} \left[\int_0^\mu f(x)\beta(t_1 - x)dx + f(\mu)\int_\mu^t \beta(t_1 - x)dx \right] dt \right] \\ &\quad + c_4 \left[\int_0^\mu [1 - \beta(t_1 - t)]f(t)dt + f(\mu)\int_\mu^{t_1} [1 - \beta(t_1 - t)]dt \right]. \end{aligned} \quad (4.13)$$

Case 6 ($\mu < \gamma < t_1 < T$). The inventory level, $I(t)$, $0 \leq t \leq T$ for this case satisfies the following differential equations:

$$\frac{dI(t)}{dt} = -f(t)\beta(t_1 - t), \quad 0 \leq t \leq \mu, \quad I(0) = 0, \quad (4.14)$$

$$\frac{dI(t)}{dt} = -f(\mu)\beta(t_1 - t), \quad \mu \leq t \leq \gamma, \quad I(\mu^-) = I(\mu^+), \quad (4.15)$$

$$\frac{dI(t)}{dt} = -g(t)\beta(t_1 - t), \quad \gamma \leq t \leq t_1, \quad I(\gamma^-) = I(\gamma^+), \quad (4.16)$$

$$\frac{dI(t)}{dt} + abt^{b-1}I(t) = -g(t), \quad t_1 \leq t \leq T, \quad I(T) = 0. \quad (4.17)$$

The solutions of (4.14), (4.15), (4.16), and (4.17), are, respectively,

$$\begin{aligned} I(t) &= -\int_0^t f(x)\beta(t_1 - x)dx, \quad 0 \leq t \leq \mu, \\ I(t) &= -\int_0^\mu f(x)\beta(t_1 - x)dx - f(\mu)\int_\mu^t \beta(t_1 - x)dx, \quad \mu \leq t \leq \gamma, \\ I(t) &= -\int_\gamma^t g(x)\beta(t_1 - x)dx - \int_0^\mu f(x)\beta(t_1 - x)dx - f(\mu)\int_\mu^\gamma \beta(t_1 - x)dx, \quad \gamma \leq t \leq t_1, \\ I(t) &= e^{-at^b} \int_t^T e^{ax^b} g(x)dx, \quad t_1 \leq t \leq T. \end{aligned} \quad (4.18)$$

The total cost of this case is obtained with a similar way of the previous cases and is,

$$\begin{aligned} TC_3(t_1) &= c_1 \left[\int_{t_1}^T e^{-at^b} \int_t^T e^{ax^b} g(x)dx dt \right] + c_3 \left[e^{-at_1^b} \int_{t_1}^T e^{ax^b} g(x)dx - \int_{t_1}^T g(x)dx \right] \\ &+ c_2 \left[\int_0^\mu \left[\int_0^t f(x)\beta(t_1 - x)dx \right] dt + \int_\mu^\gamma \left[\int_0^\mu f(x)\beta(t_1 - x)dx + f(\mu)\int_\mu^t \beta(t_1 - x)dx \right] dt \right] \\ &+ c_2 \left[\int_\gamma^{t_1} \left(\int_\gamma^t g(x)\beta(t_1 - x)dx + \int_0^\mu f(x)\beta(t_1 - x)dx + f(\mu)\int_\mu^\gamma \beta(t_1 - x)dx \right) dt \right] \\ &+ c_4 \left[\int_0^\mu [1 - \beta(t_1 - t)]f(t)dt + f(\mu)\int_\mu^\gamma [1 - \beta(t_1 - t)]dt + \int_\gamma^{t_1} [1 - \beta(t_1 - t)]g(t)dt \right]. \end{aligned} \quad (4.19)$$

Finally the total cost function of the system over $[0, T]$ takes the following form:

$$TC(t_1) = \begin{cases} TC_1(t_1), & \text{if } t_1 \leq \mu, \\ TC_2(t_1), & \text{if } \mu < t_1 < \gamma, \\ TC_3(t_1), & \text{if } \gamma \leq t_1. \end{cases} \quad (4.20)$$

It is easy to check that this function is continuous at μ and γ . The problem now is the minimization of this, three branches, function $TC(t_1)$. This requires, separately, studying each of these branches and then combining the results to state the algorithm giving the optimal policy.

4.1. The Optimal Replenishment Policy

In this subsection we derive the optimal replenishment policy, that is, we calculate the value, say t_1^* , which minimizes the total cost function. Taking the first-order derivative of $TC_1(t_1)$, say $K_1(t_1)$, and equating it to zero gives:

$$K_1(t_1) = -\left(c_1 + c_3abt_1^{b-1}\right)e^{-at_1^b} \left[\int_{t_1}^{\mu} e^{\alpha x^b} f(x) dx + f(\mu) \int_{\mu}^{\gamma} e^{\alpha x^b} dx + \int_{\gamma}^T e^{\alpha x^b} g(x) dx \right] \\ + \int_0^{t_1} [c_2\beta(t_1 - t) + c_2(t_1 - t)\beta'(t_1 - t) - c_4\beta'(t_1 - t)] f(t) dt = 0. \quad (4.21)$$

If t_1^* is a root of (4.21), for this root the second-order condition for minimum is

$$\left[c_1 + c_3ab(t_1^*)^{b-1} - c_3(b-1)(t_1^*)^{-1} \right] ab(t_1^*)^{b-1} e^{-at_1^{*b}} \\ \times \left[\int_{t_1^*}^{\mu} e^{\alpha x^b} f(x) dx + f(\mu) \int_{\mu}^{\gamma} e^{\alpha x^b} dx + \int_{\gamma}^T e^{\alpha x^b} g(x) dx \right] \\ + [c_1 + c_3ab(t_1^*)^{b-1}] f(t_1^*) + [c_2 - c_4\beta'(0)] f(t_1^*) \\ + \int_0^{t_1^*} [2c_2\beta'(t_1^* - t) + c_2(t_1^* - t)\beta''(t_1^* - t) - c_4\beta''(t_1^* - t)] f(t) dt > 0. \quad (4.22)$$

So, if (4.22) holds and $t_1^* \leq \mu$, then the value of order level, S , is

$$S^* = I(t_1^*) = e^{-at_1^{*b}} \left[\int_{t_1^*}^{\mu} e^{\alpha x^b} f(x) dx + f(\mu) \int_{\mu}^{\gamma} e^{\alpha x^b} dx + \int_{\gamma}^T e^{\alpha x^b} g(x) dx \right], \quad (4.23)$$

the ordering quantity is

$$Q^* = \int_0^{t_1^*} f(x)\beta(t_1^* - x)dx + S^*, \quad (4.24)$$

and the total cost is $TC_1(t_1^*)$.

Equating the first-order derivative of $TC_2(t_1)$, say $K_2(t_1)$, to zero gives

$$\begin{aligned} K_2(t_1) = & -\left(c_1 + c_3\alpha bt_1^{b-1}\right)e^{-\alpha t_1^b} \left(f(\mu) \int_t^\gamma e^{\alpha x^b} dx + \int_\gamma^T e^{\alpha x^b} g(x) dx \right) \\ & + \int_0^\mu [c_2\beta(t_1 - t) + c_2(t_1 - t)\beta'(t_1 - t) - c_4\beta'(t_1 - t)] f(t) dt \\ & + f(\mu) \int_\mu^{t_1} [c_2\beta(t_1 - t) + c_2(t_1 - t)\beta'(t_1 - t) - c_4\beta'(t_1 - t)] dt = 0. \end{aligned} \quad (4.25)$$

If t_1^* is a root of (4.25), for this root the second-order condition for minimum is

$$\begin{aligned} & [c_1 + c_3\alpha bt_1^{*b-1}] \alpha bt_1^{*b-1} e^{-\alpha(t_1^*)^b} \left(f(\mu) \int_{t_1^*}^\gamma e^{\alpha x^b} dx + \int_\gamma^T e^{\alpha x^b} g(x) dx \right) \\ & - c_3\alpha b(b-1)t_1^{*(b-2)} e^{-\alpha(t_1^*)^b} \left(f(\mu) \int_{t_1^*}^\gamma e^{\alpha x^b} dx + \int_\gamma^T e^{\alpha x^b} g(x) dx \right) + [c_1 + c_3\alpha b(t_1^*)^{b-1}] f(\mu) \\ & + f(\mu)(c_2 - c_4\beta'(0)) + \int_0^\mu [2c_2\beta'(t_1^* - t) + c_2(t_1^* - t)\beta''(t_1^* - t) - c_4\beta''(t_1^* - t)] f(t) dt \\ & + f(\mu) \int_\mu^{t_1^*} [2c_2\beta'(t_1^* - t) + c_2(t_1^* - t)\beta''(t_1^* - t) - c_4\beta''(t_1^* - t)] dt > 0. \end{aligned} \quad (4.26)$$

So, if (4.26) holds and $\mu < t_1^* < \gamma$, then the value of S is

$$S^* = I(t_1^*) = e^{-\alpha t_1^{*b}} \left[f(\mu) \int_{t_1^*}^\gamma e^{\alpha x^b} dx + \int_\gamma^T e^{\alpha x^b} g(x) dx \right]. \quad (4.27)$$

the ordering quantity is

$$Q^* = \int_0^\mu f(x)\beta(t_1^* - x)dx + f(\mu) \int_\mu^{t_1^*} \beta(t_1^* - x)dx + S^*, \quad (4.28)$$

and the total cost is $TC_2(t_1^*)$.

Equating the first-order derivative of $TC_3(t_1)$, say $K_3(t_1)$, to zero gives

$$\begin{aligned}
K_3(t_1) = & -\left(c_1 + c_3 \alpha b t_1^{b-1}\right) e^{-\alpha t_1^b} \int_{t_1}^T e^{\alpha x^b} g(x) dx \\
& + \int_0^\mu [c_2 \beta(t_1 - t) + c_2(t_1 - t) \beta'(t_1 - t) - c_4 \beta'(t_1 - t)] f(t) dt \\
& + f(\mu) \int_\mu^\gamma [c_2 \beta(t_1 - t) + c_2(t_1 - t) \beta'(t_1 - t) - c_4 \beta'(t_1 - t)] dt \\
& + \int_\gamma^{t_1} [c_2 \beta(t_1 - t) + c_2(t_1 - t) \beta'(t_1 - t) - c_4 \beta'(t_1 - t)] g(t) dt = 0.
\end{aligned} \tag{4.29}$$

If t_1^* is a root of (4.29), for this root the second-order condition for minimum is

$$\begin{aligned}
& \left[c_1 + c_3 \alpha b t_1^{*b-1} \right] \alpha b t_1^{*b-1} e^{-\alpha (t_1^*)^b} \int_{t_1^*}^T e^{\alpha x^b} g(x) dx \\
& - c_3 \alpha b (b-1) t_1^{*(b-2)} e^{-\alpha (t_1^*)^b} \int_{t_1^*}^T e^{\alpha x^b} g(x) dx + \left[c_1 + c_3 \alpha b (t_1^*)^{b-1} \right] g(t_1^*) \\
& + g(t_1^*) (c_2 - c_4 \beta'(0)) + \int_0^\mu [2c_2 \beta'(t_1^* - t) + c_2(t_1^* - t) \beta''(t_1^* - t) - c_4 \beta''(t_1^* - t)] f(t) dt \\
& + f(\mu) \int_\mu^\gamma [2c_2 \beta'(t_1^* - t) + c_2(t_1^* - t) \beta''(t_1^* - t) - c_4 \beta''(t_1^* - t)] dt \\
& + \int_\gamma^{t_1^*} [2c_2 \beta'(t_1^* - t) + c_2(t_1^* - t) \beta''(t_1^* - t) - c_4 \beta''(t_1^* - t)] g(t) dt > 0.
\end{aligned} \tag{4.30}$$

So, if (4.30) holds and $t_1^* > \gamma$, then the value of S is

$$S^* = I(t_1^*) = e^{-\alpha (t_1^*)^b} \int_{t_1^*}^T e^{\alpha x^b} g(x) dx, \tag{4.31}$$

the ordering quantity is

$$Q^* = \int_0^\mu f(x) \beta(t_1^* - x) dx + f(\mu) \int_\mu^\gamma \beta(t_1^* - x) dx + \int_\gamma^{t_1^*} g(x) \beta(t_1^* - x) dx + S^*, \tag{4.32}$$

and the total cost is $TC_3(t_1^*)$.

Remark 4.1. Due to (4.21), (4.25), and (4.29) the function $TC(t_1)$ is differentiable at the point μ and γ .

In the previous analysis there is no guarantee that t_1^* exists and corresponds to the minimum. Its uniqueness is also another issue. The proposition, which follows, provides sufficient conditions for existence, uniqueness, and validity of t_1^* .

Let us set

$$\begin{aligned}
 \Delta_1 &= K_1(\mu) \\
 &= -\left(c_1 + c_3 \alpha b \mu^{b-1}\right) e^{-\alpha \mu^b} \left[f(\mu) \int_{\mu}^{\gamma} e^{\alpha x^b} dx + \int_{\gamma}^T e^{\alpha x^b} g(x) dx \right] \\
 &\quad + \int_0^{\mu} [c_2 \beta(\mu-t) + c_2(\mu-t) \beta'(\mu-t) - c_4 \beta'(\mu-t)] f(t) dt, \\
 \Delta_2 &= K_2(\gamma) \\
 &= -\left(c_1 + c_3 \alpha b \gamma^{b-1}\right) e^{-\alpha \gamma^b} \left(\int_{\gamma}^T e^{\alpha x^b} g(x) dx \right) \\
 &\quad + \int_0^{\mu} [c_2 \beta(\gamma-t) + c_2(\gamma-t) \beta'(\gamma-t) - c_4 \beta'(\gamma-t)] f(t) dt \\
 &\quad + f(\mu) \int_{\mu}^{\gamma} [c_2 \beta(\gamma-t) + c_2(\gamma-t) \beta'(\gamma-t) - c_4 \beta'(\gamma-t)] dt
 \end{aligned} \tag{4.33}$$

and $h(t) = c_2 \beta(t) + c_2 t \beta'(t) - c_4 \beta'(t)$.

The following proposition can be easily proved observing that $K_1(0) < 0$ and $K_3(T) > 0$.

Proposition 4.2. *If $b < 1$, $h'(t) > 0$, $t \in [0, T]$ and*

- (1) $\Delta_1 > 0$, then (4.21) has one root, say t_1^* , which is the unique optimal value of the problem.
- (2) $\Delta_1 < 0$, and $\Delta_2 > 0$ then (4.25) has one root, say t_1^* , which is the unique optimal value of the problem.
- (3) $\Delta_1 < 0$, and $\Delta_2 < 0$ then (4.29) has one root, say t_1^* , which is the unique optimal value of the problem.

We note that the above proposition ensures the existence, uniqueness, and validity of t_1 . If the conditions of this proposition do not hold, then the following procedure can be used to calculate the optimal replenishment policy.

Step 1. (1) Find the global minimizing point, t_1^* , for $TC_1(t_1)$. This will be one of the following points:

- (a) root of (4.21), (an interior point of $[0, \mu]$) which satisfies (4.22)
- (b) $t_1^* = 0$,
- (c) $t_1^* = \mu$.

Then calculate $TC_1(t_1^*)$.

(2) Find the global minimizing, t_1^* , for $TC_2(t_1)$. This will be one of the following points:

- (a) a root of (4.25), (an interior point of $[\mu, T]$) which satisfies (4.26),
- (b) $t_1^* = \mu$,
- (c) $t_1^* = \gamma$.

Then calculate $TC_2(t_1^*)$.

(3) Find the global minimizing, t_1^* , for $TC_3(t_1)$. This will be one of the following points:

- (a) a root of (4.29), (an interior point of $[\mu, T]$) which satisfies (4.30),
- (b) $t_1^* = \gamma$,
- (c) $t_1^* = T$.

Then calculate $TC_3(t_1^*)$.

Step 2. Find the min $\{TC_1(t_1^*), TC_2(t_1^*), TC_3(t_1^*)\}$ and accordingly select the optimum t_1^* .

Remark 4.3. The analysis shows that, in this model, t_1^* is dependent from the demand rate $D(t)$.

5. Numerical Examples

The examples, which follow, illustrate the results obtained.

Example 5.1. The input parameters are $c_1 = \$3$ per unit per year, $c_2 = \$15$ per unit per year, $c_3 = \$5$ per unit, $c_4 = \$20$ per unit, $\mu = 0.12$ year, $\gamma = 0.9$, $a = 0.001$, $b = 2$, $T = 1$ year, $f(t) = 3e^{4.5t}$, $g(t) = 3e^{4.5\mu - 0.8(t-\gamma)}$, and $\beta(x) = e^{-0.2x}$.

Model Starting with No Shortages

Using (3.37) the optimal value of t_1 is $\mu < t_1^* = 0.860 < \gamma$, and consequently the optimal ordering quantity is $Q^* = 4.98$ (from (3.39)) and the minimum cost is $TC(t_1^*) = 6.631$ (from (3.23)).

Model Starting with Shortages

Using (4.25) the optimal value of t_1 is $\mu < t_1^* = 0.163 < \gamma$, and consequently the optimal ordering quantity is $Q^* = 4.98$ (from (4.28)), and the minimum cost is $TC(t_1^*) = 6.329$ (from (4.13)).

Example 5.2. This example is identical to Example 5.1, except that $\gamma = 0.7$, $f(t) = 20e^{3.5t}$, and $g(t) = 20e^{3.5\mu} - 50(t - \gamma)$.

Model Starting with No Shortages

Again $t_1^* = 0.860$ but now $\gamma < t_1^*$, the optimal ordering quantity is $Q^* = 27.49$ (from (3.40)), and the minimum cost is $TC(t_1^*) = 35.855$ (from (3.34)).

Table 1: Sensitivity analysis for model starting without shortages.

The initial cost parameters: $c_1 = 3, c_2 = 15, c_3 = 5, c_4 = 20$						
$D(t) = \begin{cases} 3e^{4.5t}, & 0 < t < 0.12, \\ f(0.12) = g(0.9), & 0.12 \leq t \leq 0.9, \\ 3e^{4.5 \times 0.12 - 0.8(t-0.9)}, & 0.9 < t. \end{cases} \quad D(t) = \begin{cases} 20e^{3.5t}, & 0 < t < 0.12, \\ f(0.12) = g(0.7), & 0.12 \leq t \leq 0.7, \\ 20e^{3.5 \times 0.12 - 50(t-0.7)}, & 0.7 < t. \end{cases}$						
	t_1^*	Q^*	$TC(t_1^*)$	Q^*	$TC(t_1^*)$	
c_1	0.75	0.962	4.99	1.851	27.52	9.686
	1.50	0.926	4.98	3.562	27.51	18.868
	4.50	0.803	4.96	9.290	27.44	51.073
	6.00	0.752	4.95	11.612	27.38	64.638
c_2	3.25	0.698	4.94	5.397	27.29	30.362
	7.50	0.787	4.97	6.075	27.42	33.535
	22.5	0.896	4.98	6.900	27.50	36.895
	30.0	0.918	4.99	7.058	27.51	37.482
c_3	1.25	0.860	4.98	6.626	27.49	35.832
	2.50	0.860	4.98	6.628	27.49	35.840
	7.50	0.860	4.98	6.633	27.49	35.870
	10.0	0.860	4.98	6.635	27.49	35.885
c_4	5.00	0.838	4.98	6.459	27.47	35.166
	10.0	0.846	4.98	6.522	27.48	35.423
	30.0	0.872	4.98	6.720	27.49	36.206
	40.0	0.882	4.98	6.795	27.50	36.497

Model Starting with Shortages

Using (4.25) the optimal value of t_1 is $\mu < t_1^* = 0.148 < \gamma$, and consequently the optimal ordering quantity is $Q^* = 27.48$ (from (4.28)) and the minimum cost is $TC(t_1^*) = 33.008$ (from (4.13)).

In Tables 1 and 2 some sensitivity analysis for the models starting without and with shortages, respectively, is performed, for the above examples, by changing the cost parameter values $-75\%, -50\%, +50\%$ and, $+100\%$ taking one at a time and keeping the remaining unchanged. From these two examples and the sensitivity analysis is evident the following.

(1) For the model starting without shortages, although the time when shortages occur (t_1^*) is identical for the two examples, the ordering quantities and costs are significant different, obviously because of the demand rate.

(2) For the two models the changes in the total optimal costs indicate that the models are highly sensitive to the error on the estimation of the parameter value c_1 , moderately sensitive to the error on c_2 , while low sensitivity is to the error on the estimation of the parameters c_3 and c_4 . In presented examples, the costs related to storage inventory (c_1 and c_3) are less than costs related to unsatisfied demand (c_2 and c_4) maybe this explains the low sensitivity to the error on the estimation of the parameter c_4 . While the small deterioration rate maybe implies low sensitivity with respect to c_3 .

Table 2: Sensitivity analysis for model starting with shortages.

The initial cost parameters: $c_1 = 3, c_2 = 15, c_3 = 5, c_4 = 20$							
		$D(t) = \begin{cases} 3e^{4.5t}, & 0 < t < 0.12, \\ f(0.12) = g(0.9), & 0.12 \leq t \leq 0.9, \\ 3e^{4.5 \times 0.12 - 0.8(t-0.9)}, & 0.9 < t. \end{cases}$			$D(t) = \begin{cases} 20e^{3.5t}, & 0 < t < 0.12, \\ f(0.12) = g(0.7), & 0.12 \leq t \leq 0.7, \\ 20e^{3.5 \times 0.12 - 50(t-0.7)}, & 0.7 < t. \end{cases}$		
	t_1^*	Q^*	$TC(t_1^*)$	t_1^*	Q^*	$TC(t_1^*)$	
c_1	0.75	0.056	4.99	1.811	0.048	27.52	9.407
	1.50	0.098	4.99	3.440	0.087	27.51	17.910
	4.50	0.218	4.97	8.835	0.200	27.43	46.126
	6.00	0.269	4.96	11.024	0.246	27.37	57.618
c_2	3.25	0.320	4.94	5.120	0.294	27.30	26.794
	7.50	0.234	4.97	5.775	0.214	27.41	30.158
	22.5	0.128	4.98	6.616	0.116	27.50	34.483
	30.0	0.107	4.99	6.798	0.095	27.51	35.410
c_3	1.25	0.162	4.98	6.323	0.148	27.48	32.979
	2.50	0.162	4.98	6.325	0.148	27.48	32.989
	7.50	0.163	4.98	6.333	0.148	27.48	33.027
	10.0	0.163	4.98	6.337	0.148	27.48	33.047
c_4	5.00	0.185	4.98	6.155	0.169	27.46	32.113
	10.0	0.176	4.98	6.219	0.161	27.46	32.440
	30.0	0.151	4.98	6.422	0.137	27.48	33.487
	40.0	0.141	4.98	6.502	0.128	27.49	33.896

(3) The errors in cost parameters are attenuated when translated into changes in the optimal ordering quantity.

6. Concluding Remarks

In this paper, an order level inventory model for deteriorating items has been studied. The basic assumption of the model is based on time dependent three branches ramp type demand rate. The demand of seasonable and fashionable products can be described well with this function, as the nature of demand of these products is increasing at the beginning of the season, steady in the mid of the season, and decreasing at the end of the season. To the best of our knowledge this demand pattern studied for the first time, at least, using so general functions for the nonsteady periods. In addition a time dependent backlogging and deterioration rate are assumed. The inventory model is studied under two different replenishment policies: (a) starting with no shortages and (b) starting with shortages. An algorithm to obtain the optimal policy is proposed.

Moreover from this model follows as special cases the following ones.

- (1) If $f(t) = D_0t, t \in [0, \mu], g(x) = f(\mu), \mu = \gamma$ and $\beta(x) = 1/(1 + \delta x)$, this model reduces to that of Wu [12].
- (2) If in addition to (1), $\beta(x) = 1$ (case of complete backlogging), then it further reduces to that of Wu et al. [10].

- (3) If $b = 1$ (i.e., constant deterioration rate), $f(t) = D_0t$, $g(x) = f(\mu)$, $\mu = \gamma$, and $\beta(x) = 1$, then the models reduce to those of Wu and Ouyang [11], Mandal and Pal [9], and Deng et al. [15].
- (4) If $g(x) = f(\mu)$, $\mu = \gamma$, this model gives the one model proposed by Skouri et al. [16].

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