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Research Article

Productivity Formulas for a Partially Penetrating Vertical Well in a Circular Cylinder Drainage Volume

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Taking a partially penetrating vertical well as a uniform line sink in three-dimensional space, by developing necessary mathematical analysis, this paper presents steady state productivity formulas for an off-center partially penetrating vertical well in a circular cylinder drainage volume with constant pressure at outer boundary. This paper also gives formulas for calculating the pseudo-skin factor due to partial penetration. If top and bottom reservoir boundaries are impermeable, the radius of the cylindrical system and off-center distance appears in the productivity formulas. If the reservoir has a gas cap or bottom water, the effects of the radius and off-center distance on productivity can be ignored. It is concluded that, for a partially penetrating vertical well, different productivity equations should be used under different reservoir boundary conditions.

1. Introduction

Well productivity is one of primary concerns in oil field development and provides the basis for oil field development strategy. To determine the economical feasibility of drilling a well, the engineers need reliable methods to estimate its expected productivity. Well productivity is often evaluated using the productivity index, which is defined as the production rate per unit pressure drawdown. Petroleum engineers often relate the productivity evaluation to the long time performance behavior of a well, that is, the behavior during pseudo-steady-state or steady-state flow.

The productivity index expresses an intuitive feeling, that is, once the well production is stabilized, the ratio of production rate to some pressure difference between the reservoir and the well must depend on the geometry of the reservoir/well system only. Indeed, a long time ago, petroleum engineers observed that in a bounded reservoir or a reservoir with strong water drive, the productivity index of a well stabilizes in a long time asymptote.

When an oil reservoir is bounded with a constant pressure boundary (such as a gas cap or an aquifer), flow reaches the steady-state regime after the pressure transient reaches the constant pressure boundary. Rate and pressure become constant with time at all points in the reservoir and wellbore once steady-state flow is established. Therefore, the productivity index during steady-state flow is a constant.

Strictly speaking, steady-state flow can occur only if the flow across the drainage boundary is equal to the flow across the wellbore wall at well radius, and the fluid properties remain constant throughout the oil reservoir. These conditions may never be met in an oil reservoir; however, in oil reservoirs produced by a strong water drive, whereby the water influx rate at reservoir outer boundary equals the well producing rate, the pressure change with time is so slight that it is practically undetectable. In such cases, the assumption of steady-state is acceptable.

In many oil reservoirs the producing wells are completed as partially penetrating wells; that is, only a portion of the pay zone is perforated. This may be done for a variety of reasons, but the most common one is to prevent or delay the unwanted fluids into the wellbore. If a vertical well partially penetrates the formation, there is an added resistance to flow in the vicinity of the wellbore. The streamlines converge and the area for flow decreases, which results in added resistance.

The problem of fluid flow into wells with partial penetration has received much attention in the past, the exact solution of the partial penetration problem presents great analytical problems. Brons and Marting [4], Papatzacos [5], Basinev [3] developed solutions to the two dimensional diffusivity equation, which included flow of fluid in the vertical direction. They only obtained semianalytical and semiempirical expressions to calculate the added resistance due to partial penetration.

The primary goal of this study is to present new steady-state productivity formulas for a partially penetrating vertical well in a circular cylinder drainage reservoir with constant pressure at outer boundary. Analytical solutions are derived by making the assumption of uniform fluid withdrawal along the portion of the wellbore open to flow. The producing portion of a partially penetrating vertical well is modeled as a uniform line sink. This paper also gives new expressions for calculating the added resistance due to partial penetration, by solving the three-dimensional Laplace equation.

2. Literature Review

Putting Darcy's equation into the equation of continuity, the productivity formula of a fully penetrating vertical well in a homogeneous, isotropic permeability reservoir is obtained [1, page 52]:

$$Q_w = F_D \frac{2\pi K H (P_e - P_w)/(\mu B)}{\ln(R_e/R_w)}, \tag{2.1}$$

where P_e is outer boundary pressure, P_w is flowing wellbore pressure, K is permeability, H is payzone thickness, μ is oil viscosity, B is oil formation volume factor, R_e is drainage radius,

R_w is wellbore radius, and F_D is the factor which allows the use of field units, and it can be found in a Table 1 at page 52 of [1].

Formula (2.1) is applicable for a fully penetrating vertical well in a circular drainage area with constant pressure outer boundary.

If a vertical well partially penetrates the formation, there is an added resistance to flow which is limited to the region around the wellbore, this added resistance is included by introducing the pseudo-skin factor, S_{ps}. Thus, Formula (2.1) may be rewritten to include the pseudo-skin factor due to partial penetration as [2, page 92]

$$Q_w = F_D \frac{2\pi KH(P_e - P_w)/(\mu B)}{\ln(R_e/R_w) + S_{ps}}. \tag{2.2}$$

Define partial penetration factor η:

$$\eta = \frac{L_p}{H}, \tag{2.3}$$

where L_p is the producing well length, that is, perforated interval.

Several authors obtained semianalytical and semiempirical expressions for evaluating pseudo-skin factor due to partial penetration.

Bervaldier's pseudo-skin factor formula [3]:

$$S_{ps} = \left(\frac{1}{\eta} - 1\right)\left[\frac{\ln(L_p/R_w)}{(1 - R_w/L_p)} - 1\right]. \tag{2.4}$$

Brons and Marting's pseudo-skin factor formula [4] is as follows:

$$S_{ps} = \left(\frac{1}{\eta} - 1\right)[\ln(h_D) - G(\eta)], \tag{2.5}$$

where

$$\begin{gathered} h_D = \left(\frac{H}{2R_w}\right)\left(\frac{K_h}{K_v}\right)^{1/2}, \\ G(\eta) = 2.948 - 7.363\eta + 11.45\eta^2 - 4.675\eta^3. \end{gathered} \tag{2.6}$$

Papatzacos's pseudo-skin factor formula [5] is as follows:

$$S_{ps} = \left(\frac{1}{\eta} - 1\right)\ln\left(\frac{\pi h_D}{2}\right) + \left(\frac{1}{\eta}\right)\ln\left[\left(\frac{\eta}{2+\eta}\right)\left(\frac{I_1 - 1}{I_2 - 1}\right)^{1/2}\right], \tag{2.7}$$

where

$$I_1 = \frac{H}{h_1 + 0.25L_p},$$
$$I_2 = \frac{H}{h_1 + 0.75L_p}, \tag{2.8}$$

and h_1 is the distance from the top of the reservoir to the top of the open interval.

It must be pointed out that the aforementioned formulas are only applicable to a reservoir with both impermeable top and bottom boundaries.

3. Partially Penetrating Vertical Well Model

Figure 1 is a schematic of an off-center partially penetrating vertical well. A partially penetrating well of drilled length L drains a circular cylinder porous volume with height H and radius R_e.

The following assumptions are made.

(1) The reservoir has constant K_x, K_y, K_z permeabilities, thickness H, porosity ϕ. During production, the partially penetrating vertical well has a circular cylinder drainage volume with height H and radius R_e. The well is located at R_0 away from the axis of symmetry of the cylindrical body.

(2) At time $t = 0$, pressure is uniformly distributed in the reservoir, equal to the initial pressure P_i. If the reservoir has constant pressure boundaries (edge water, gas cap, bottom water), the pressure is equal to the initial value at such boundaries during production.

(3) The production occurs through a partially penetrating vertical well of radius R_w, represented in the model by a uniform line sink, the drilled well length is L, the producing well length is L_p.

(4) A single phase fluid, of small and constant compressibility C_f, constant viscosity μ, and formation volume factor B, flows from the reservoir to the well. Fluid properties are independent of pressure. Gravity forces are neglected.

(5) There is no water encroachment and no water/gas coning. Edge water, gas cap, and bottom water are taken as constant pressure boundaries, multiphase flow effects are ignored.

(6) Any additional pressure drops caused by formation damage, stimulation, or perforation are ignored, we only consider pseudo-skin factor due to partial penetration.

The porous media domain is:

$$\Omega = \left\{ (x, y, z) \mid x^2 + y^2 < R_e^2,\ 0 < z < H \right\}, \tag{3.1}$$

where R_e is cylinder radius, Ω is the cylindrical body.

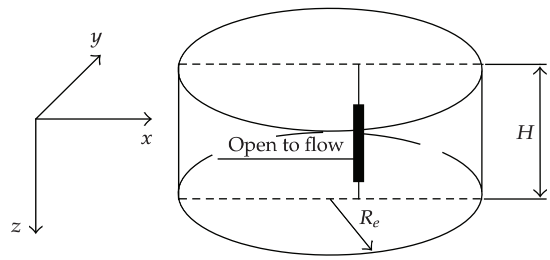

Figure 1: Partially penetrating vertical well model.

Located at R_0 away from the center of the cylindrical body, the coordinates of the top and bottom points of the well line are $(R_0, 0, 0)$ and $(R_0, 0, L)$, respectively, while point $(R_0, 0, L_1)$ and point $(R_0, 0, L_2)$ are the beginning point and end point of the producing portion of the well, respectively. The well is a uniform line sink between $(R_0, 0, L_1)$ and $(R_0, 0, L_2)$, and there hold

$$L_p = L_2 - L_1, \quad L_p \leq L \leq H. \tag{3.2}$$

We assume

$$K_x = K_y = K_h, \qquad K_z = K_v, \tag{3.3}$$

and define average permeability:

$$K_a = (K_x K_y K_z)^{1/3} = K_h^{2/3} K_v^{1/3}. \tag{3.4}$$

The reservoir initial pressure is a constant:

$$P|_{t=0} = P_i. \tag{3.5}$$

The pressure at constant pressure boundaries (edge water, gas cap, bottom water) is assumed to be equal to the reservoir initial pressure during production:

$$P_e = P_i. \tag{3.6}$$

Suppose point $(R_0, 0, z')$ is in the producing portion, and its point convergence intensity is q, in order to obtain the pressure at point (x, y, z) caused by the point $(R_0, 0, z')$, according to mass conservation law and Darcy's law, we have to obtain the basic solution of the diffusivity equation in Ω [6, 7]:

$$K_h \frac{\partial^2 P}{\partial x^2} + K_h \frac{\partial^2 P}{\partial y^2} + K_v \frac{\partial^2 P}{\partial z^2} = \mu q B \delta(x - R_0)\delta(y)\delta(z - z'), \quad \text{in } \Omega, \tag{3.7}$$

where C_t is total compressibility coefficient of porous media, $\delta(x - R_0)$, $\delta(y)$, $\delta(z - z')$ are Dirac functions.

In order to simplify the equations, we take the following dimensionless transforms:

$$x_D = \left(\frac{x}{L}\right)\left(\frac{K_a}{K_h}\right)^{1/2}, \qquad y_D = \left(\frac{y}{L}\right)\left(\frac{K_a}{K_h}\right)^{1/2}, \qquad z_D = \left(\frac{z}{L}\right)\left(\frac{K_a}{K_v}\right)^{1/2}, \tag{3.8}$$

$$L_D = \left(\frac{K_a}{K_v}\right)^{1/2}, \qquad H_D = \left(\frac{H}{L}\right)\left(\frac{K_a}{K_v}\right)^{1/2}, \tag{3.9}$$

$$L_{1D} = \left(\frac{L_1}{L}\right)\left(\frac{K_a}{K_v}\right)^{1/2}, \qquad L_{2D} = \left(\frac{L_2}{L}\right)\left(\frac{K_a}{K_v}\right)^{1/2}, \tag{3.10}$$

$$L_{pD} = L_{2D} - L_{1D} = \left(\frac{L_p}{L}\right)\left(\frac{K_a}{K_v}\right)^{1/2}, \tag{3.11}$$

$$R_{eD} = \left(\frac{R_e}{L}\right)\left(\frac{K_a}{K_h}\right)^{1/2}, \qquad R_{0D} = \left(\frac{R_0}{L}\right)\left(\frac{K_a}{K_h}\right)^{1/2}. \tag{3.12}$$

The dimensionless wellbore radius is [8]

$$R_{wD} \approx \left[\left(\frac{K_h}{K_v}\right)^{1/4} + \left(\frac{K_h}{K_v}\right)^{-1/4}\right]\left(\frac{R_w}{2L}\right). \tag{3.13}$$

Assume q is the point convergence intensity at the point sink $(R_0, 0, z')$, the partially penetrating well is a uniform line sink, the total productivity of the well is Q, and there holds

$$q = \frac{Q}{L_{pD}}. \tag{3.14}$$

Define the dimensionless pressures:

$$P_D = \frac{K_a L(P_e - P)}{\mu q B}, \qquad P_{wD} = \frac{K_a L(P_e - P_w)}{\mu q B}. \tag{3.15}$$

Then (3.7) becomes [6, 7]

$$\frac{\partial^2 P_D}{\partial x_D{}^2} + \frac{\partial^2 P_D}{\partial y_D{}^2} + \frac{\partial^2 P_D}{\partial z_D{}^2} = -\delta(x_D - R_{0D})\delta(y_D)\delta(z_D - z'_D), \quad \text{in } \Omega_D, \tag{3.16}$$

where

$$\Omega_D = \left\{(x_D, y_D, z_D) \mid x_D^2 + y_D^2 < R_{eD}^2,\ 0 < z_D < H_D\right\}. \tag{3.17}$$

If point $\mathbf{r}_0$ and point $\mathbf{r}$ are with distances ρ_0 and ρ, respectively, from the axis of symmetry of the cylindrical body, then the dimensionless off-center distances are

$$\rho_{0D} = \left(\frac{\rho_0}{L}\right)\left(\frac{K_a}{K_h}\right)^{1/2}, \qquad \rho_D = \left(\frac{\rho}{L}\right)\left(\frac{K_a}{K_h}\right)^{1/2}. \tag{3.18}$$

There holds

$$\begin{aligned}\left(\frac{\pi}{H_D}\right)\left(2R_{eD} - \rho_{0D} - \rho_D - \sqrt{\rho_{0D}\rho_D}\right) &= \left(\frac{K_v}{K_h}\right)^{1/2}\left(\frac{\pi L}{2H}\right)\left(\frac{4R_e}{L} - \frac{2\rho_0}{L} - \frac{2\rho}{L} - \frac{2\sqrt{\rho_0\rho}}{L}\right)\\ &= \left(\frac{K_v}{K_h}\right)^{1/2}\left(\frac{\pi R_e}{H}\right)\left(2 - \frac{\rho_0}{R_e} - \frac{\rho}{R_e} - \frac{\sqrt{\rho_0\rho}}{R_e}\right)\\ &= \left(\frac{K_v}{K_h}\right)^{1/2}\left(\frac{\pi R_e}{H}\right)\left(2 - \vartheta_0 - \vartheta - \sqrt{\vartheta_0\vartheta}\right),\end{aligned} \tag{3.19}$$

where

$$\vartheta_0 = \frac{\rho_0}{R_e}, \qquad \vartheta = \frac{\rho}{R_e}. \tag{3.20}$$

Since the reservoir is with constant pressure outer boundary (edge water), in order to delay water encroachment, a producing well must keep a sufficient distance from the outer boundary. Thus in this paper, it is reasonable to assume that

$$\vartheta_0 \leq 0.6, \qquad \vartheta \leq 0.6 \tag{3.21}$$

If

$$\vartheta_0 = \vartheta = 0.6, \qquad \frac{K_v}{K_h} = 0.25, \qquad \frac{R_e}{H} = 15, \tag{3.22}$$

then

$$\begin{aligned}&\left(\frac{K_v}{K_h}\right)^{1/2}\left(\frac{\pi R_e}{H}\right)\left(2 - \vartheta_0 - \vartheta - \sqrt{\vartheta_0\vartheta}\right)\\ &\quad= 0.25^{1/2} \times (\pi \times 15) \times \left(2.0 - 0.6 - 0.6 - \sqrt{0.6 \times 0.6}\right) = 4.7124,\\ &\qquad\qquad \exp(-4.7124) = 8.983 \times 10^{-3}.\end{aligned} \tag{3.23}$$

Moreover if

$$\vartheta_0 = \vartheta = 0.5, \qquad \frac{K_v}{K_h} = 0.5, \qquad \frac{R_e}{H} = 10, \tag{3.24}$$

then

$$\begin{aligned}
&\left(\frac{K_v}{K_h}\right)^{1/2}\left(\frac{\pi R_e}{H}\right)\left(2-\vartheta_0-\vartheta-\sqrt{\vartheta_0\vartheta}\right)\\
&\quad = 0.5^{1/2}\times(\pi\times 10)\times\left(2.0-0.5-0.5-\sqrt{0.5\times 0.5}\right) = 11.107,\\
&\qquad \exp(-11.107) = 1.501\times 10^{-5}\ .
\end{aligned} \tag{3.25}$$

Recall (3.19), there holds

$$\exp\left[-\left(\frac{\pi}{H_D}\right)\left(2R_{eD}-\rho_{0D}-\rho_D-\sqrt{\rho_{0D}\rho_D}\right)\right] = \left(\frac{K_v}{K_h}\right)^{1/2}\left(\frac{\pi R_e}{H}\right)\left(2-\vartheta_0-\vartheta-\sqrt{\vartheta_0\vartheta}\right), \tag{3.26}$$

since there holds (3.21), and according to the aforementioned calculations in (3.22), (3.23), (3.24), and (3.25), we obtain

$$\exp\left[-\left(\frac{\pi}{H_D}\right)\left(2R_{eD}-\rho_{0D}-\rho_D-\sqrt{\rho_{0D}\rho_D}\right)\right]\approx 0. \tag{3.27}$$

Because

$$0<\left(\frac{\pi}{H_D}\right)\left(2R_{eD}-\rho_{0D}-\rho_D-\sqrt{\rho_{0D}\rho_D}\right)<\left(\frac{\pi}{H_D}\right)\left(2R_{eD}-\rho_{0D}-\rho_D\right), \tag{3.28}$$

thus

$$\exp\left[-\left(\frac{\pi}{H_D}\right)\left(2R_{eD}-\rho_{0D}-\rho_D-\sqrt{\rho_{0D}\rho_D}\right)\right]>\exp\left[-\left(\frac{\pi}{H_D}\right)\left(2R_{eD}-\rho_{0D}-\rho_D\right)\right]. \tag{3.29}$$

Combining (3.27) and (3.29), we have

$$\exp\left[-\left(\frac{\pi}{H_D}\right)\left(2R_{eD}-\rho_{0D}-\rho_D\right)\right]\approx 0. \tag{3.30}$$

4. Boundary Conditions

In this paper, we always assume constant pressure lateral boundary:

$$P(x,y,z)=P_e=P_i, \tag{4.1}$$

on cylindrical lateral surface:

$$\Gamma=\left\{(x,y,z)\mid x^2+y^2=R_e^2,\ 0<z<H\right\}. \tag{4.2}$$

Recall (3.15), the dimensionless form of constant pressure lateral boundary condition is

$$P_D(x_D, y_D, z_D) = 0, \tag{4.3}$$

on

$$\Gamma_D = \left\{ (x_D, y_D, z_D) \mid x_D^2 + y_D^2 = R_{eD}^2,\ 0 < z_D < H_D \right\}. \tag{4.4}$$

Also we have the following dimensionless equations for top and bottom boundary conditions:

(i) If the circular cylinder drainage volume is with top and bottom impermeable boundaries, that is, the boundaries at $z = 0$ and $z = H$ are both impermeable (e.g., the reservoir does not have gas cap drive or bottom water drive), then

$$\left.\frac{\partial P_D}{\partial z_D}\right|_{z_D=0} = 0; \qquad \left.\frac{\partial P_D}{\partial z_D}\right|_{z_D=H_D} = 0. \tag{4.5}$$

(ii) If the circular cylinder drainage volume is with impermeable boundary at $z = H$, constant pressure boundary at $z = 0$, (e.g., the reservoir has gas cap drive), then

$$P_D|_{z_D=0} = 0; \qquad \left.\frac{\partial P_D}{\partial z_D}\right|_{z_D=H_D} = 0. \tag{4.6}$$

(iii) If the circular cylinder drainage volume is with impermeable boundary at $z = 0$, constant pressure boundary at $z = H$ (e.g., the reservoir has bottom water drive), then

$$P_D|_{z_D=H_D} = 0; \qquad \left.\frac{\partial P_D}{\partial z_D}\right|_{z_D=0} = 0. \tag{4.7}$$

(iv) If the circular cylinder drainage volume is with top and bottom constant pressure boundaries, that is, the boundaries at $z = 0$ and $z = H$ are both constant pressure boundaries (e.g., the reservoir has both gas cap drive and bottom water drive), then

$$P_D|_{z_D=0} = 0; \qquad P_D|_{z_D=H_D} = 0. \tag{4.8}$$

5. Point Sink Solutions

For convenience, we use dimensionless variables given by (3.8) through (3.13), but we drop the subscript D. In order to obtain the dimensionless pressure of a point sink in a circular cylinder reservoir, we need to solve a dimensionless Laplace equation in dimensionless space:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = -\delta(x - x')\delta(y)\delta(z - z'), \quad \text{in } \Omega, \tag{5.1}$$

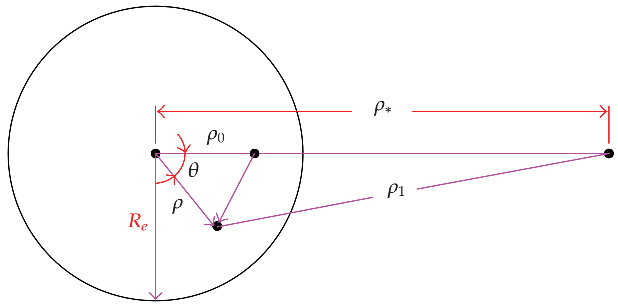

Figure 2: Geometric representation of a circular system.

where

$$\Omega = \left\{ (x, y, z) \mid x^2 + y^2 < R_e^2,\ 0 < z < H \right\}. \tag{5.2}$$

The following initial reservoir condition and lateral reservoir boundary condition will be used to obtain point sink pressure in a circular cylinder reservoir with constant pressure outer boundary:

$$\begin{aligned} P(t, x, y, z)\big|_{t=0} &= 0, \quad \text{in } \Omega, \\ P(t, x, y, z) &= 0, \quad \text{on } \Gamma, \end{aligned} \tag{5.3}$$

where $\Gamma = \{(x, y, z) \mid x^2 + y^2 = R_e^2,\ 0 < z < H\}$.

The problem under consideration is that of fluid flow toward a point sink from an off-center position within a circular of radius R_e. We want to determine the pressure change at an observation point with a distance ρ from the center of circle.

Figure 2 is a geometric representation of the system. In Figure 2, the point sink $\mathbf{r}_0$ and the observation point $\mathbf{r}$ are with distances ρ_0 and ρ, respectively, from the circular center; and the two points are separated at the center by an angle θ. The inverse point of the point sink $\mathbf{r}_0$ with respect to the circle is point $\mathbf{r}_*$. Point $\mathbf{r}_*$ is with a distance ρ_* from the center, and ρ_1 from the observation point. The inverse point is the point outside the circle, on the extension of the line connecting the center and the point sink, and such that

$$\rho_* = \frac{R_e^2}{\rho_0}. \tag{5.4}$$

Assume R' is the distance between point $\mathbf{r}$ and point $\mathbf{r}_0$, then

$$R' = \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\theta}. \tag{5.5}$$

If the observation point $\mathbf{r}$ is on the drainage circle, $\rho = R_e$, then

$$R' = \sqrt{R_e^2 + \rho_0^2 - 2R_e\rho_0 \cos\theta}, \quad R_e > \rho_0 > 0. \tag{5.6}$$

If the observation point **r** is on the wellbore, then

$$R' = R_w. \tag{5.7}$$

Define

$$\lambda_n = \frac{n\pi}{H}. \tag{5.8}$$

5.1. Impermeable Upper and Lower Boundaries

If upper and lower boundaries are impermeable,

$$\left.\frac{\partial P}{\partial z}\right|_{z=0} = 0, \qquad \left.\frac{\partial P}{\partial z}\right|_{z=H} = 0, \tag{5.9}$$

obviously for such impermeable boundary conditions, we have

$$\delta(z - z') = \frac{\sum_{n=0}^{\infty} \cos(\lambda_n z') \cos(\lambda_n z)}{(H d_n)}, \tag{5.10}$$

where

$$d_n = \begin{cases} 1, & \text{if } n = 0, \\ \dfrac{1}{2}, & \text{if } n > 0. \end{cases} \tag{5.11}$$

Let

$$P(x, y, z) = \sum_{n=0}^{\infty} \varphi_n(x, y) \cos(\lambda_n z), \tag{5.12}$$

and substituting (5.12) into (5.1) and compare the coefficients of $\cos(\lambda_n z)$, we obtain

$$\Delta\varphi_n - \lambda_n^2 \varphi_n = -\frac{\cos(\lambda_n z')\delta(x - x')\delta(y)}{(H d_n)}, \tag{5.13}$$

in circular area $\Omega_1 = \{(x, y) \mid x^2 + y^2 < R_e^2\}$, and

$$\varphi_n = 0, \tag{5.14}$$

on circumference $\Gamma_1 = \{(x, y) \mid x^2 + y^2 = R_e^2\}$, and Δ is two-dimensional Laplace operator,

$$\Delta\varphi_n = \frac{\partial^2 \varphi_n}{\partial x^2} + \frac{\partial^2 \varphi_n}{\partial y^2}. \tag{5.15}$$

Case 1. If $n = 0$, then

$$\begin{aligned} \Delta\varphi_0 &= -\frac{\delta(x-x')\delta(y)}{H}, \quad \text{in } \Omega_1, \\ \varphi_0 &= 0, \quad \text{on } \Gamma_1. \end{aligned} \tag{5.16}$$

Using Green's function of Laplace problem in a circular domain, we obtain [9–11]

$$\varphi_0(x,y;x',0) = \left(\frac{1}{2\pi H}\right)\left\{\ln\left[\frac{|\rho-\rho_*|}{|\rho-\rho_0|}\frac{|\rho_0|}{R_e}\right]\right\}. \tag{5.17}$$

Case 2. If $n > 0$, then φ_n satisfies (5.13). Since $[-1/(2\pi)]K_0(\lambda_n R')$ satisfies the equations:

$$\begin{aligned} \Delta u - \lambda_n^2 u &= \delta(x-x')\delta(y), \\ u &= 0, \quad \text{on } \Gamma_1. \end{aligned} \tag{5.18}$$

So $[-\alpha_n/(2\pi)]K_0(\lambda_n R')$ is a basic solution of (5.13), where

$$\alpha_n = \left(-\frac{1}{Hd_n}\right)\cos\left(\frac{n\pi z'}{H}\right) = \left(-\frac{2}{H}\right)\cos(\lambda_n z'), \tag{5.19}$$

$$\beta_n = \frac{\alpha_n}{(2\pi)} = \left(\frac{-2}{2\pi H}\right)\cos\left(\frac{n\pi z'}{H}\right) = \left(\frac{-1}{\pi H}\right)\cos(\lambda_n z'). \tag{5.20}$$

Define

$$\psi_n = \varphi_n + \beta_n K_0(\lambda_n R'), \tag{5.21}$$

thus

$$\varphi_n = \psi_n - \beta_n K_0(\lambda_n R'), \tag{5.22}$$

then ψ_n satisfies homogeneous equation

$$\begin{aligned} \Delta\psi_n - \lambda_n^2\psi_n &= 0, \quad \text{in } \Omega_1, \\ \psi_n &= \beta_n K_0(\lambda_n R'), \quad \text{on } \Gamma_1, \end{aligned} \tag{5.23}$$

and R' has the same meaning as in (5.6).

Under polar coordinates representation of Laplace operator and by using methods of separation of variables, we obtain a general solution [9–11]:

$$\psi_n = \left[A_{0n} I_0(\lambda_n \rho) + B_{0n} K_0(\lambda_n \rho)\right][a_{0n}\theta + b_{0n}] + \sum_{m=1}^{\infty}[A_{mn} I_m(\lambda_n \rho) + B_{mn} K_m(\lambda_n \rho)] \times [a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta)], \quad (5.24)$$

where A_{in}, B_{in}, a_{in}, b_{in}, $i = 0, 1, \ldots$ are undetermined coefficients.

Because ψ_n is continuously bounded within Ω_1, but $K_i(0) = \infty$, so there holds

$$B_{in} = 0, \quad i = 0, 1, \ldots. \quad (5.25)$$

There hold [7, 12]

$$K_\upsilon(z) = \left(\frac{\pi i}{2}\right) e^{\upsilon\pi i/2} H_\upsilon^{(1)}(zi), \qquad I_\upsilon(z) = e^{-\upsilon\pi i/2} J_\upsilon(zi), \quad (5.26)$$

where $K_\upsilon(z)$ is modified Bessel function of second kind and order υ, $I_\upsilon(z)$ is modified Bessel function of first kind and order υ, $J_\upsilon(z)$ is Bessel function of first kind and order υ, $H_\upsilon^{(1)}(z)$ is Hankel function of first kind and order υ, and $i = \sqrt{-1}$.

Also there hold [13, page 979]

$$H_0^{(1)}(\sigma R') = J_0(\sigma\rho_0) H_0^{(1)}(\sigma R_e) + 2\sum_{m=1}^{\infty} J_m(\sigma\rho_0) H_m^{(1)}(\sigma R_e)\cos(m\theta), \quad (5.27)$$

$$K_0(\lambda_n R') = \left(\frac{\pi i}{2}\right) H_0^{(1)}(i\lambda_n R'). \quad (5.28)$$

Let $\sigma = i\lambda_n$, (note that $i^2 = -1$) putting (5.26) into (5.27), and using (5.28), we have the following Cosine Fourier expansions of $K_0(\lambda_n R')$ [13, page 952]:

$$\begin{aligned} K_0(\lambda_n R') &= \left(\frac{\pi i}{2}\right)\left[J_0(i\lambda_n\rho_0) H_0^{(1)}(i\lambda_n R_e) + 2\sum_{m=1}^{\infty} J_m(i\lambda_n\rho_0) H_m^{(1)}(i\lambda_n R_e)\cos(m\theta)\right] \\ &= J_0(i\lambda_n\rho_0) K_0(\lambda_n R_e) + 2\sum_{m=1}^{\infty} e^{-m\pi i/2} J_m(i\lambda_n\rho_0) K_m(\lambda_n R_e)\cos(m\theta) \\ &= I_0(\lambda_n\rho_0) K_0(\lambda_n R_e) + 2\sum_{m=1}^{\infty} I_m(\lambda_n\rho_0) K_m(\lambda_n R_e)\cos(m\theta). \end{aligned} \quad (5.29)$$

Note that $\psi_n = \beta_n K_0(\lambda_n R')$ on Γ_1, comparing coefficients of Cosine Fourier expansions of $K_0(\lambda_n R')$ in (5.29) and (5.24), we obtain

$$a_{0n} = 0, \qquad b_{0n} = 1, \qquad b_{in} = 0, \quad i = 1, 2, \ldots. \tag{5.30}$$

Define

$$\Upsilon_{mn} = a_{mn} A_{mn}, \quad n = 0, 1, 2, \ldots, \tag{5.31}$$

and recall (5.24), then we have

$$\psi_n = \sum_{m=0}^{\infty} \Upsilon_{mn} I_m(\lambda_n \rho) \cos(m\theta), \quad n = 0, 1, 2, \ldots, \tag{5.32}$$

where

$$\begin{aligned} \Upsilon_{0n} &= \frac{\beta_n K_0(\lambda_n R_e) I_0(\lambda_n \rho_0)}{I_0(\lambda_n R_e)}, \\ \Upsilon_{mn} &= \frac{2\beta_n K_m(\lambda_n R_e) I_m(\lambda_n \rho_0)}{I_m(\lambda_n R_e)}. \end{aligned} \tag{5.33}$$

There hold [13, page 919]

$$I_m(x) \simeq \frac{\exp(x)}{(2\pi x)^{1/2}}, \quad K_m(x) \simeq \frac{[\pi/(2x)]^{1/2}}{\exp(x)}, \quad x \gg 1, \forall m \geq 0. \tag{5.34}$$

Note that H in Formula (5.8) is in dimensionless form, recall Formulas (3.9), (3.12) and (3.18), for dimensionless H, R_e, ρ_0, ρ, there hold

$$\lambda_n R_e \gg 1, \qquad \lambda_n \rho_0 \gg 1, \qquad \lambda_n \rho \gg 1, \tag{5.35}$$

thus

$$\frac{K_m(\lambda_n R_e)}{I_m(\lambda_n R_e)} \approx \pi \exp(-2\lambda_n R_e), \tag{5.36}$$

$$\begin{aligned} I_m(\lambda_n \rho_0) I_m(\lambda_n \rho) &\approx \left[\frac{\exp(\lambda_n \rho_0)}{(2\pi\lambda_n \rho_0)^{1/2}}, \frac{\exp(\lambda_n \rho)}{(2\pi\lambda_n \rho)^{1/2}} \right] \\ &= \frac{\exp[\lambda_n(\rho + \rho_0)]}{(2\pi\lambda_n)(\rho\rho_0)^{1/2}}, \end{aligned} \tag{5.37}$$

$$\begin{aligned} \Upsilon_{mn} I_m(\lambda_n \rho) &= \frac{2\beta_n K_m(\lambda_n R_e) I_m(\lambda_n \rho_0) I_m(\lambda_n \rho)}{I_m(\lambda_n R_e)} \\ &\approx (2\beta_n)[\pi \exp(-2\lambda_n R_e)] \left\{ \frac{\exp[\lambda_n(\rho + \rho_0)]}{(2\pi\lambda_n)(\rho\rho_0)^{1/2}} \right\} \\ &= (2\beta_n) \left[\frac{\pi}{(2\pi\lambda_n)(\rho\rho_0)^{1/2}} \right] \exp[-\lambda_n(2R_e - \rho_0 - \rho)] \\ &= \left[\frac{\beta_n}{\lambda_n(\rho\rho_0)^{1/2}} \right] \exp[-\lambda_n(2R_e - \rho_0 - \rho)]. \end{aligned} \tag{5.38}$$

There holds

$$\begin{aligned} |\varphi_n| &= \sum_{m=0}^{\infty} |\Upsilon_{mn} I_m(\lambda_n \rho) \cos(m\theta)| \\ &< \sum_{m=0}^{\infty} |\Upsilon_{mn} I_m(\lambda_n \rho)| \\ &= \sum_{m=0}^{\infty} \left| \frac{2\beta_n K_m(\lambda_n R_e) I_m(\lambda_n \rho_0) I_m(\lambda_n \rho)}{I_m(\lambda_n R_e)} \right|. \end{aligned} \tag{5.39}$$

Combining Formulas (3.18), (5.38), and (5.39), we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} \left| \varphi_n \cos\left(\frac{n\pi z}{H}\right) \right| \\ &\quad \leq \sum_{n=1}^{\infty} |\varphi_n| \\ &\quad = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left| \frac{2\beta_n K_m(\lambda_n R_e) I_m(\lambda_n \rho_0) I_m(\lambda_n \rho)}{I_m(\lambda_n R_e)} \right| \end{aligned}$$

$$\begin{aligned}
&\le \sum_{n=1}^{\infty}\left[\left|\frac{2\beta_n K_0(\lambda_n R_e) I_0(\lambda_n \rho_0) I_0(\lambda_n \rho)}{I_0(\lambda_n R_e)}\right| + \sum_{m=1}^{\infty}\left|\frac{2\beta_n K_m(\lambda_n R_e) I_m(\lambda_n \rho_0) I_m(\lambda_n \rho)}{I_m(\lambda_n R_e)}\right|\right] \\
&= \sum_{n=1}^{\infty}\left|\frac{2\beta_n K_0(\lambda_n R_e) I_0(\lambda_n \rho_0) I_0(\lambda_n \rho)}{I_0(\lambda_n R_e)}\right| + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\frac{2\beta_n K_m(\lambda_n R_e) I_m(\lambda_n \rho_0) I_m(\lambda_n \rho)}{I_m(\lambda_n R_e)}\right| \\
&= \Xi_1 + \Xi_2,
\end{aligned} \tag{5.40}$$

where

$$\Xi_1 = \sum_{n=1}^{\infty}\left|\frac{2\beta_n K_0(\lambda_n R_e) I_0(\lambda_n \rho_0) I_0(\lambda_n \rho)}{I_0(\lambda_n R_e)}\right|, \tag{5.41}$$

$$\Xi_2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\frac{2\beta_n K_m(\lambda_n R_e) I_m(\lambda_n \rho_0) I_m(\lambda_n \rho)}{I_m(\lambda_n R_e)}\right|. \tag{5.42}$$

There holds

$$\begin{aligned}
\Xi_1 &= \sum_{n=1}^{\infty}\left|\frac{2\beta_n K_0(\lambda_n R_e) I_0(\lambda_n \rho_0) I_0(\lambda_n \rho)}{I_0(\lambda_n R_e)}\right| \\
&\approx \sum_{n=1}^{\infty}\left[\frac{|\beta_n|}{\lambda_n (\rho\rho_0)^{1/2}}\right]\exp\left[-\lambda_n (2R_e - \rho_0 - \rho)\right] \\
&= \sum_{n=1}^{\infty}\left[\frac{1}{n\pi^2 (\rho\rho_0)^{1/2}}\right]\exp\left[-\left(\frac{n\pi}{H}\right)(2R_e - \rho_0 - \rho)\right] \\
&< \sum_{n=1}^{\infty}\left[\frac{1}{\pi^2 (\rho\rho_0)^{1/2}}\right]\exp\left[-\left(\frac{n\pi}{H}\right)(2R_e - \rho_0 - \rho)\right] \\
&\approx \left[\frac{1}{\pi^2 (\rho\rho_0)^{1/2}}\right]\left\{\frac{\exp\left[-(\pi/H)(2R_e - \rho_0 - \rho)\right]}{1 - \exp\left[-(\pi/H)(2R_e - \rho_0 - \rho)\right]}\right\} \\
&\approx 0,
\end{aligned} \tag{5.43}$$

where we use Formula (3.26),

$$\exp\left[-\left(\frac{\pi}{H}\right)(2R_e - \rho_0 - \rho)\right] \approx 0, \tag{5.44}$$

$$x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{x}{1-x}, \quad 0 < x < 1. \tag{5.45}$$

If $m > -1/2$, there holds [13, page 916]

$$I_m(z) = \left[\frac{(z/2)^m}{\Gamma(m+1/2)\Gamma(1/2)}\right]\int_{-1}^{1}(1-t^2)^{m-1/2}\cosh(zt)\,dt, \tag{5.46}$$

thus for $m \geq 1$,

$$\begin{aligned}
I_m(\lambda_n\rho) &\leq \left[\frac{(\lambda_n\rho/2)^m}{\Gamma(m+1/2)\Gamma(1/2)}\right]\int_{-1}^{1}\cosh(\lambda_n\rho t)\,dt \\
&= \left[\frac{2(\lambda_n\rho/2)^m}{(\lambda_n\rho)\Gamma(m+1/2)\Gamma(1/2)}\right]\sinh(\lambda_n\rho) \\
&= \left[\frac{(\lambda_n\rho/2)^{m-1}}{\Gamma(m+1/2)\Gamma(1/2)}\right]\sinh(\lambda_n\rho) \\
&< \left[\frac{(\lambda_n\rho/2)^{m-1}}{2\Gamma(m+1/2)\Gamma(1/2)}\right]\exp(\lambda_n\rho),
\end{aligned} \tag{5.47}$$

where we use

$$\begin{aligned}
\int_{-1}^{1}\cosh(\lambda_n\rho t)\,dt &= \frac{2\sinh(\lambda_n\rho)}{\lambda_n\rho}, \\
\sinh(\lambda_n\rho) &< \frac{\exp(\lambda_n\rho)}{2},
\end{aligned} \tag{5.48}$$

and if $-1 < t < 1$, $m \geq 1$, then

$$(1-t^2)^{m-1/2} \leq 1. \tag{5.49}$$

Putting Formula (5.47) into Formula (5.42), we obtain

$$\begin{aligned}
\Xi_2 &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\frac{2\beta_n K_m(\lambda_n R_e) I_m(\lambda_n\rho_0) I_m(\lambda_n\rho)}{I_m(\lambda_n R_e)}\right| \\
&< \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(2\pi|\beta_n|)\exp[-\lambda_n(2R_e-\rho_0-\rho)]\left[\frac{(\lambda_n\rho/2)^{m-1}}{2\Gamma(m+1/2)\Gamma(1/2)}\right]\left[\frac{(\lambda_n\rho_0/2)^{m-1}}{2\Gamma(m+1/2)\Gamma(1/2)}\right] \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{2}{H}\right)\frac{(\lambda_n^2\rho\rho_0/4)^{m-1}}{[2\Gamma(m+1/2)\Gamma(1/2)]^2}\exp[-\lambda_n(2R_e-\rho_0-\rho)] \\
&= \sum_{n=1}^{\infty}\left(\frac{2}{H}\right)\exp[-\lambda_n(2R_e-\rho_0-\rho)]\sum_{m=1}^{\infty}\frac{(\lambda_n^2\rho\rho_0/4)^{m-1}}{[2\Gamma(m+1/2)\Gamma(1/2)]^2}.
\end{aligned} \tag{5.50}$$

Note that

$$\begin{aligned}\Gamma(m+1/2) &= \frac{1\times 3\times 5\times\cdots\times(2m-1)\sqrt{\pi}}{2^m}\\ &> \frac{1\times 2\times 6\times\cdots\times(2m-2)\sqrt{\pi}}{2^m}\\ &= \frac{2^{m-1}(m-1)!\sqrt{\pi}}{2^m}\\ &= \frac{(m-1)!\sqrt{\pi}}{2},\end{aligned} \tag{5.51}$$

then we obtain

$$\begin{aligned}\Xi_2 &< \sum_{n=1}^{\infty}\left(\frac{2}{H}\right)\exp[-\lambda_n(2R_e-\rho_0-\rho)]\sum_{m=1}^{\infty}\frac{(\lambda_n^2\rho\rho_0/4)^{m-1}}{[2\Gamma(m+1/2)\Gamma(1/2)]^2}\\ &< \sum_{n=1}^{\infty}\left(\frac{2}{H}\right)\exp[-\lambda_n(2R_e-\rho_0-\rho)]\sum_{m=1}^{\infty}\frac{(\lambda_n^2\rho\rho_0/4)^{m-1}}{[(m-1)!\pi]^2}\\ &= \sum_{n=1}^{\infty}\left(\frac{2}{H}\right)\exp[-\lambda_n(2R_e-\rho_0-\rho)]\sum_{k=0}^{\infty}\frac{(\lambda_n\sqrt{\rho\rho_0}/2)^{2k}}{(k!\pi)^2}\\ &= \sum_{n=1}^{\infty}\left(\frac{2}{\pi^2 H}\right)\exp[-\lambda_n(2R_e-\rho_0-\rho)]I_0(\lambda_n\sqrt{\rho\rho_0}),\end{aligned} \tag{5.52}$$

where we use [13, page 919]

$$I_0(z)=\sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{(k!)^2},\qquad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}. \tag{5.53}$$

Note that $\lambda_n\sqrt{\rho\rho_0}\gg 1$, and we have

$$I_0(\lambda_n\sqrt{\rho\rho_0})\approx\frac{\exp(\lambda_n\sqrt{\rho\rho_0})}{(2\pi\lambda_n\sqrt{\rho\rho_0})^{1/2}}, \tag{5.54}$$

thus Formula (5.52) can be simplified as follows:

$$\begin{aligned}
\Xi_2 &< \sum_{n=1}^{\infty}\left(\frac{2}{\pi^2 H}\right)\exp\left[-\lambda_n(2R_e-\rho_0-\rho)\right]I_0\left(\lambda_n\sqrt{\rho\rho_0}\right)\\
&\approx \sum_{n=1}^{\infty}\left(\frac{2}{\pi^2 H}\right)\exp\left[-\lambda_n(2R_e-\rho_0-\rho)\right]\left[\frac{\exp\left(\lambda_n\sqrt{\rho\rho_0}\right)}{\left(2\pi\lambda_n\sqrt{\rho\rho_0}\right)^{1/2}}\right]\\
&< \sum_{n=1}^{\infty}\left[\frac{2}{\pi^2 H(2\pi)^{1/2}(\rho\rho_0)^{1/4}}\right]\frac{\exp\left[-\lambda_n\left(2R_e-\rho_0-\rho-\sqrt{\rho\rho_0}\right)\right]}{\lambda_n^{1/2}}\\
&= \sum_{n=1}^{\infty}\left[\frac{2}{\pi^3(2nH)^{1/2}(\rho\rho_0)^{1/4}}\right]\exp\left[-\left(\frac{n\pi}{H}\right)\left(2R_e-\rho_0-\rho-\sqrt{\rho\rho_0}\right)\right]\\
&< \sum_{n=1}^{\infty}\left[\frac{2}{\pi^3(2H)^{1/2}(\rho\rho_0)^{1/4}}\right]\exp\left[-\left(\frac{n\pi}{H}\right)\left(2R_e-\rho_0-\rho-\sqrt{\rho\rho_0}\right)\right]\\
&= \left[\frac{2}{\pi^3(2H)^{1/2}(\rho\rho_0)^{1/4}}\right]\left\{\frac{\exp\left[-(\pi/H)\left(2R_e-\rho_0-\rho-\sqrt{\rho\rho_0}\right)\right]}{1-\exp\left[-(\pi/H)\left(2R_e-\rho_0-\rho-\sqrt{\rho\rho_0}\right)\right]}\right\}\\
&\approx 0,
\end{aligned} \tag{5.55}$$

where we use Formulas (5.45) and (3.26)

$$\exp\left[-\left(\frac{\pi}{H}\right)\left(2R_e-\rho_0-\rho-\sqrt{\rho\rho_0}\right)\right]\approx 0. \tag{5.56}$$

Combining Formulas (5.40), (5.43), and (5.55), we prove

$$\sum_{n=1}^{\infty}\varphi_n\cos\left(\frac{n\pi z}{H}\right)\approx 0. \tag{5.57}$$

Combining Formulas (5.4), (5.12), (5.17), (5.20), (5.22), (5.29), and (5.57), the point convergence pressure of point $(x',0,z')$ is

$$\begin{aligned}
P(x,y,z;x',0,z') &= \left(\frac{1}{2\pi H}\right)\left\{\ln\left[\frac{\left|R_e^2-\rho\rho_0\right|}{|\rho-\rho_0|R_e}\right]\right\}\\
&\quad+\sum_{n=1}^{\infty}\left[\frac{\beta_n K_0(\lambda_n R_e)I_0(\lambda_n R')}{I_0(\lambda_n R_e)}-\beta_n K_0(\lambda_n R')\right]\cos(\lambda_n z)\\
&= \left(\frac{-1}{\pi H}\right)\sum_{n=1}^{\infty}\left[\frac{K_0(\lambda_n R_e)I_0(\lambda_n R')}{I_0(\lambda_n R_e)}-K_0(\lambda_n R')\right]\cos(\lambda_n z)\cos(\lambda_n z')\\
&\quad+\left(\frac{1}{2\pi H}\right)\left\{\ln\left[\frac{\left|R_e^2-\rho\rho_0\right|}{|\rho-\rho_0|R_e}\right]\right\}.
\end{aligned} \tag{5.58}$$

5.2. Constant Pressure Upper or Lower Boundaries

If the reservoir is with gas cap and impermeable bottom boundary, then

$$P|_{z=0} = 0, \qquad \left.\frac{\partial P}{\partial z}\right|_{z=H} = 0, \tag{5.59}$$

and assume the outer boundary is at constant pressure

$$P = 0, \tag{5.60}$$

on cylindrical surface $\Gamma = \{(x, y) \mid x^2 + y^2 = R_e^2,\ 0 < z < H\}$.

Define

$$\begin{gathered} \omega_n = \frac{[(2n-1)\pi]}{(2H)}, \\ g_n(z) = \sqrt{\frac{2}{H}} \sin(\omega_n z), \quad (n = 1, 2, 3, \ldots), \end{gathered} \tag{5.61}$$

then under the boundary condition of (5.59), we have

$$\delta(z - z') = \sum_{n=1}^{\infty} g_n(z) g_n(z'). \tag{5.62}$$

Let

$$P(x, y, z) = \sum_{n=1}^{\infty} \varphi_n(x, y) \sin(\omega_n z), \tag{5.63}$$

where φ_n satisfies

$$\Delta\varphi_n - \omega_n^2 \varphi_n = \left(-\frac{2}{H}\right) \delta(x - x') \delta(y) \sin(\omega_n z'), \tag{5.64}$$

in Ω_1, and

$$\varphi_n = 0, \tag{5.65}$$

on Γ_1.

Let

$$\psi_n = \varphi_n + \zeta_n K_0(\omega_n R'), \tag{5.66}$$

where

$$\zeta_n = \left(\frac{-1}{\pi H}\right)\sin(\omega_n z'), \tag{5.67}$$

and R' has the same meaning as in Formula (5.6).

Thus ψ_n satisfies homogeneous equation:

$$\Delta\psi_n - \omega_n^2\psi_n = 0, \tag{5.68}$$

in Ω_1, and

$$\psi_n = \zeta_n K_0(\omega_n R'), \tag{5.69}$$

on Γ_1.

Using polar coordinates, we have

$$\psi_n = \frac{\zeta_n K_0(\omega_n R_e) I_0(\omega_n R')}{I_0(\omega_n R_e)}, \tag{5.70}$$

then

$$\varphi_n = \psi_n - \zeta_n K_0(\omega_n R'), \tag{5.71}$$

and point convergence pressure of point $(x', 0, z')$ is:

$$\begin{aligned} P(x, y, z; x', 0, z') &= \sum_{n=1}^{\infty}\left[\frac{\zeta_n K_0(\omega_n R_e) I_0(\omega_n R')}{I_0(\omega_n R_e)} - \zeta_n K_0(\omega_n R')\right]\sin(\omega_n z) \\ &= \left(\frac{-1}{\pi H}\right)\sum_{n=1}^{\infty}\left[\frac{K_0(\omega_n R_e) I_0(\omega_n R')}{I_0(\omega_n R_e)} - K_0(\omega_n R')\right]\sin(\omega_n z)\sin(\omega_n z'). \end{aligned} \tag{5.72}$$

If the reservoir is with bottom water and impermeable top boundary, then

$$P|_{z=H} = 0, \qquad \left.\frac{\partial P}{\partial z}\right|_{z=0} = 0, \tag{5.73}$$

and recall Formula (5.60), the outer boundary is at constant pressure.

Define

$$h_n(z) = \sqrt{\frac{2}{H}}\cos(\omega_n z), \quad (n = 1, 2, 3, \ldots), \tag{5.74}$$

then under the boundary condition of (5.73), we have

$$\delta(z-z') = \sum_{n=1}^{\infty} h_n(z) h_n(z'). \tag{5.75}$$

Let

$$P(x,y,z) = \sum_{n=1}^{\infty} \varphi_n(x,y)\cos(\omega_n z), \tag{5.76}$$

where φ_n satisfies

$$\Delta\varphi_n - \omega_n^2\varphi_n = \left(-\frac{2}{H}\right)\delta(x-x')\delta(y)\cos(\omega_n z'), \tag{5.77}$$

in Ω_1, and

$$\varphi_n = 0, \tag{5.78}$$

on Γ_1.

Let

$$\psi_n = \varphi_n + \eta_n K_0(\omega_n R'), \tag{5.79}$$

where

$$\eta_n = \left(\frac{-1}{\pi H}\right)\cos(\omega_n z'), \tag{5.80}$$

thus ψ_n satisfies

$$\Delta\psi_n - \omega_n^2\psi_n = 0, \tag{5.81}$$

in Ω_1, and

$$\psi_n = \eta_n K_0(\omega_n R'), \tag{5.82}$$

on Γ_1.

Using polar coordinates, we have

$$\psi_n = \frac{\eta_n K_0(\omega_n R_e) I_0(\omega_n R')}{I_0(\omega_n R_e)}, \tag{5.83}$$

then

$$\varphi_n = \psi_n - \eta_n K_0(\omega_n R'), \tag{5.84}$$

and point convergence pressure of point $(x', 0, z')$ is

$$\begin{aligned} P(x, y, z; x', 0, z') &= \sum_{n=1}^{\infty}\left[\frac{\eta_n K_0(\omega_n R_e) I_0(\omega_n R')}{I_0(\omega_n R_e)} - \eta_n K_0(\omega_n R')\right]\cos(\omega_n z) \\ &= \left(\frac{-1}{\pi H}\right)\sum_{n=1}^{\infty}\left[\frac{K_0(\omega_n R_e) I_0(\omega_n R')}{I_0(\omega_n R_e)} - K_0(\omega_n R')\right]\cos(\omega_n z)\cos(\omega_n z'). \end{aligned} \tag{5.85}$$

If the reservoir is with gas cap and bottom water, then

$$P|_{z=0} = 0, \qquad P|_{z=H} = 0, \tag{5.86}$$

and recall Formula (5.60), the outer boundary is at constant pressure.

Define

$$f_n(z) = \sqrt{\frac{2}{H}}\sin(\lambda_n z), \quad (n = 1, 2, 3, \ldots), \tag{5.87}$$

where λ_n has the same meaning as in Formula (5.8).

Under the boundary condition of (5.86), we have

$$\delta(z - z') = \sum_{n=1}^{\infty} f_n(z) f_n(z'). \tag{5.88}$$

Let

$$P(x, y, z) = \sum_{n=1}^{\infty} \varphi_n(x, y)\sin(\lambda_n z), \tag{5.89}$$

where φ_n satisfies

$$\Delta\varphi_n - \lambda_n^2 \varphi_n = \left(-\frac{2}{H}\right)\delta(x - x')\delta(y)\sin(\lambda_n z'), \tag{5.90}$$

in Ω_1, and

$$\varphi_n = 0, \tag{5.91}$$

on Γ_1.

Let

$$\psi_n = \varphi_n + \iota_n K_0(\lambda_n R'), \tag{5.92}$$

where

$$\iota_n = \left(\frac{-1}{\pi H}\right)\sin(\lambda_n z'), \tag{5.93}$$

thus ψ_n satisfies

$$\Delta\psi_n - \lambda_n^2\psi_n = 0, \tag{5.94}$$

in Ω_1, and

$$\psi_n = \iota_n K_0(\lambda_n R'), \tag{5.95}$$

on Γ_1.

Using polar coordinates, we have

$$\psi_n = \frac{\iota_n K_0(\lambda_n R_e) I_0(\lambda_n R')}{I_0(\lambda_n R_e)}, \tag{5.96}$$

then

$$\varphi_n = \psi_n - \iota_n K_0(\lambda_n R'), \tag{5.97}$$

and point convergence pressure of point $(x', 0, z')$ is

$$\begin{aligned} P(x, y, z; x', 0, z') &= \sum_{n=1}^{\infty}\left[\frac{\iota_n K_0(\lambda_n R_e) I_0(\lambda_n R')}{I_0(\lambda_n R_e)} - \iota_n K_0(\lambda_n R')\right]\sin(\lambda_n z) \\ &= \left(\frac{-1}{\pi H}\right)\sum_{n=1}^{\infty}\left[\frac{K_0(\lambda_n R_e) I_0(\lambda_n R')}{I_0(\lambda_n R_e)} - K_0(\lambda_n R')\right]\sin(\lambda_n z)\sin(\lambda_n z'). \end{aligned} \tag{5.98}$$

6. Uniform Line Sink Solutions

For convenience, we drop the subscript D for dimensionless variables. In order to calculate the pressure at wellbore, assume the observation point $\mathbf{r}$ is on the wellbore, and recall Formula (5.7), $R' = R_w$ and

$$\left|\rho - \rho_0\right| = R_w, \qquad \rho\rho_0 = \left(\rho_0 + R_w\right)\rho_0 \approx \rho_0^2 = R_0^2. \tag{6.1}$$

The partially penetrating vertical well is taken as a uniform line sink. Recall Formula (5.9), integrate z' at both sides of Formula (5.58) from L_1 to L_2, if the upper and lower boundaries are impermeable, pressure at wellbore point (R_w, z) is

$$\begin{aligned} P(R_w, z) &= \int_{L_1}^{L_2} P\left(R_w, z; R_w, z'\right) dz' \\ &= \left(\frac{L_p}{2\pi H}\right) \ln\left(\frac{R_e^2 - R_0^2}{R_e R_w}\right) + \left(\frac{-1}{\pi H}\right) \sum_{n=1}^{\infty} \left[\frac{K_0(\lambda_n R_e) I_0(\lambda_n R_w)}{I_0(\lambda_n R_e)} - K_0(\lambda_n R_w)\right] \\ &\quad \times \cos\left(\frac{n\pi z}{H}\right) \int_{L_1}^{L_2} \cos\left(\frac{n\pi z'}{H}\right) dz'. \end{aligned} \tag{6.2}$$

Recall Formula (5.34), there hold

$$K_0(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}, \qquad \frac{K_0(x)}{I_0(x)} \approx \pi e^{-2x}, \tag{6.3}$$

if $x > 1$, and

$$I_0(x) \approx 1.0, \tag{6.4}$$

if $0 < x < 0.5$.

Combining to Formulas (3.9), (3.12), (3.13), and (5.8), we obtain

$$\lambda_n R_e > 1, \qquad \lambda_n R_w < 0.5, \tag{6.5}$$

so

$$\frac{K_0(\lambda_n R_e)}{I_0(\lambda_n R_e)} \approx \pi e^{-2\lambda_n R_e}, \qquad I_0(\lambda_n R_w) \approx 1.0. \tag{6.6}$$

According to Formula (6.6), we have

$$
\begin{aligned}
&\left|\left(\frac{-1}{\pi H}\right)\sum_{n=1}^{\infty}\left[\frac{K_0(\lambda_n R_e)I_0(\lambda_n R_w)}{I_0(\lambda_n R_e)}\right]\cos\left(\frac{n\pi z}{H}\right)\int_{L_1}^{L_2}\cos\left(\frac{n\pi z'}{H}\right)dz'\right| \\
&\quad=\left|\left(\frac{-1}{\pi H}\right)\sum_{n=1}^{\infty}\left[\frac{K_0(\lambda_n R_e)I_0(\lambda_n R_w)}{I_0(\lambda_n R_e)}\right]\left(\frac{H}{n\pi}\right)\cos\left(\frac{n\pi z}{H}\right)\left[\sin\left(\frac{n\pi L_2}{H}\right)-\sin\left(\frac{n\pi L_1}{H}\right)\right]\right| \\
&\quad\le\left|\left(\frac{-1}{\pi H}\right)\sum_{n=1}^{\infty}\left[\frac{K_0(\lambda_n R_e)I_0(\lambda_n R_w)}{I_0(\lambda_n R_e)}\right]\left(\frac{H}{n\pi}\right)\right| \\
&\quad=\left(\frac{1}{\pi^2}\right)\sum_{n=1}^{\infty}\frac{1}{n}\left|\frac{K_0(\lambda_n R_e)I_0(\lambda_n R_w)}{I_0(\lambda_n R_e)}\right| \\
&\quad\le\left(\frac{1}{\pi^2}\right)\sum_{n=1}^{\infty}\left|\frac{K_0(\lambda_n R_e)I_0(\lambda_n R_w)}{I_0(\lambda_n R_e)}\right| \\
&\quad\approx\left(\frac{1}{\pi^2}\right)\sum_{n=1}^{\infty}\pi e^{-2\lambda_n R_e} \\
&\quad=\frac{e^{-2\pi R_e/H}}{\pi\left(1-e^{-2\pi R_e/H}\right)} \\
&\quad=\frac{1}{\pi\left(e^{2\pi R_e/H}-1\right)} \\
&\quad\approx 0,
\end{aligned}
\tag{6.7}
$$

because the value of R_e/H is very big.

Combining Formulas (6.2) and (6.7) yields

$$
\begin{aligned}
P(R_w,z)&=\left(\frac{L_p}{2\pi H}\right)\ln\left(\frac{R_e^2-R_0^2}{R_eR_w}\right) \\
&\quad+\left(\frac{1}{\pi H}\right)\sum_{n=1}^{\infty}K_0(\lambda_n R_w)\cos\left(\frac{n\pi z}{H}\right)\int_{L_1}^{L_2}\cos\left(\frac{n\pi z'}{H}\right)dz' \\
&=\left(\frac{L_p}{2\pi H}\right)\ln\left(\frac{R_e^2-R_0^2}{R_eR_w}\right) \\
&\quad+\sum_{n=1}^{\infty}\left(\frac{1}{n\pi^2}\right)K_0(\lambda_n R_w)\cos\left(\frac{n\pi z}{H}\right)\left[\sin\left(\frac{n\pi L_2}{H}\right)-\sin\left(\frac{n\pi L_1}{H}\right)\right],
\end{aligned}
\tag{6.8}
$$

where we use

$$\int_{L_1}^{L_2} \cos\left(\frac{n\pi z'}{H}\right)dz' = \left(\frac{H}{n\pi}\right)\left[\sin\left(\frac{n\pi L_2}{H}\right) - \sin\left(\frac{n\pi L_1}{H}\right)\right]. \tag{6.9}$$

In order to obtain the average wellbore pressure, integrate both sides of Formula (6.2) with respect to z from L_1 to L_2, then divided by L_p, we obtain

$$\begin{aligned}
P_a(R_w) &= \frac{1}{L_p}\int_{L_1}^{L_2} P(R_w, z)dz \\
&= \left(\frac{L_p}{2\pi H}\right)\ln\left(\frac{R_e^2 - R_0^2}{R_e R_w}\right) \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{1}{n\pi^2}\right)K_0(\lambda_n R_w)\left[\sin\left(\frac{n\pi L_2}{H}\right) - \sin\left(\frac{n\pi L_1}{H}\right)\right]\left[\frac{1}{L_p}\int_{L_1}^{L_2}\cos\left(\frac{n\pi z}{H}\right)dz\right] \\
&= \left(\frac{L_p}{2\pi H}\right)\ln\left(\frac{R_e^2 - R_0^2}{R_e R_w}\right) \\
&\quad + \left(\frac{H}{\pi^3 L_p}\right)\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)K_0(\lambda_n R_w)\left[\sin\left(\frac{n\pi L_2}{H}\right) - \sin\left(\frac{n\pi L_1}{H}\right)\right]^2 \\
&= \left(\frac{L_p}{2\pi H}\right)\ln\left(\frac{R_e^2 - R_0^2}{R_e R_w}\right) \\
&\quad + \left(\frac{4H}{\pi^3 L_p}\right)\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)K_0(\lambda_n R_w)\sin^2\left(\frac{n\pi L_p}{2H}\right)\cos^2\left[\frac{n\pi(L_2 + L_1)}{2H}\right].
\end{aligned} \tag{6.10}$$

Recall Formula (5.59), let $R' = R_w$, integrate z' at both sides of Formula (5.72) from L_1 to L_2, if the reservoir is with gas cap and impermeable bottom boundary, pressure at wellbore point (R_w, z) is

$$\begin{aligned}
P(R_w, z) &= \int_{L_1}^{L_2} P(R_w, z; R_w, z')dz' \\
&= \left(\frac{-1}{\pi H}\right)\sum_{n=1}^{\infty}\left[\frac{K_0(\omega_n R_e)I_0(\omega_n R_w)}{I_0(\omega_n R_e)} - K_0(\omega_n R_w)\right]\Xi_1,
\end{aligned} \tag{6.11}$$

where

$$\begin{aligned}
\Xi_1 &= \sin(\omega_n z)\int_{L_1}^{L_2}\sin(\omega_n z')dz' \\
&= \sin\left[\frac{(2n-1)\pi z}{2H}\right]\left[\frac{(-2H)}{(2n-1)\pi}\right]\left\{\cos\left[\frac{(2n-1)\pi L_2}{2H}\right] - \cos\left[\frac{(2n-1)\pi L_1}{2H}\right]\right\}.
\end{aligned} \tag{6.12}$$

With a similar procedure in Formula (6.7), there holds

$$\begin{aligned}
&\left|\left(\frac{-1}{\pi H}\right)\sum_{n=1}^{\infty}\left[\frac{K_0(\omega_n R_e)I_0(\omega_n R_w)}{I_0(\omega_n R_e)}\right]\sin\left[\frac{(2n-1)\pi z}{2H}\right]\left[\frac{(-2H)}{(2n-1)\pi}\right]\right| \\
&\quad\times\left|\left\{\cos\left[\frac{(2n-1)\pi L_2}{2H}\right]-\cos\left[\frac{(2n-1)\pi L_1}{2H}\right]\right\}\right| \\
&\approx 0,
\end{aligned} \tag{6.13}$$

thus

$$\begin{aligned}
P(R_w,z) &= \int_{L_1}^{L_2} P(R_w,z;R_w,z')dz' \\
&= \left(\frac{1}{\pi H}\right)\sum_{n=1}^{\infty}K_0(\omega_n R_w)\sin\left[\frac{(2n-1)\pi z}{2H}\right]\left[\frac{-2H}{(2n-1)\pi}\right] \\
&\quad\times\left\{\cos\left[\frac{(2n-1)\pi L_2}{2H}\right]-\cos\left[\frac{(2n-1)\pi L_1}{2H}\right]\right\} \\
&= \sum_{n=1}^{\infty}\left[\frac{2}{(2n-1)\pi^2}\right]K_0(\omega_n R_w)\sin\left[\frac{(2n-1)\pi z}{2H}\right] \\
&\quad\times\left\{\cos\left[\frac{(2n-1)\pi L_1}{2H}\right]-\cos\left[\frac{(2n-1)\pi L_2}{2H}\right]\right\}.
\end{aligned} \tag{6.14}$$

In order to obtain the average wellbore pressure, integrate both sides of Formula (6.14) with respect to z from L_1 to L_2, then divided by L_p, we obtain

$$\begin{aligned}
P_a(R_w) &= \frac{1}{L_p}\int_{L_1}^{L_2} P(R_w,z)dz \\
&= \frac{1}{L_p}\sum_{n=1}^{\infty}\left[\frac{2}{(2n-1)\pi^2}\right]K_0(\omega_n R_w)\left\{\cos\left[\frac{(2n-1)\pi L_1}{2H}\right]-\cos\left[\frac{(2n-1)\pi L_2}{2H}\right]\right\} \\
&\quad\times\int_{L_1}^{L_2}\sin\left[\frac{(2n-1)\pi z}{2H}\right]dz \\
&= \left(\frac{4H}{\pi^3 L_p}\right)\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}K_0(\omega_n R_w)\left\{\cos\left[\frac{(2n-1)\pi L_2}{2H}\right]-\cos\left[\frac{(2n-1)\pi L_1}{2H}\right]\right\}^2 \\
&= \left(\frac{16H}{\pi^3 L_p}\right)\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}K_0(\omega_n R_w)\sin^2\left[\frac{(2n-1)\pi L_p}{4H}\right]\sin^2\left[\frac{(2n-1)\pi (L_2+L_1)}{4H}\right].
\end{aligned} \tag{6.15}$$

Recall Formula (5.73), let $R' = R_w$, integrate z' at both sides of Formula (5.85) from L_1 to L_2, if the reservoir is with bottom water and impermeable top boundary, pressure at wellbore point (R_w, z) is

$$\begin{aligned} P(R_w, z) &= \int_{L_1}^{L_2} P(R_w, z; R_w, z')\,dz' \\ &= \left(\frac{-1}{\pi H}\right)\sum_{n=1}^{\infty}\left[\frac{K_0(\omega_n R_e) I_0(\omega_n R_w)}{I_0(\omega_n R_e)} - K_0(\omega_n R_w)\right]\Xi_2, \end{aligned} \tag{6.16}$$

where

$$\begin{aligned} \Xi_2 &= \cos(\omega_n z)\int_{L_1}^{L_2} \cos(\omega_n z')\,dz' \\ &= \cos\left[\frac{(2n-1)\pi z}{2H}\right]\left[\frac{(2H)}{(2n-1)\pi}\right]\left\{\sin\left[\frac{(2n-1)\pi L_2}{2H}\right] - \sin\left[\frac{(2n-1)\pi L_1}{2H}\right]\right\}. \end{aligned} \tag{6.17}$$

With a similar procedure in Formula (6.7), there hold

$$\begin{aligned} &\left|\left(\frac{-1}{\pi H}\right)\sum_{n=1}^{\infty}\left[\frac{K_0(\omega_n R_e) I_0(\omega_n R_w)}{I_0(\omega_n R_e)}\right]\cos\left[\frac{(2n-1)\pi z}{2H}\right]\left[\frac{2H}{(2n-1)\pi}\right]\right| \\ &\quad\times\left|\left\{\sin\left[\frac{(2n-1)\pi L_2}{2H}\right] - \sin\left[\frac{(2n-1)\pi L_1}{2H}\right]\right\}\right| \approx 0, \end{aligned} \tag{6.18}$$

$$\begin{aligned} P(R_w, z) &= \int_{L_1}^{L_2} P(R_w, z; R_w, z')\,dz' \\ &= \left(\frac{1}{\pi H}\right)\sum_{n=1}^{\infty} K_0(\omega_n R_w)\cos\left[\frac{(2n-1)\pi z}{2H}\right]\left[\frac{2H}{(2n-1)\pi}\right] \\ &\quad\times\left\{\sin\left[\frac{(2n-1)\pi L_2}{2H}\right] - \sin\left[\frac{(2n-1)\pi L_1}{2H}\right]\right\} \\ &= \sum_{n=1}^{\infty}\left[\frac{2}{(2n-1)\pi^2}\right] K_0(\omega_n R_w)\cos\left[\frac{(2n-1)\pi z}{2H}\right] \\ &\quad\times\left\{\sin\left[\frac{(2n-1)\pi L_2}{2H}\right] - \sin\left[\frac{(2n-1)\pi L_1}{2H}\right]\right\}. \end{aligned} \tag{6.19}$$

In order to obtain the average wellbore pressure, integrate both sides of Formula (6.19) with respect to z from L_1 to L_2, then divided by L_p, we obtain

$$\begin{aligned} P_a(R_w) &= \frac{1}{L_p}\int_{L_1}^{L_2} P(R_w, z)dz \\ &= \frac{1}{L_p}\sum_{n=1}^{\infty}\left[\frac{2}{(2n-1)\pi^2}\right]K_0(\omega_n R_w)\left\{\sin\left[\frac{(2n-1)\pi L_2}{2H}\right] - \sin\left[\frac{(2n-1)\pi L_1}{2H}\right]\right\} \\ &\quad\times\int_{L_1}^{L_2}\cos\left[\frac{(2n-1)\pi z}{2H}\right]dz \\ &= \left(\frac{4H}{\pi^3 L_p}\right)\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}K_0(\omega_n R_w)\left\{\sin\left[\frac{(2n-1)\pi L_2}{2H}\right] - \sin\left[\frac{(2n-1)\pi L_1}{2H}\right]\right\}^2 \\ &= \left(\frac{16H}{\pi^3 L_p}\right)\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}K_0(\omega_n R_w)\sin^2\left[\frac{(2n-1)\pi L_p}{4H}\right]\cos^2\left[\frac{(2n-1)\pi(L_2+L_1)}{4H}\right]. \end{aligned} \tag{6.20}$$

Recall Formula (5.86), let $R' = R_w$, integrate z' at both sides of Formula (5.98) from L_1 to L_2, if the reservoir is with bottom water and gas cap, pressure at wellbore point (R_w, z) is

$$\begin{aligned} P(R_w, z) &= \int_{L_1}^{L_2} P(R_w, z; R_w, z')dz' \\ &= \left(\frac{-1}{\pi H}\right)\sum_{n=1}^{\infty}\left[\frac{K_0(\lambda_n R_e)I_0(\lambda_n R_w)}{I_0(\lambda_n R_e)} - K_0(\lambda_n R_w)\right]\Xi_3, \end{aligned} \tag{6.21}$$

where

$$\begin{aligned} \Xi_3 &= \sin(\lambda_n z)\int_{L_1}^{L_2}\sin(\lambda_n z')dz' \\ &= \sin\left(\frac{n\pi z}{H}\right)\left(\frac{-H}{n\pi}\right)\left[\cos\left(\frac{n\pi L_2}{H}\right) - \cos\left(\frac{n\pi L_1}{H}\right)\right]. \end{aligned} \tag{6.22}$$

With a similar procedure in Formula (6.7), there holds

$$\begin{aligned} &\left|\left(\frac{-1}{\pi H}\right)\sum_{n=1}^{\infty}\left[\frac{K_0(\lambda_n R_e)I_0(\lambda_n R_w)}{I_0(\lambda_n R_e)}\right]\sin\left(\frac{n\pi z}{H}\right)\left(\frac{-H}{n\pi}\right)\right| \\ &\quad\times\left|\left[\cos\left(\frac{n\pi L_2}{H}\right) - \cos\left(\frac{n\pi L_1}{H}\right)\right]\right| \approx 0, \end{aligned} \tag{6.23}$$

thus

$$\begin{aligned} P(R_w, z) &= \int_{L_1}^{L_2} P(R_w, z; R_w, z')dz' \\ &= \left(\frac{1}{\pi H}\right)\sum_{n=1}^{\infty} K_0(\lambda_n R_w)\sin\left(\frac{n\pi z}{H}\right)\left(\frac{-H}{n\pi}\right)\left[\cos\left(\frac{n\pi L_2}{H}\right) - \cos\left(\frac{n\pi L_1}{H}\right)\right] \\ &= \sum_{n=1}^{\infty}\left(\frac{-1}{n\pi^2}\right)K_0(\lambda_n R_w)\sin\left(\frac{n\pi z}{H}\right)\left[\cos\left(\frac{n\pi L_2}{H}\right) - \cos\left(\frac{n\pi L_1}{H}\right)\right]. \end{aligned} \tag{6.24}$$

In order to obtain the average pressure, integrate both sides of Formula (6.24) with respect to z from L_1 to L_2, then divided by L_p, we obtain

$$\begin{aligned} P_a(R_w) &= \frac{1}{L_p}\int_{L_1}^{L_2} P(R_w, z)dz \\ &= \frac{1}{L_p}\sum_{n=1}^{\infty}\left(\frac{-1}{n\pi^2}\right)K_0(\lambda_n R_w)\left[\cos\left(\frac{n\pi L_2}{H}\right) - \cos\left(\frac{n\pi L_1}{H}\right)\right]\int_{L_1}^{L_2}\sin\left(\frac{n\pi z}{H}\right)dz \\ &= \left(\frac{H}{\pi^3 L_p}\right)\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)K_0(\lambda_n R_w)\left[\cos\left(\frac{n\pi L_2}{H}\right) - \cos\left(\frac{n\pi L_1}{H}\right)\right]^2 \\ &= \left(\frac{4H}{\pi^3 L_p}\right)\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)K_0(\lambda_n R_w)\sin^2\left(\frac{n\pi L_p}{2H}\right)\sin^2\left[\frac{n\pi(L_2 + L_1)}{2H}\right]. \end{aligned} \tag{6.25}$$

7. Partially Penetrating Vertical Well Productivity

Note that Formula (6.10) is in dimensionless form, substitute Formulas (3.14) and (3.15) into Formula (6.10), then rearrange and simplify the resulting formula, the productivity formula for a partially penetrating vertical well is obtained:

$$Q_w = F_D\frac{2\pi K_h H(P_e - P_w)/(\mu B)}{\ln(R_{eD}/R_{wD}) + S_{\text{ps}}}, \tag{7.1}$$

where P_w is average wellbore pressure, S_{ps} is pseudo-skin factor due to partial penetration:

$$S_{\text{ps}} = \left(\frac{8H_D^2}{\pi^2 L_{pD}^2}\right)\sum_{n=1}^{N}\left(\frac{1}{n^2}\right)K_0\left[\left(\frac{n\pi}{H_D}\right)R_{wD}\right]\sin^2\left(\frac{n\pi L_{pD}}{2H_D}\right)\cos^2\left[\frac{n\pi(L_{2D} + L_{1D})}{2H_D}\right]. \tag{7.2}$$

However, (7.1) is only applicable to a circular cylinder drainage reservoir has both impermeable top and bottom boundaries, and has constant pressure lateral boundary.

If the well is fully penetrating, ($L_p = L = H$,) then $S_{\text{ps}} = 0$, and if the reservoir is an isotropic permeability reservoir, that is, $K_a = K_h = K_v = K$, then Formula (7.1) reduces to Formula (2.1).

Substituting Formulas (3.14) and (3.15) into Formula (6.15), then rearranging and simplifing the resulting formula, we obtain the productivity formula for a partially penetrating vertical well in a reservoir with impermeable bottom boundary and constant pressure top boundary (gas cap):

$$Q_w = F_D \frac{\pi^3 K_h L_p^2 (P_e - P_w)/(\mu B)}{16 H \Theta_1}, \tag{7.3}$$

where

$$\Theta_1 = \sum_{n=1}^{N} \frac{1}{(2n-1)^2} K_0 \left\{ \left[\frac{(2n-1)\pi}{2H_D} \right] R_{wD} \right\} \times \sin^2\left[\frac{(2n-1)\pi L_{pD}}{4H_D} \right] \sin^2\left[\frac{(2n-1)\pi (L_{2D} + L_{1D})}{4H_D} \right]. \tag{7.4}$$

Substituting Formulas (3.14) and (3.15) into Formula (6.20), then rearranging and simplifing the resulting formula, we obtain the productivity formula for a partially penetrating vertical well in a reservoir with impermeable top boundary and constant pressure bottom boundary (bottom water):

$$Q_w = F_D \frac{\pi^3 K_h L_p^2 (P_e - P_w)/(\mu B)}{16 H \Theta_2}, \tag{7.5}$$

where

$$\Theta_2 = \sum_{n=1}^{N} \frac{1}{(2n-1)^2} K_0 \left\{ \left[\frac{(2n-1)\pi}{2H_D} \right] R_{wD} \right\} \times \sin^2\left[\frac{(2n-1)\pi L_{pD}}{4H_D} \right] \cos^2\left[\frac{(2n-1)\pi (L_{2D} + L_{1D})}{4H_D} \right]. \tag{7.6}$$

Substituting Formulas (3.14) and (3.15) into Formula (6.25), then rearranging and simplifing the resulting formula, we obtain the productivity formula for a partially penetrating vertical well in a reservoir with both bottom water and gas cap:

$$Q_w = F_D \frac{\pi^3 K_h L_p^2 (P_e - P_w)/(\mu B)}{4 H \Theta_3}, \tag{7.7}$$

where

$$\Theta_3 = \sum_{n=1}^{N} \frac{1}{n^2} K_0 \left[\left(\frac{n\pi}{H_D} \right) R_{wD} \right] \sin^2\left(\frac{n\pi L_{pD}}{2H_D} \right) \sin^2\left[\frac{n\pi (L_{2D} + L_{1D})}{2H_D} \right]. \tag{7.8}$$

It must be pointed out that $N = 500$ in the summation of the series in the aforementioned formulas is sufficient to reach engineering accuracy.

8. Conclusions and Discussions

Comparing Formulas (7.1), (7.3), (7.5), and (7.7), we can reach the the following conclusions.

If top and bottom reservoir boundaries are impermeable, the radius of the cylindrical system and off-center distance appears in the productivity formulas; if the reservoir has a gas cap or bottom water, the effects of the radius and off-center distance on productivity can be ignored.

Because the pay zone thickness H is very small compared with the circular cylinder drainage radius R_e, when the reservoir has gas cap or bottom water which provides the main drive mechanism, the lateral boundary has little influence on productivity, the well is producing as if the reservoir is infinite. The performance is the same for a centered well and an off-center well, even the reservoir is with an edge water boundary (constant pressure outer boundary). The effects of drainage radius and off-center distance on productivity are negligible, thus R_e and R_0 do not show up in Formulas (7.3), (7.5), and (7.7). However, (7.3), (7.5), and (7.7) are applicable to both centered and off-center wells.

If both top and bottom boundaries are impermeable, then R_e and R_0 play important roles in well productivity, as indicated by Formula (7.1).

Formula (7.2) for pseudo-skin factor is obtained by solving three-dimensional Laplace equation, thus it is more reliable than semianalytical and semiempirical expressions of pseudo-skin factor, that is, Formulas (2.4) and (2.7).

Field data will be provided in forthcoming papers in the near future, which can show that the proposed formulas are reliable and accurate, they are fast analytical tools to evaluate well performance.

In this paper, we always assume the lateral boundary is at constant pressure, so the proposed formulas are only applicable for steady-state, this study can be improved by considering:

(1) pseudo-steady-state productivity formulas for an oil reservoir with impermeable upper, lower, and lateral boundaries;

(2) steady-state productivity formulas for an oil reservoir with impermeable lateral boundary, but constant pressure upper or lower boundary;

(3) productivity formulas for a partially penetrating vertical well in a box-shaped reservoir, and the corresponding equation of pseudo-skin factor due to partial penetration;

(4) non-Darcy flow effect on an oil well productivity.

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