Hindawi Publishing Corporation
Journal of Probability and Statistics
Volume 2012, Article ID 850608, 12 pages
doi:10.1155/2012/850608

Research Article

On Complete Moment Convergence of Weighted Sums for Arrays of Rowwise Negatively Associated Random Variables

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Received 8 May 2012; Revised 29 July 2012; Accepted 13 August 2012

Academic Editor: Shein-chung Chow

The complete moment convergence of weighted sums for arrays of rowwise negatively associated random variables is investigated. Some sufficient conditions for complete moment convergence of weighted sums for arrays of rowwise negatively associated random variables are established. Moreover, the results of Baek et al. (2008), are complemented. As an application, the complete moment convergence of moving average processes based on a negatively associated random sequences is obtained, which improves the result of Li et al. (2004).

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables and, as usual, set $S_n = \sum_{i=1}^{n} X_i, n \geq 1$. When $\{X_n, n \geq 1\}$ are independent and identically distributed (i.i.d.), Baum and Katz [1] proved the following remarkable result concerning the convergence rate of the tail probabilities $P(|S_n| > \epsilon n^{1/p})$ for any $\epsilon > 0$.

Theorem A (see [1]). *Let $0 < p < 2$ and $r \geq p$. Then*

$$\sum_{n=1}^{\infty} n^{(r/p)-2} P\left(|S_n| > \epsilon n^{1/p}\right) < \infty \quad \forall \epsilon > 0, \tag{1.1}$$

if and only if $E|X|^r < \infty$, where $EX_1 = 0$ whenever $1 \leq p < 2$.

There is an interesting and substantial literature of investigation apropos of extending the Baum-Katz theorems along a variety of different paths. One of these extensions is due to Chow [2] who established the following refinement which is a complete moment convergence result for sums of i.i.d. random variables.

Theorem B (see [2]). *Let $EX_1 = 0, 1 \le p < 2$, and $r \ge p$. Suppose that $E[|X_1|^r + |X_1| \log(1+|X_1|)] < \infty$. Then*

$$\sum_{n=1}^{\infty} n^{(r/p)-2-(1/p)} E\left(|S_n| - \epsilon n^{1/p}\right)^{+} < \infty \quad \forall \epsilon > 0. \tag{1.2}$$

Recently, Baum-Katz theorem is extended to the case of dependence random variables. Liang [3] obtained some general results on the complete convergence of weighted sums of negatively associated random variables. Li and Zhang [4] showed complete moment convergence for moving average processes under negative association as follows.

Theorem C (see [4]). *Suppose that $X_n = \sum_{i=-\infty}^{\infty} a_{i+n} Y_i, n \ge 1$, where $\{a_i, -\infty < i < \infty\}$ is a sequence of real numbers with $\sum_{-\infty}^{\infty} |a_i| < \infty$ and $\{Y_i, -\infty < i < \infty\}$ is a sequence of identically distributed and negatively associated random variables with $EY_1 = 0,\ EY_1^2 < \infty$. Let $l(x) > 0$ be a slowly varying function and $1 \le p < 2,\ r > 1 + (p/2)$. Then $E|Y_1|^r l(|Y_1|^p) < \infty$ implies that*

$$\sum_{n=1}^{\infty} n^{(r/p)-2-(1/p)} l(n) E\left(|S_n| - \epsilon n^{1/p}\right)^{+} < \infty, \quad \text{where } S_n = \sum_{i=1}^{n} X_i,\ n \ge 1. \tag{1.3}$$

Kuczmaszewska [5] proposed a very general result for complete convergence of rowwise negatively associated arrays of random variables which is stated in Theorem D.

Theorem D (see [5]). *Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise negatively associated random variables and let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers. Let $\{b_n, n \ge 1\}$ be an increasing sequence of positive integers and let $\{c_n, n \ge 1\}$ be a sequence of positive real numbers. If for some $q > 2,\ 0 < t < 2$ and any $\epsilon > 0$ the following conditions are fulfilled:*

(a) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\{|a_{ni} X_{ni}| \ge \epsilon b_n^{1/t}\} < \infty$,

(b) $\sum_{n=1}^{\infty} c_n b_n^{-q/t} \sum_{i=1}^{b_n} |a_{ni}|^q E|X_{ni}|^q I(|a_{ni} X_{ni}| < \epsilon b_n^{1/t}) < \infty$,

(c) $\sum_{n=1}^{\infty} c_n b_n^{-q/t} (\sum_{i=1}^{b_n} |a_{ni}|^2 E|X_{ni}|^2 I(|a_{ni} X_{ni}| < \epsilon b_n^{1/t}))^{q/2} < \infty$,

then

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1 \le k \le b_n} \left| \sum_{i=1}^{k} \left(a_{ni} X_{ni} - a_{ni} E X_{ni} I\left(|a_{ni} X_{ni}| < \epsilon b_n^{1/t}\right)\right) \right| > \epsilon b_n^{1/t} \right\} < \infty. \tag{1.4}$$

Baek et al. [6] discussed complete convergence of weighted sums for arrays of rowwise negatively associated random variables and obtained the following results.

Theorem E (see [6]). *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise negatively associated random variables with $EX_{ni} = 0$ and $P\{|X_{ni}| > x\} \leq CP\{|X| > x\}$ for all n, i and $x \geq 0$. Suppose that $\beta \geq -1$, and that $\{a_{ni}, i \geq 1,\ n \geq 1\}$ is an array of constants such that*

$$\sup_{i\geq 1}|a_{ni}| = O(n^{-r}) \quad \text{for some } r > 0, \tag{1.5}$$

$$\sum_{i=1}^{\infty}|a_{ni}| = O(n^{\alpha}) \quad \text{for some } \alpha \in [0, r). \tag{1.6}$$

(a) *If $\alpha + \beta + 1 > 0$ and there exists some $\delta > 0$ such that $\alpha/r + 1 < \delta \leq 2$, and $s = \max(1 + ((\alpha + \beta + 1)/r), \delta)$, then, under $E|X|^s < \infty$, one has*

$$\sum_{n=1}^{\infty} n^{\beta} P\left\{\left|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right| > \epsilon\right\} < \infty \quad \forall\epsilon > 0. \tag{1.7}$$

(b) *If $\alpha + \beta + 1 = 0$, then, under $E|X|\log(1 + |X|) < \infty$,* (1.7) *remain true.*

In this paper, the authors take the inspiration in [5, 6] and discuss the complete moment convergence of weighted sums for arrays of rowwise negatively associated random variables by applying truncation methods, which extend the results of [5, 6]. As an application, the complete moment convergence of moving average processes based on a negatively associated random sequences is obtained, which extend the result of Li and Zhang [4].

For the proof of the main results, we need to restate a few definitions and lemmas for easy reference. Throughout this paper, C will represent positive constants whom their value may change from one place to another. The symbol $I(A)$ denotes the indicator function of A, $[x]$ indicate the maximum integer not larger than x. For a finite set B, the symbol $\#B$ denotes the number of elements in the set B.

Definition 1.1. A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (abbreviated to NA in the following), if for every pair disjoint subsets A and B of $\{1, 2, \ldots, n\}$ and any real nondecreasing coordinate-wise functions f_1 on $\mathbb{R}^A$ and f_2 on $\mathbb{R}^B$

$$\operatorname{Cov}(f_1(X_i, i \in A), f_2(X_i, i \in B)) \leq 0 \tag{1.8}$$

whenever the covariance exists.

An infinite family of random variables $\{X_i, -\infty < i < \infty\}$ is NA if every finite subfamily is NA.

The definition of negatively associated was introduced by Alam and Saxena [7] and was studied by Joag-Dev and Proschan [8] and Block et al. [9]. As pointed out and proved by Joag-Dev and Proschan, a number of well-known multivariate distributions possess the NA property. Negative association has found important and wide applications in multivariate statistical analysis and reliability. Many investigators discuss applications of negative association to probability, stochastic processes, and statistics.

Definition 1.2. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X (write $\{X_i\} \prec X$) if there exists a constant C, such that $P\{|X_n| > x\} \leq CP\{|X| > x\}$ for all $x \geq 0$ and $n \geq 1$.

The following lemma is a well-known result.

Lemma 1.3. *Let the sequence $\{X_n, n \geq 1\}$ of random variables be stochastically dominated by a random variable X. Then for any $p > 0,\ x > 0$*

$$\begin{gathered} E|X_n|^p I(|X_n| \leq x) \leq C\left[E|X|^p I(|X| \leq x) + x^p P\{|X| > x\}\right], \\ E|X_n|^p I(|X_n| > x) \leq CE|X|^p I(|X| > x). \end{gathered} \tag{1.9}$$

Definition 1.4. A real-valued function $l(x)$, positive and measurable on $[A, \infty)$ for some $A > 0$, is said to be slowly varying if $\lim_{x\to\infty}(l(x\lambda)/l(x)) = 1$ for each $\lambda > 0$.

By the properties of slowly varying function, we can easily prove the following two lemmas. Here we omit the details of the proof.

Lemma 1.5. *Let $l(x) > 0$ be a slowly varying function as $x \to \infty$.*

(i) $Ck^{r+1}l(k) \leq \sum_{n=1}^{k} n^r l(n) \leq Ck^{r+1}l(k)$ *for any $r > -1$ and positive integer k.*

(ii) $Ck^{r+1}l(k) \leq \sum_{n=k}^{\infty} n^r l(n) \leq Ck^{r+1}l(k)$ *for any $r < -1$ and positive integer k.*

Lemma 1.6. *Let X be a random variable and let $l(x) > 0$ be a slowly varying function as $x \to \infty$.*

(i) *If $\alpha > 0$, $r > 0$, $\beta > -1$, then $E|X|^{\alpha+((\beta+1)/r)}l(|X|^{1/r}) < \infty$ if and only if $\sum_{n=1}^{\infty} n^\beta l(n) E|X|^\alpha I(|X| > n^r) < \infty$.*

(ii) *If $\alpha > 0$, $r > 0$, $\beta < -1$, then $E|X|^{\alpha+((\beta+1)/r)}l(|X|^{1/r}) < \infty$ if and only if $\sum_{n=1}^{\infty} n^\beta l(n) E|X|^\alpha I(|X| \leq n^r) < \infty$.*

The following lemma will play an important role in the proof of our main results. The proof is according to Shao [10].

Lemma 1.7. *Let $\{X_i, 1 \leq i \leq n\}$ be a sequence of NA random variables with mean zero and $E|X_i|^q < \infty$ for every $1 \leq i \leq n,\ q \geq 2$. Then*

$$E\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^q \leq C\left(\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{q/2}\right). \tag{1.10}$$

By monotone convergence and (1.10), we have the following lemma.

Lemma 1.8. *Let $\{X_i, i \geq 1\}$ be a sequence of NA random variables with mean zero and $E|X_i|^q < \infty$ for every $i \geq 1,\ q \geq 2$. Then*

$$E\max_{k\geq 1}\left|\sum_{i=1}^{k} X_i\right|^q \leq C\left(\sum_{i=1}^{\infty} E|X_i|^q + \left(\sum_{i=1}^{\infty} EX_i^2\right)^{q/2}\right). \tag{1.11}$$

Using Lemma 1.4, Lemma 1.5, and Theorem 2.11 in Sung [11], we obtain the following lemmas.

Lemma 1.9. *Let $\{X_{ni}, 1 \le i \le n,\ n \ge 1\}$ be an array of rowwise NA random variables with $E|X_{ni}| < \infty$ for $1 \le i \le n$, $n \ge 1$. Let $\{b_n, n \ge 1\}$ be a sequence of real numbers. If for some $q \ge 2$, the following conditions are fulfilled:*

(a) $\sum_{n=1}^{\infty} b_n \sum_{i=1}^{n} E|X_{ni}| I(|X_{ni}| > 1) < \infty$;

(b) $\sum_{n=1}^{\infty} b_n \sum_{i=1}^{n} E|X_{ni}|^q I(|X_{ni}| \le 1) < \infty$;

(c) $\sum_{n=1}^{\infty} b_n (\sum_{i=1}^{n} E|X_{ni}|^2 I(|X_{ni}| \le 1))^{q/2} < \infty$.

Then

$$\sum_{n=1}^{\infty} b_n E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(X_{ni} - EX_{ni})\right| - \epsilon\right)^{+} < \infty \quad \forall \epsilon > 0. \tag{1.12}$$

Lemma 1.10. *Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise NA random variables with $E|X_{ni}| < \infty$ for $i \ge 1, n \ge 1$. Let $\{b_n, n \ge 1\}$ be a sequence of real numbers. If for some $q > 2$, the following conditions are fulfilled:*

(a) $\sum_{n=1}^{\infty} b_n \sum_{i=1}^{\infty} E|X_{ni}| I(|X_{ni}| > 1) < \infty$;

(b) $\sum_{n=1}^{\infty} b_n \sum_{i=1}^{\infty} E|X_{ni}|^q I(|X_{ni}| \le 1) < \infty$;

(c) $\sum_{n=1}^{\infty} b_n (\sum_{i=1}^{\infty} E|X_{ni}|^2 I(|X_{ni}| \le 1))^{q/2} < \infty$.

Then

$$\sum_{n=1}^{\infty} b_n E\left(\sup_{k\ge 1}\left|\sum_{i=1}^{k}(X_{ni} - EX_{ni})\right| - \epsilon\right)^{+} < \infty \quad \forall \epsilon > 0. \tag{1.13}$$

2. Main Results

Now we state our main results. The proofs will be given in Section 3.

Theorem 2.1. *Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise NA random variables with $EX_{ni} = 0$ and stochastically dominated by a random variable X. Suppose that $l(x) > 0$ is a slowing varying function and that $\{a_{ni}, 1 \le i \le n,\ n \ge 1\}$ is an array of constants such that*

$$\sup_{1\le i\le n} |a_{ni}| = O(n^{-r}) \quad \textit{for some } r > 0, \tag{2.1}$$

$$\sum_{i=1}^{n} |a_{ni}| = O(n^{\alpha}) \quad \textit{for some } \alpha \in [0, r). \tag{2.2}$$

(i) *If* $\alpha+\beta+1>0$ *and there exists some* $\delta>0$ *such that* $(\alpha/r)+1<\delta\leq 2$, *and* $s=\max(1+((\alpha+\beta+1)/r),\delta)$, *then* $E|X|^s l(|X|^{1/r})<\infty$ *implies*

$$\sum_{n=1}^{\infty} n^{\beta} l(n) E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\epsilon\right)^{+}<\infty \quad \forall\epsilon>0. \tag{2.3}$$

(ii) *If* $\alpha+\beta+1=0$, *assume also* $l(x)\leq Cl(y)$ *for all* $0<x<y$. *Then* $E|X|\log(1+|X|)l(|X|^{1/r})<\infty$ *implies*

$$\sum_{n=1}^{\infty} n^{\beta} l(n) E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\epsilon\right)^{+}<\infty \quad \forall\epsilon>0. \tag{2.4}$$

Theorem 2.2. *Let* $\{X_{ni}, i\geq 1,\ n\geq 1\}$ *be an array of rowwise NA random variables with* $EX_{ni}=0$ *and stochastically dominated by a random variable* X. *Suppose that* $l(x)>0$ *is a slowing varying function and that* $\{a_{ni}, i\geq 1,\ n\geq 1\}$ *is an array of constants such that*

$$\sup_{i\geq 1}|a_{ni}|=O(n^{-r}) \quad \text{for some } r>0, \tag{2.5}$$

$$\sum_{i=1}^{\infty}|a_{ni}|=O(n^{\alpha}) \quad \text{for some } \alpha\in[0,r). \tag{2.6}$$

(i) *If* $\alpha+\beta+1>0$ *and there exists some* $\delta>0$ *such that* $(\alpha/r)+1<\delta\leq 2$, *and* $s=\max(1+(\alpha+\beta+1/r),\delta)$. *Then* $E|X|^s l(|X|^{1/r})<\infty$ *implies*

$$\sum_{n=1}^{\infty} n^{\beta} l(n) E\left(\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\epsilon\right)^{+}<\infty \quad \forall\epsilon>0. \tag{2.7}$$

(ii) *If* $\alpha+\beta+1=0$, *assume also* $l(x)\leq Cl(y)$ *for all* $0<x<y$. *Then* $E|X|\log(1+|X|)l(|X|^{1/r})<\infty$ *implies*

$$\sum_{n=1}^{\infty} n^{\beta} l(n) E\left(\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\epsilon\right)^{+}<\infty \quad \forall\epsilon>0. \tag{2.8}$$

Remark 2.3. If (2.7) and (2.8) hold, then for all $\epsilon>0$, we have

$$\sum_{n=1}^{\infty} n^{\beta} l(n) P\left\{\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|>\epsilon\right\}<\infty, \tag{2.9}$$

$$\sum_{n=1}^{\infty} n^{-1} l(n) P\left\{\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|>\epsilon\right\}<\infty. \tag{2.10}$$

Thus, we improve the results of Baek et al. [6] to supreme value of partial sums.

Remark 2.4. If $\alpha+\beta+1<0$, then $E|X|<\infty$ implies that (2.10) holds. In fact,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\beta} l(n) E\left[\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\epsilon\right]^{+} &\leq \sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{i=1}^{\infty}|a_{ni}|E|X_{ni}|+\epsilon\sum_{n=1}^{\infty} n^{\beta} l(n) \\ &\leq C\sum_{n=1}^{\infty} n^{\beta+\alpha} l(n) E|X|+\epsilon\sum_{n=1}^{\infty} n^{\beta} l(n)<\infty.\end{aligned} \tag{2.11}$$

Corollary 2.5. *Under the conditions of Theorem 2.2,*

$$\sum_{n=1}^{\infty} n^{\beta} l(n) E\left[\left|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right|-\epsilon\right]^{+}<\infty \quad \forall\epsilon>0. \tag{2.12}$$

Corollary 2.6. *Let $\{X_{ni}, 1\leq i\leq n,\ n\geq 1\}$ be an array of rowwise NA random variables with $EX_{ni}=0$ and stochastically dominated by a random variable X. Suppose that $l(x)>0$ is a slowing varying function.*

(1) *Let $p>1$ and $1\leq t<2$. If $E|X|^{pt}l(|X|^{t})<\infty$, then*

$$\sum_{n=1}^{\infty} n^{p-2-(1/t)} l(n) E\left[\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_{ni}\right|-\epsilon n^{1/t}\right]^{+}<\infty \quad \forall\epsilon>0. \tag{2.13}$$

(2) *Let $1<t<2$. If $E|X|^{t}l(|X|^{t})<\infty$, then*

$$\sum_{n=1}^{\infty} n^{-1-(1/t)} l(n) E\left[\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_{ni}\right|-\epsilon n^{1/t}\right]^{+}<\infty \quad \forall\epsilon>0. \tag{2.14}$$

Theorem 2.7. *Suppose that $X_n=\sum_{i=-\infty}^{\infty} a_{i+n}Y_i,\ n\geq 1$, where $\{a_i, -\infty<i<\infty\}$ is a sequence of real numbers with $\sum_{-\infty}^{\infty}|a_i|<\infty$, and $\{Y_i, -\infty<i<\infty\}$ is a NA random sequence with $EY_i=0$ and is stochastically dominated by a random variable Y. Let $l(x)$ be a slowly varying function.*

(1) *Let $1\leq t<2,\ r\geq 1+(t/2)$. If $E|Y|^{r}l(|Y|^{t})<\infty$, then*

$$\sum_{n=1}^{\infty} n^{(r/t)-2-(1/t)} l(n) E\left[\left|\sum_{i=1}^{n} X_i\right|-\epsilon n^{1/t}\right]^{+}<\infty \quad \forall\epsilon>0. \tag{2.15}$$

(2) *Let $1<t<2$. If $E|Y|^{t}l(|Y|^{t})<\infty$, then*

$$\sum_{n=1}^{\infty} n^{-1-(1/t)} l(n) E\left[\left|\sum_{i=1}^{n} X_i\right|-\epsilon n^{1/t}\right]^{+}<\infty \quad \forall\epsilon>0. \tag{2.16}$$

Remark 2.8. Theorem 2.7 obtains the result about the complete moment convergence of moving average processes based on an NA random sequence with different distributions. The result of Li and Zhang [4] is a special case of Theorem 2.7 (1). Moreover, our result covers the case of $r = t$, which was not considered by Li and Zhang.

3. Proofs of the Main Results

Proof of Theorem 2.1. Since $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$, where $a_{ni}^{+} = \max(a_{ni}, 0)$ and $a_{ni}^{-} = \max(-a_{ni}, 0)$, we have

$$E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| - \epsilon\right)^{+} \le E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}^{+}X_{ni}\right| - \frac{\epsilon}{2}\right)^{+} + E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}^{-}X_{ni}\right| - \frac{\epsilon}{2}\right)^{+}. \quad (3.1)$$

So, without loss of generality, we can assume $a_{ni} > 0$. From (2.1) and (2.2), without loss of generality, we assume

$$\sup_{1\le i\le n} a_{ni} \le n^{-r}, \qquad \sum_{i=1}^{n} a_{ni} \le n^{\alpha}. \quad (3.2)$$

Put $b_n = n^{\beta}l(n)$, $n = 1, 2, \ldots$ in Lemma 1.9. Noting that $\alpha + \beta + 1 > 0$, by Lemma 1.3 and Lemma 1.7, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\beta}l(n)\sum_{i=1}^{n} E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}| > 1) &\le C\sum_{n=1}^{\infty} n^{\beta}l(n)\sum_{i=1}^{n} Ea_{ni}|X|I(a_{ni}|X| > 1)\\
&\le C\sum_{n=1}^{\infty} n^{\beta}l(n)\sum_{i=1}^{n} a_{ni}E|X|I(|X| > n^{r})\\
&\le C\sum_{n=1}^{\infty} n^{\beta}l(n)\sum_{i=1}^{n} a_{ni}E|X|I(|X| > n^{r})\\
&\le C\sum_{n=1}^{\infty} n^{\alpha+\beta}l(n)E|X|I(|X| > n^{r})\\
&\le CE|X|^{1+((\alpha+\beta+1)/r)}l\left(|X|^{1/r}\right) < \infty.
\end{aligned} \quad (3.3)$$

Since $\alpha < r$, we can take some t such that $1 + (\alpha/r) < t \le \min(s, 2)$. Observe that

$$\begin{aligned}
\sum_{i=1}^{n} E(a_{ni}X_{ni})^{2}I(|a_{ni}X_{ni}| \le 1) &\le \sum_{i=1}^{n} E|a_{ni}X_{ni}|^{t}I(|a_{ni}X_{ni}| \le 1)\\
&\le C\sum_{i=1}^{n} a_{ni}^{t}\\
&= C\sum_{i=1}^{n} a_{ni}a_{ni}^{t-1} \le Cn^{\alpha-r(t-1)}.
\end{aligned} \quad (3.4)$$

Hence, choosing q large enough such that $\beta + (q/2)(\alpha - r(t-1)) < -1$, we have

$$\sum_{n=1}^{\infty} n^{\beta} l(n) \left(\sum_{i=1}^{n} E|a_{ni}X_{ni}|^2 I(|a_{ni}X_{ni}| \le 1) \right)^{q/2} \le C \sum_{n=1}^{\infty} n^{\beta + (q/2)(\alpha - r(t-1))} l(n) < \infty. \tag{3.5}$$

By (3.3) and Lemma 1.3, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{i=1}^{n} E|a_{ni}X_{ni}|^q I(|a_{ni}X_{ni}| \le 1) \\
&\quad \le C \sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{i=1}^{n} \left\{ E|a_{ni}X|^q I(|a_{ni}X| \le 1) + P(|a_{ni}X| > 1) \right\} \\
&\quad \le C \sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{i=1}^{n} \left\{ E|a_{ni}X|^q I(|a_{ni}X| \le 1) + E|a_{ni}X| I(|a_{ni}X| > 1) \right\} \\
&\quad \le C \sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{i=1}^{n} E|a_{ni}X|^q I(|a_{ni}X| \le 1) + C.
\end{aligned} \tag{3.6}$$

Set $I_{nj} = \{1 \le i \le n \mid (n(j+1))^{-r} < a_{ni} \le (nj)^{-r}\}$, $j = 1, 2, \ldots$. Then $\cup_{j \ge 1} I_{nj} = \{1, 2, \ldots, n\}$. Note also that for all $k \ge 1$, $n \ge 1$,

$$n^{\alpha} \ge \sum_{i=1}^{n} a_{ni} = \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} a_{ni} \ge \sum_{j=1}^{\infty} (\#I_{nj}) (n(j+1))^{-r} \ge n^{-r} \sum_{j=k}^{\infty} (\#I_{nj}) (j+1)^{-rq} (k+1)^{rq-r}. \tag{3.7}$$

Hence, we have

$$\sum_{j=k}^{\infty} (\#I_{nj}) j^{-rq} \le C n^{\alpha + r} k^{r - rq}. \tag{3.8}$$

Note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{i=1}^{n} E|a_{ni}X|^q I(|a_{ni}X| \le 1) \\
&\quad = \sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} E|a_{ni}X|^q I(|a_{ni}X| \le 1) \\
&\quad \le \sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{j=1}^{\infty} \#I_{nj} (nj)^{-rq} E|X|^q I(|X| \le (n(j+1)^r))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{j=1}^{\infty} \# I_{nj} (nj)^{-rq} E|X|^{q} I\left(|X| \le (2n)^{r}\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{j=1}^{\infty} \# I_{nj} (nj)^{-rq} \sum_{k=2n+1}^{n(j+1)} E|X|^{q} I\left((k-1)^{r} < |X| \le k^{r}\right) \\
&=: J_1 + J_2.
\end{aligned}
\tag{3.9}
$$

Choosing q large enough such that $\alpha + \beta + r - rq < -1$, we obtain by Lemma 1.6 and (3.8) that

$$
\begin{aligned}
J_1 &= \sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{j=1}^{\infty} \# I_{nj} (nj)^{-rq} E|X|^{q} I\left(|X| \le (2n)^{r}\right) \\
&\le \sum_{n=1}^{\infty} n^{\alpha+\beta+r-rq} l(n) E|X|^{q} I\left(|X| \le (2n)^{r}\right) \le C E|X|^{1+((\alpha+\beta+1)/r)} l\left(|X|^{1/r}\right) < \infty.
\end{aligned}
\tag{3.10}
$$

Noting that $\alpha + \beta > -1$, by (3.8) and Lemma 1.5, we see

$$
\begin{aligned}
J_2 &= \sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{j=1}^{\infty} \# I_{nj} (nj)^{-rq} \sum_{k=2n+1}^{n(j+1)} E|X|^{q} I\left((k-1)^{r} |X| \le k^{r}\right) \\
&\le \sum_{n=1}^{\infty} n^{\beta-rq} l(n) \sum_{k=2n+1}^{\infty} E|X|^{q} I\left((k-1)^{r} < |X| \le k^{r}\right) \sum_{j=[(k/n)-1]}^{\infty} \# I_{nj} j^{-rq} \\
&\le C \sum_{n=1}^{\infty} n^{\beta-rq} l(n) \sum_{k=2n+1}^{\infty} n^{\alpha+r} \left(\frac{k}{n}\right)^{r-rq} E|X|^{q} I\left((k-1)^{r} < |X| \le k^{r}\right) \\
&\le C \sum_{k=2}^{\infty} k^{r-rq} E|X|^{q} I\left((k-1)^{r} < |X| \le k^{r}\right) \sum_{n=1}^{[k/2]} n^{\alpha+\beta} l(n) \\
&\le C \sum_{k=2}^{\infty} k^{\alpha+\beta+1+r-rq} l(k) E|X|^{q} I\left((k-1)^{r} < |X| \le k^{r}\right) \\
&\le C E|X|^{1+((\alpha+\beta+1)/r)} l\left(|X|^{1/r}\right) < \infty.
\end{aligned}
\tag{3.11}
$$

From (3.6), (3.9), (3.10), and (3.11), we know that

$$
\sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{i=1}^{n} E|a_{ni} X_{ni}|^{q} I(|a_{ni} X_{ni}| \le 1) < \infty. \tag{3.12}
$$

By (3.3), (3.5), and (3.12), we see that (a), (b), and (c) in Lemma 1.9 with X_{ni} replaced by $a_{ni} X_{ni}$ are fulfilled. Since $\{a_{ni} X_{ni}, 1 \le i \le n, n \ge 1\}$ is also an array of rowwise NA random variables, by Lemma 1.9, we complete the proof of (2.3).

Next, we prove (2.4). If $\alpha + \beta + 1 = 0$, then $\sum_{n=1}^{k} n^{\alpha+\beta} \le C \log(1 + n)$. Similarly for the proof of (3.3), noting that $l(x) \le Cl(y)$, $0 < x < y$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{i=1}^{n} E|a_{ni}X_{ni}| I(|a_{ni}X_{ni}| > 1) &\le C \sum_{n=1}^{\infty} n^{\alpha+\beta} l(n) E|X| I(|X| > n^{r}) \\
&= C \sum_{n=1}^{\infty} n^{-1} l(n) E|X| I(|X| > n^{r}) \\
&= C \sum_{n=1}^{\infty} n^{-1} l(n) \sum_{k=n}^{\infty} E|X| I\left(k^{r} < |X| \le (k+1)^{r}\right) \\
&= C \sum_{k=1}^{\infty} E|X| I\left(k^{r} < |X| \le (k+1)^{r}\right) \sum_{n=1}^{k} n^{-1} l(n) \\
&\le C \sum_{k=1}^{\infty} \log(1+k) l(k) E|X| I\left(k^{r} < |X| \le (k+1)^{r}\right) \\
&\le C E|X| \log\Big((1+|X|) l|X|^{1/r}\Big) < \infty.
\end{aligned} \tag{3.13}$$

Taking $q = 2$, from the proof of (3.9), (3.10), and (3.11), we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{i=1}^{n} E|a_{ni}X|^{2} I(|a_{ni}X| \le 1) \\
&\quad \le C \sum_{n=1}^{\infty} n^{\alpha+\beta+r-2r} l(n) E|X|^{2} I\left(|X| \le (2n)^{r}\right) \\
&\qquad + C \sum_{n=1}^{\infty} n^{\alpha+\beta} l(n) \sum_{k=2n+1}^{\infty} k^{r-2r} E|X|^{2} I\left((k-1)^{r} < |X| \le k^{r}\right) \\
&\quad = C \sum_{n=1}^{\infty} n^{-1-r} l(n) E|X|^{2} I\left(|X| \le (2n)^{r}\right) \\
&\qquad + C \sum_{n=1}^{\infty} n^{-1} l(n) \sum_{k=2n+1}^{\infty} k^{-r} E|X|^{2} I\left((k-1)^{r} < |X| \le k^{r}\right) \\
&\quad \le C E|X| l\Big(|X|^{1/r}\Big) + C E|X| \log(1+|X|) l\Big(|X|^{1/r}\Big) < \infty.
\end{aligned} \tag{3.14}$$

Thus, for $q = 2$, (a), (b), and (c) in Lemma 1.9 with X_{ni} replaced by $a_{ni}X_{ni}$ are fulfilled. So (2.4) holds. □

Proof of Theorem 2.2. By Lemma 1.10, the rest of the proof is similar to that of Theorem 2.1 and is omitted. □

Proof of Corollary 2.5. Note that

$$\left[\left|\sum_{i=1}^{\infty}a_{ni}X_{ni}\right|-\epsilon\right]^{+}\leq\left[\sup_{k\geq1}\left|\sum_{i=1}^{k}a_{ni}X_{ni}\right|-\epsilon\right]^{+}. \tag{3.15}$$

Therefore, by (2.7) and (2.8), we prove that (2.12) holds. □

Proof of Corollary 2.6. By applying Theorem 2.1, taking $\beta = p-2$, $a_{ni} = n^{-1/t}$ for $1 \leq i \leq n$, and $a_{ni} = 0$ for $i > n$, then we obtain (2.13). Similarly, taking $\beta = -1$, $a_{ni} = n^{-1/t}$ for $1 \leq i \leq n$, and $a_{ni} = 0$ for $i > n$, we obtain (2.14) by Theorem 2.1. □

Proof of Theorem 2.7. Let $X_{ni} = Y_i$ and $a_{ni} = n^{-1/t}\sum_{j=1}^{n}a_{i+j}$ for all $n \geq 1, -\infty < i < \infty$. Since $\sum_{-\infty}^{\infty}|a_i| < \infty$, we have $\sup_i|a_{ni}| = O(n^{-1/t})$ and $\sum_{i=-\infty}^{\infty}|a_{ni}| = O(n^{1-1/t})$. By applying Corollary 2.5, taking $\beta = (r/t)-2, r = 1/t$, $\alpha = 1-(1/t)$, we obtain

$$\sum_{n=1}^{\infty}n^{(r/t)-2-(1/t)}l(n)E\left[\left|\sum_{i=1}^{n}X_i\right|-\epsilon n^{1/t}\right]^{+}=\sum_{n=1}^{\infty}n^{\beta}l(n)E\left[\left|\sum_{i=-\infty}^{\infty}a_{ni}X_{ni}\right|-\epsilon\right]^{+}<\infty\quad\forall\epsilon>0. \tag{3.16}$$

Therefore, (2.15) and (2.16) hold. □

Acknowledgments

This paper is supported by the National Natural Science Foundation of China (Grant no. 10901003), the Key Project of Chinese Ministry of Education (Grant no. 211077), and the Anhui Provincial Natural Science Foundation (Grant no. 10040606Q30).

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