

Research Article

On the Steady Flow of a Second-Grade Fluid between Two Coaxial Porous Cylinders

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An exact solution of an incompressible second-grade fluid for flow between two coaxial porous cylinders is given. The velocity profiles for various values of the cross-Reynolds number and the elastic number are plotted. It is found that for large values of the cross-Reynolds number, the velocity variation near boundaries shows a different behaviour than that of the Newtonian fluid.

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1. Introduction

The exact solution given in this paper is connected with flow-over-porous boundaries. The flow of fluids over boundaries of porous materials has many applications in practice, such as control of the flow. For Newtonian flows, there are many exact solutions. A simple exact solution for flow over a porous plane boundary, where the suction velocity is uniform, was found by Griffith and Meredith and given in [1]. There is no solution for flow over a porous plane boundary with a uniform injection velocity. However, if the porous plane boundary is bounded by side walls, a solution of the Navier-Stokes equation can be found for the injection case [2]. The flow in a duct of rectangular cross-section with uniform suction and injection has been examined by Mehta and Jain [3], Sai and Rao [4], and Erdoğan [2]. For large values of the cross-Reynolds number near the suction region, the flow shows a boundary-layer character. Fully developed nonswirling laminar flow through a porous pipe and a discussion of previous research have been given by Ter- ril and Thomas [5]. The flow with swirl in a porous pipe with injection along the pipe is three-dimensional [6]. Recently, a three-dimensional flow in a porous channel has been

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investigated in [7]. The flow in a porous pipe with uniform injection and suction shows a boundary layer character near the suction region [8].

The problem considered in this paper is an extension of the flow of a viscous fluid in an annulus with uniform porous walls, investigated by Berman [9], to second-grade fluid flows. The fluid injection rate at one wall is taken equal to the withdrawal rate at the other wall. The axial flow of a non-Newtonian fluid between two coaxial porous cylinders with a discussion of previous research has been investigated by Mishra and Roy [10]. A perturbation method is used for the axial velocity. The perturbation parameter used is the product of the cross-Reynolds number and the elastic number. Although the solution is obtained for small values of the perturbation parameters, the results are given for very small elastic numbers and for very large Reynolds numbers. However, there is no comparison with the Newtonian flow. Unsteady flow of an viscoelastic fluid between two coaxial circular cylinders has been investigated in [11]. A number of unidirectional transient flows of a second-grade fluid using the method of integral transforms have been studied in [12]. Some unsteady unidirectional flows in unbounded domains, which are axially symmetric have been investigated in [13]. Some steady and unsteady solutions of the equations of motion for an incompressible second-grade fluids have been given by applying different methods in [14].

It is well known that the equation of motion of incompressible second-grade fluids, in general, is of higher order than the Navier-Stokes equation. The Navier-Stokes equation is a second-order partial differential equation, but the equation of motion of a second-grade fluid is a third-order partial differential equation. A marked difference between the case of Navier-Stokes theory and that for fluids of second grade is that ignoring the non-linearity in the Navier-Stokes does not lower the order of equation, however, ignoring the higher order nonlinearities in the case of the second-grade fluid reduces the order of the equation. The no-slip boundary condition is sufficient for a Newtonian fluid, but based on the previous experience with partial differential equations, it may not be sufficient for a second-grade fluid and therefore needs an additional boundary condition [15–17]. If one uses a perturbation expansion in terms of the coefficient appearing in the higher order derivative of the governing equation, the no-slip boundary condition is sufficient. However, Frater [18], considering the asymptotic suction flow, has shown that this type of perturbation expansion may lead to erroneous results. This arises from the consideration of singular perturbation problem as a regular one. He exposes an extra condition that the solution tends to the Newtonian value as the coefficient of the higher derivative in the governing equation approaches zero.

In this paper, the solution is obtained in terms of the confluent hypergeometric functions, and it is valid for all values of the cross-Reynolds number and the elastic number. The solution has three coefficients: two of them can be determined by the no-slip boundary condition and the other can be determined by using the properties of the confluent hypergeometric functions. A comparison of the solution with that of the Newtonian fluid shows that there is a different behaviour near boundaries. The velocity distribution is given for positive, negative, and infinite values of the cross-Reynolds numbers. The velocity profiles for various values of the cross-Reynolds number and the elastic number are plotted in Figure 1.1.

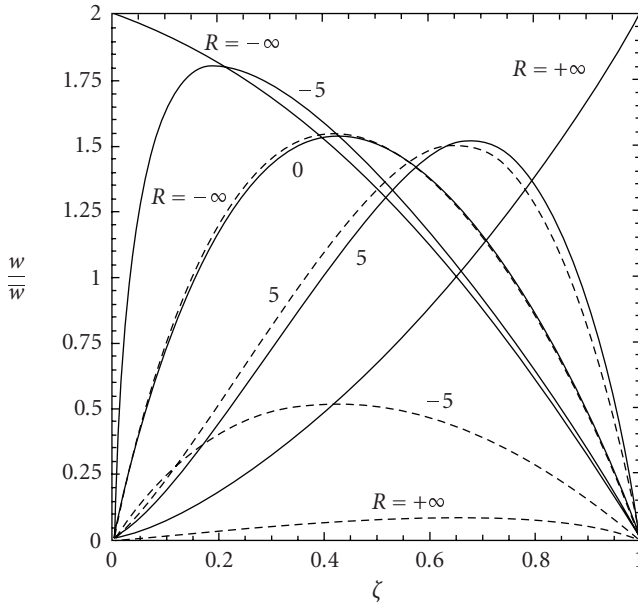


Figure 1.1. The variation of the axial velocity for various values of the cross-Reynolds number (—) ($\varepsilon = 0$); (---) ($\varepsilon = 1$). $\zeta = (\xi - \sigma)/(1 - \sigma)$; $\sigma \leq \xi \leq r/b_1$, $\sigma = 0.2$.

2. Basic equations

The equation of motion of a fluid in the absence of body forces is

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \boldsymbol{\sigma}, \quad (2.1)$$

where ρ is the density of the fluid, \mathbf{v} is the velocity, $\boldsymbol{\sigma}$ is the stress tensor, and D/Dt represents the material derivative. The continuity equation for the velocity is

$$\nabla \cdot \mathbf{v} = 0. \quad (2.2)$$

Equations (2.1) and (2.2) can be applied to all types of fluids, Newtonian and non-Newtonian. The stress depends on the local properties of the fluid. The relation which is called the constitutive equation is in the following form for an incompressible second-grade fluid [19]:

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \quad (2.3)$$

where μ , α_1 , and α_2 are material constants, and \mathbf{A}_n represents the Rivlin-Ericksen tensor defined as

$$\begin{aligned} \mathbf{A}_0 &= \mathbf{I}, & \mathbf{A}_1 &= \nabla\mathbf{v} + (\nabla\mathbf{v})^T, \\ \mathbf{A}_2 &= \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{A}_1 + \mathbf{A}_1 \cdot (\nabla\mathbf{v}) + (\nabla\mathbf{v})^T \cdot \mathbf{A}_1, \end{aligned} \quad (2.4)$$

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where t is time, p is pressure, and \mathbf{I} is the identity tensor. The Clausius-Duhem inequality and the condition that Helmholtz free energy is minimum at equilibrium provide the following restrictions [20].

$$\mu \geq 0, \quad \alpha_1 + \alpha_2 = 0, \quad \alpha_1 \geq 0. \quad (2.5)$$

A comprehensive discussion on the restrictions for μ , α_1 , and α_2 can be found in the work by Dunn and Rajagopal [20]. The sign of the material moduli α_1 and α_2 is the subject of much controversy [21]. The experiments have not confirmed these restrictions on α_1 and α_2 . Thus, the conclusion is that the fluids which have been tested are not fluids of second grade and are characterized by a different constitutive structure.

Fully developed laminar flow of an incompressible fluid of second grade between two coaxial porous cylinders is considered. The cylindrical polar coordinates are used. The radii of the porous cylinders are a_1 and b_1 ($a_1 < b_1$). The rate of fluid withdrawal at one wall of the annulus is assumed to be equal to the rate of injection of fluid at the other wall, and that these rates are independent of axial position in the annulus.

The velocity field is assumed to be in the following form:

$$v_r = \frac{\alpha}{r}, \quad v_\theta = 0, \quad v_z = w(r), \quad (2.6)$$

where v_r , v_θ , v_z are components of the velocity in cylindrical polar coordinates, α is positive for injection at the inner cylinder and negative for suction at the inner one. Equation (2.2) is satisfied identically by the velocity. Inserting the velocity given by (2.6) into the expression for the stress, the components of the stress tensor, in cylindrical polar coordinates, can be written in the following forms:

$$\begin{aligned} \sigma_{rr} &= -p - \frac{2\alpha\mu}{r^2} + \alpha_1 \left(\frac{8\alpha^2}{r^4} + 2w'^2 \right) + \alpha_2 \left(\frac{4\alpha^2}{r^4} + w'^2 \right), \\ \sigma_{r\theta} &= 0, \\ \sigma_{rz} &= \mu w' + \alpha_1 \alpha \left(\frac{w'}{r} \right)' - \frac{2\alpha\alpha_2}{r^2} w', \\ \sigma_{\theta\theta} &= -p + \frac{2\alpha\mu}{r^2} + \frac{4\alpha_2\alpha^2}{r^4}, \\ \sigma_{\theta z} &= 0, \\ \sigma_{zz} &= -p + \alpha_2 w'^2, \end{aligned} \quad (2.7)$$

where $\sigma_{r\theta} = \sigma_{\theta r}$, $\sigma_{rz} = \sigma_{zr}$, $\sigma_{\theta z} = \sigma_{z\theta}$; the primes denote differentiation with respect to r . Inserting the stress components and the velocity given by (2.6) into (2.1), one obtains

$$\alpha_1 \alpha \left(\frac{w'''}{r} + \frac{w''}{r^2} - \frac{w'}{r^3} \right) + \mu \left(w'' + \frac{1}{r} w' \right) - \rho \frac{\alpha}{r} w' = \frac{dp}{dz}, \quad (2.8)$$

where dp/dz is the axial pressure gradient which is constant. The boundary conditions are

$$w(a_1) = 0, \quad w(b_1) = 0, \quad (2.9)$$

where a_1 is the radius of the inner cylinder and b_1 is the radius of the outer cylinder.

3. Solution of the problem

After some manipulations, (2.8) takes the form

$$xy'' + (2-x)y' - \left(1 - \frac{R}{2}\right)y = k, \quad (3.1)$$

where

$$w' = \frac{r}{b_1^2} y(x), \quad x = -\frac{\xi^2}{2\varepsilon R}, \quad \xi = \frac{r}{b_1}, \quad R = \frac{\alpha}{\nu}, \quad \varepsilon = \frac{\alpha_1/\rho}{b_1^2}, \quad k = -\frac{b_1^2}{2\mu} \frac{dp}{dz}. \quad (3.2)$$

The solution of (3.1) can be written in the following form:

$$y = -\frac{k}{1-R/2} + C_1 M\left(1 - \frac{R}{2}, 2, x\right) + C_2 U\left(1 - \frac{R}{2}, 2, x\right), \quad (3.3)$$

where $M(a, b, x)$ and $U(a, b, x)$ are the confluent hypergeometric functions [22, 23]. One needs three boundary conditions in order to determine three arbitrary constants. Thus, unless an additional condition is prescribed over the conditions (2.9), one has a parametric solution. For $R > 0$, x becomes negative, then for $x < 0$, $U(a, b, x)$ is not acceptable and therefore, C_2 must be zero and (3.3) takes the form

$$\frac{dw}{dr} = \frac{r}{b_1^2} \left[C_1 M\left(1 - \frac{R}{2}, 2, -\frac{\xi^2}{2\varepsilon R}\right) - \frac{2k}{2-R} \right]. \quad (3.4)$$

Using the identity [23]

$$M\left(1 - \frac{R}{2}, 2, -\frac{\xi^2}{2\varepsilon R}\right) = e^{-\xi^2/2\varepsilon R} M\left(1 + \frac{R}{2}, 2, \frac{\xi^2}{2\varepsilon R}\right), \quad (3.5)$$

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the integration gives

$$w = C_1 \varepsilon R \int e^{-z} M(1 + (R/2), 2, z) dz - \frac{k}{2 - R} \xi^2 + C_3. \quad (3.6)$$

Using the identity [23]

$$(1 - a) \int e^{-z} M(a, 2, z) dz = -e^{-z} M(a, 1, z) + C, \quad (3.7)$$

and the boundary conditions (2.9), one obtains

$$\frac{w}{k} = \frac{1}{2 - R} [1 - \xi^2 + (1 - \sigma^2)E], \quad (3.8)$$

where

$$E = \frac{e^{-\xi^2/2\varepsilon R} M(1 + (R/2), 1, \xi^2/2\varepsilon R) - e^{1/2\varepsilon R} M(1 + (R/2), 1, 1/2\varepsilon R)}{e^{-1/2\varepsilon R} M(1 + (R/2), 1, 1/2\varepsilon R) - e^{-\sigma^2/2\varepsilon R} M(1 + (R/2), 1, \sigma^2/2\varepsilon R)}, \quad (3.9)$$

and $\sigma = a_1/b_1$. When ε goes to zero, using the asymptotic expression of $M(a, b, x)$ [23], E becomes

$$\lim_{\varepsilon \rightarrow 0} E = \lim_{\varepsilon \rightarrow 0} \frac{(\xi^2/2\varepsilon R)^{R/2} - (1/2\varepsilon R)^{R/2}}{(1/2\varepsilon R)^{R/2} - (\sigma^2/2\varepsilon R)^{R/2}} = \frac{\xi^R - 1}{1 - \sigma^R}, \quad (3.10)$$

and (3.8) can be written as

$$\frac{w}{k} = \frac{1}{2 - R} \left[1 - \xi^2 + (1 - \sigma^2) \frac{1 - \xi^R}{1 - \sigma^R} \right], \quad (3.11)$$

which is the expression of the velocity of a Newtonian fluid [9].

Since the volume flux across a plane normal to the flow is given by

$$Q = 2\pi \int_{a_1}^{b_1} w r dr, \quad (3.12)$$

the mean velocity can be written as

$$\bar{w} = \frac{2}{1 - \sigma^2} \int_0^1 w \xi d\xi. \quad (3.13)$$

Inserting the expression for w given by (3.8) into (3.13) and using the identity [23]

$$\int e^{-z} M(a, 1, z) dz = z e^{-z} M(a + 1, 2, z) + C, \quad (3.14)$$

one finds

$$\bar{w} = \frac{1}{2 - R} \left(\frac{1 - \sigma^2}{2} + F \right), \quad (3.15)$$

where

$$F = \frac{e^{-1/2\varepsilon R} M(2 + (R/2), 2, 1/2\varepsilon R) - \sigma^2 e^{-\sigma^2/2\varepsilon R} M(2 + (R/2), 2, \sigma^2/2\varepsilon R)}{e^{-1/2\varepsilon R} M(1 + (R/2), 1, 1/2\varepsilon R) - e^{-\sigma^2/2\varepsilon R} M(1 + (R/2), 1, \sigma^2/2\varepsilon R)} - \frac{(1 - \sigma^2) e^{-1/2\varepsilon R} M(1 + (R/2), 1, 1/2\varepsilon R)}{e^{-1/2\varepsilon R} M(1 + (R/2), 1, 1/2\varepsilon R) - e^{-\sigma^2/2\varepsilon R} M(1 + (R/2), 1, \sigma^2/2\varepsilon R)}. \quad (3.16)$$

When ε goes to zero, using the asymptotic expression of $M(a, b, x)$ [23], F becomes

$$\lim_{\varepsilon \rightarrow 0} F = \lim_{\varepsilon \rightarrow 0} \frac{(2/(2+R))[(1/2\varepsilon R)^{R/2} - \sigma^2(\sigma^2/2\varepsilon R)^{R/2}] - (1 - \sigma^2)(1/2\varepsilon R)^{R/2}}{(1/2\varepsilon R)^{R/2} - (\sigma^2/2\varepsilon R)^{R/2}} = \frac{-R(1 - \sigma^2) + 2\sigma^2(1 - \sigma^R)}{(2+R)(1 - \sigma^R)}, \quad (3.17)$$

and (3.15) can be written as

$$\frac{\bar{w}}{k} = \frac{1}{4 - R^2} \left[\frac{(2+R) + (2-R)\sigma^2}{2} - \frac{R(1 - \sigma^2)}{1 - \sigma^R} \right], \quad (3.18)$$

which is the expression of the mean velocity of a Newtonian fluid [9]. Using the expression of (3.8) and (3.13), the velocity becomes

$$\frac{w}{\bar{w}} = \frac{1 - \xi^2 + (1 - \sigma^2)E}{(1/2)(1 - \sigma^2) + F}. \quad (3.19)$$

The variation of w/\bar{w} with respect to $\zeta = (\xi - \sigma)/(1 - \sigma)$ for various values of R and ε is illustrated in Figure 1.1. The value of σ is taken as 0.2 and the values of ε are taken as 0 and 1. Equation (3.19) is valid for all values of R and ε . When R goes to infinity, using the expression given in [23] which is

$$\lim_{a \rightarrow \infty} \frac{1}{\Gamma(b)} M(a, b, x/a) = x^{1/2 - (1/2)b} I_{b-1}(2\sqrt{x}), \quad (3.20)$$

equation (3.19) takes the following form

$$\frac{w}{\bar{w}} = \frac{1 - \xi^2 + (1 - \sigma^2) [I_0(\varepsilon^{-1/2\xi}) - I_0(\varepsilon^{-1/2})] / [I_0(\varepsilon^{-1/2}) - I_0(\sigma\varepsilon^{-1/2})]}{(1 - \sigma^2)/2 + (2\varepsilon^{1/2} I_1(\varepsilon^{-1/2}) - 2\sigma\varepsilon^{1/2} I_1(\sigma\varepsilon^{-1/2}) - (1 - \sigma^2) I_0(\varepsilon^{-1/2})) / (I_0(\varepsilon^{-1/2}) - I_0(\sigma\varepsilon^{-1/2}))}, \quad (3.21)$$

where $\Gamma(x)$ is the gamma function and $I_0(x)$ and $I_1(x)$ are the modified Bessel functions of the first kind of orders zero and one. The variation of w/\bar{w} with respect to ζ for various values of ε is illustrated in Figure 1.1. Since the asymptotic form of $I_n(x)$ is $e^x/\sqrt{2\pi x}$ when ε goes to zero, (3.21) becomes

$$\frac{w}{\bar{w}} = 2 \frac{\xi^2 - \sigma^2}{1 - \sigma^2}. \quad (3.22)$$

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For $R < 0$, x is positive, then for $x > 0$, $M(a, b, x)$ becomes an increasing function of x , therefore, C_1 must be zero and (3.3) takes the form

$$\frac{dw}{dr} = \frac{r}{b_1^2} \left[C_2 U(1 + (N/2), 2, \xi^2/2\epsilon N) - \frac{2k}{2+N} \right], \quad (3.23)$$

where $N = -R$. The integration gives

$$w = C_2 \epsilon N \int U(1 + (N/2), 2, z) dz - \frac{k}{2+N} \xi^2 + C_4. \quad (3.24)$$

Using the identity [21]

$$\int U(a, b, z) dz = \frac{1}{1-a} U(a-1, b-1, z) + C \quad (3.25)$$

and the boundary conditions (2.9), one obtains

$$\frac{w}{k} = \frac{1}{2+N} [(1 - \xi^2) - (1 - \sigma^2)G], \quad (3.26)$$

where

$$G = \frac{U(N/2, 1, \xi^2/2\epsilon N) - U(N/2, 1, 1/2\epsilon N)}{U(N/2, 1, 1/2\epsilon N) - U(N/2, 1, \sigma^2/2\epsilon N)}. \quad (3.27)$$

When ϵ goes to zero, using the asymptotic expression of $U(a, b, x)$ in [23], G becomes

$$\lim_{\epsilon \rightarrow 0} G = \lim_{\epsilon \rightarrow 0} \frac{(\xi^2/2\epsilon R)^{-N/2} - (1/2\epsilon R)^{-N/2}}{(1/2\epsilon R)^{-N/2} - (\sigma^2/2\epsilon R)^{-N/2}} = \frac{\xi^{-N} - 1}{1 - \sigma^{-N}} \quad (3.28)$$

and (3.26) reduces to

$$\lim_{\epsilon \rightarrow 0} \frac{w}{k} = \frac{1}{2+N} \left[1 - \xi^2 - (1 - \sigma^2) \frac{1 - \xi^{-N}}{1 - \sigma^{-N}} \right], \quad (3.29)$$

which is the expression for the velocity of a Newtonian fluid [9]. The mean velocity is given by (3.13). Inserting the expression for w given by (3.26) into (3.13) and using the identity

$$\int U(a, 1, z) dz = \frac{1}{1-a} U(a-1, 0, z) + C, \quad (3.30)$$

one obtains

$$\bar{w} = \frac{1}{2+N} \left(\frac{1 - \sigma^2}{2} + 2\epsilon NH \right), \quad (3.31)$$

where

$$H = \frac{(2/(2-N))[U(N/2-1, 0, 1/2\epsilon N) - U(N/2, 0, \sigma^2/2\epsilon N)] - ((1-\sigma^2)/2\epsilon N)U(N/2, 1, 1/2\epsilon N)}{U(N/2, 1, 1/2\epsilon N) - U(N/2, 1, \sigma^2/2\epsilon N)}. \quad (3.32)$$

When ε goes to zero, using the asymptotic expression of $U(a, b, x)$, H becomes

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} H &= \lim_{\varepsilon \rightarrow 0} \frac{(2/(2-N))[(1/2\varepsilon N)^{1-N/2} - (\sigma^2/2\varepsilon N)^{1-N/2}] - ((1-\sigma^2)/2\varepsilon N)(1/2\varepsilon N)^{-N/2}}{(1/2\varepsilon N)^{-N/2} - (\sigma^2/2\varepsilon N)^{-N/2}} \\ &= \frac{(2/(2-N))(1-\sigma^{2-N}) - (1-\sigma^2)}{1-\sigma^{-N}} \end{aligned} \quad (3.33)$$

and (3.31) can be written as

$$\frac{\bar{w}}{k} = \frac{1}{2+N} \left[\frac{1-\sigma^2}{2} + \frac{(2/(2-N))(1-\sigma^{2-N}) - (1-\sigma^2)}{1-\sigma^{-N}} \right], \quad (3.34)$$

which is the expression for the mean velocity of a Newtonian fluid [9]. Using the expressions in (3.26) and (3.31), the velocity becomes

$$\frac{w}{\bar{w}} = \frac{1-\xi^2 + (1-\sigma^2)G}{(1-\sigma^2)/2 + 2\varepsilon NH}. \quad (3.35)$$

The variation of w/\bar{w} with respect to $\zeta = (\xi - \sigma)/(1 - \sigma)$ for various values of $-R$ and ε is illustrated in Figure 1.1. The value of σ is taken as 0.2 and the values of ε are 0 and 1. Equation (3.35) is valid for all values of $-R$ and ε . When $-R$ goes to infinity, using the expression in [23] which is

$$\lim_{\varepsilon \rightarrow 0} \Gamma(1+a-b) U(a, b, x/a) = 2x^{1/2-(1/2)b} K_{b-1}(2\sqrt{x}), \quad (3.36)$$

equation (3.35) takes the following form:

$$\frac{w}{\bar{w}} = \frac{1-\xi^2 + (1-\sigma^2)[K_0(\varepsilon^{-1/2}\xi) - K_0(\varepsilon^{-1/2})]/[K_0(\varepsilon^{-1/2}) - K_0(\sigma\varepsilon^{-1/2})]}{K_0(\varepsilon^{-1/2}) - K_0(\sigma\varepsilon^{-1/2})}, \quad (3.37)$$

where $\Gamma(x)$ is the gamma function and $K_0(x)$ and $K_1(x)$ are the modified Bessel function of the second kind of orders zero and one. The variation of w/\bar{w} with respect to ζ for various values of ε is illustrated in Figure 1.1. When ε goes to zero, since the asymptotic form of $K_n(x)$ is $e^{-x}\sqrt{\pi}/2x$, (3.37) becomes

$$\frac{w}{\bar{w}} = 2 \frac{1-\xi^2}{1-\sigma^2}. \quad (3.38)$$

This expression of the velocity satisfies the boundary condition at $\xi = 1$, but it does not satisfy the boundary condition at $\xi = \sigma$.

4. Conclusions

The aim of this paper was to obtain an exact solution of the governing equation for the axial flow of a second-grade fluid between two coaxial porous cylinders. The solution has three coefficients: two of them can be determined by the no-slip boundary condition. Thus, unless an additional condition is prescribed over the boundaries, one has a parametric solution. The other coefficient can be determined by using the properties of the

confluent hypergeometric functions. The exact solution is valid for all values of the cross-Reynolds numbers and the elastic numbers. A comparison of the solution with that of the Newtonian fluid shows that there is a different behaviour near the boundaries.

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References

- [1] L. Rosenhead, Ed., *Laminar Boundary Layers*, Clarendon Press, Oxford, UK, 1963.
- [2] M. E. Erdoğan, "The effects of side walls on axial flow in rectangular ducts with suction and injection," *Acta Mechanica*, vol. 162, no. 1–4, pp. 157–166, 2003.
- [3] K. N. Mehta and R. K. Jain, "Laminar hydromagnetic flow in a rectangular channel with porous walls," *Proceedings of the National Institute of Sciences of India, Part A*, vol. 28, pp. 846–856, 1962.
- [4] K. S. Sai and B. N. Rao, "Magnetohydrodynamic flow in a rectangular duct with suction and injection," *Acta Mechanica*, vol. 140, no. 1–2, pp. 57–64, 2000.
- [5] R. M. Terrill and P. W. Thomas, "On laminar flow through a uniform porous pipe," *Applied Scientific Research*, vol. 21, no. 1, pp. 37–67, 1969.
- [6] R. M. Terrill and P. W. Thomas, "Spiral flow in a porous pipe," *Physics of Fluids*, vol. 16, no. 3, pp. 356–359, 1973.
- [7] C. L. Taylor, W. H. H. Banks, M. B. Zatorska, and P. G. Drazin, "Three-dimensional flow in a porous channel," *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 44, no. 1, pp. 105–133, 1991.
- [8] M. E. Erdoğan and C. E. İmrak, "On the axial flow of an incompressible viscous fluid in a pipe with a porous boundary," *Acta Mechanica*, vol. 178, no. 3–4, pp. 187–197, 2005.
- [9] A. S. Berman, "Laminar flow in an annulus with porous walls," *Journal of Applied Physics*, vol. 29, pp. 71–75, 1958.
- [10] S. P. Mishra and J. S. Roy, "Laminar elasticoviscous flow in an annulus with porous walls," *Physics of Fluids*, vol. 10, no. 11, pp. 2300–2304, 1967.
- [11] E. Rukmangadachari, "Unsteady flow of an elastico-viscous fluid between two coaxial cylinders," *Rheologica Acta*, vol. 21, no. 3, pp. 223–227, 1982.
- [12] R. Bandelli and K. R. Rajagopal, "Start-up flows of second grade fluids in domains with one finite dimension," *International Journal of Non-Linear Mechanics*, vol. 30, no. 6, pp. 817–839, 1995.
- [13] C. Fetecau, "Analytical solutions for non-Newtonian fluid flows in pipe-like domains," *International Journal of Non-Linear Mechanics*, vol. 39, no. 2, pp. 225–231, 2004.
- [14] M. R. Mohyuddin, T. Hayat, F. M. Mahomed, S. Asghar, and A. M. Siddiqui, "On solutions of some non-linear differential equations arising in Newtonian and non-Newtonian fluids," *Non-linear Dynamics*, vol. 35, no. 3, pp. 229–248, 2004.
- [15] K. R. Rajagopal and P. N. Kaloni, "Some remarks on boundary conditions for flows of fluids of the differential type," in *Continuum Mechanics and Its Applications*, G. A. C. Graham and S. K. Malik, Eds., pp. 935–942, Hemisphere, New York, NY, USA, 1989.
- [16] K. R. Rajagopal, "On boundary conditions for fluids of the differential type," in *Navier-Stokes Equations and Related Nonlinear Problems*, pp. 273–278, Plenum Press, New York, NY, USA, 1995.
- [17] R. L. Fosdick and B. Bernstein, "Nonuniqueness of second-order fluids under steady radial flow in annuli," *International Journal of Engineering Science*, vol. 7, no. 6, pp. 555–569, 1969.

- [18] K. R. Frater, "On the solution of some boundary-value problems arising in elasto-viscous fluid mechanics," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 21, no. 1, pp. 134–137, 1970.
- [19] K. R. Rajagopal, "A note on unsteady unidirectional flows of a non-Newtonian fluid," *International Journal of Non-Linear Mechanics*, vol. 17, no. 5-6, pp. 369–373, 1982.
- [20] J. E. Dunn and K. R. Rajagopal, "Fluids of differential type: critical review and thermodynamic analysis," *International Journal of Engineering Science*, vol. 33, no. 5, pp. 689–729, 1995.
- [21] R. L. Fosdick and K. R. Rajagopal, "Uniqueness and drag for fluids of second grade in steady motion," *International Journal of Non-Linear Mechanics*, vol. 13, no. 3, pp. 131–137, 1978.
- [22] L. J. Slater, *Confluent Hypergeometric Functions*, Cambridge University Press, New York, NY, USA, 1960.
- [23] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, NY, USA, 1965.

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