Hindawi Publishing Corporation
Journal of Probability and Statistics
Volume 2011, Article ID 181409, 13 pages
doi:10.1155/2011/181409

Research Article

A Limit Theorem for Random Products of Trimmed Sums of i.i.d. Random Variables

Fa-mei Zheng

School of Mathematical Science, Huaiyin Normal University, Huaian 223300, China

Correspondence should be addressed to Fa-mei Zheng, 16032@hytc.edu.cn

Received 13 May 2011; Revised 25 July 2011; Accepted 11 August 2011

Academic Editor: Man Lai Tang

Let $\{X, X_i;\ i \geq 1\}$ be a sequence of independent and identically distributed positive random variables with a continuous distribution function F, and F has a medium tail. Denote $S_n = \sum_{i=1}^{n} X_i, S_n(a) = \sum_{i=1}^{n} X_i I(M_n - a < X_i \leq M_n)$ and $V_n^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2$, where $M_n = \max_{1\leq i\leq n} X_i$, $\overline{X} = (1/n)\sum_{i=1}^{n} X_i$, and $a > 0$ is a fixed constant. Under some suitable conditions, we show that $(\prod_{k=1}^{[nt]}(T_k(a)/\mu k))^{\mu/V_n} \xrightarrow{d} \exp\{\int_0^t (W(x)/x)dx\}$ *in* $D[0,1]$, as $n \to \infty$, where $T_k(a) = S_k - S_k(a)$ is the trimmed sum and $\{W(t); t \geq 0\}$ is a standard Wiener process.

1. Introduction

Let $\{X_n; n \geq 1\}$ be a sequence of random variables and define the partial sum $S_n = \sum_{i=1}^{n} X_i$ and $V_n^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2$ for $n \geq 1$, where $\overline{X} = 1/n\sum_{i=1}^{n} X_i$. In the past years, the asymptotic behaviors of the products of various random variables have been widely studied. Arnold and Villaseñor [1] considered sums of records and obtained the following form of the central limit theorem (CLT) for independent and identically distributed (i.i.d.) exponential random variables with the mean equal to one,

$$\frac{\sum_{k=1}^{n} \log S_k - n\log(n) + n}{\sqrt{2n}} \xrightarrow{d} \mathcal{N} \quad \text{as } n \longrightarrow \infty. \tag{1.1}$$

Here and in the sequel, $\mathcal{N}$ is a standard normal random variable, and $\xrightarrow{d}$ ($\xrightarrow{p}$, $\xrightarrow{\text{a.s.}}$) stands for convergence in distribution (in probability, almost surely). Observe that, via the Stirling formula, the relation (1.1) can be equivalently stated as

$$\left(\prod_{k=1}^{n} \frac{S_k}{k}\right)^{1/\sqrt{n}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}}. \tag{1.2}$$

In particular, Rempała and Wesołowski [2] removed the condition that the distribution is exponential and showed the asymptotic behavior of products of partial sums holds for any sequence of i.i.d. positive random variables. Namely, they proved the following theorem.

Theorem A. *Let $\{X_n;\ n \geq 1\}$ be a sequence of i.i.d. positive square integrable random variables with $\mathsf{E}X_1 = \mu$, $\operatorname{Var} X_1 = \sigma^2 > 0$ and the coefficient of variation $\gamma = \sigma/\mu$. Then, one has*

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{1/\gamma\sqrt{n}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}} \quad \text{as } n \longrightarrow \infty. \tag{1.3}$$

Recently, the above result was extended by Qi [3], who showed that whenever $\{X_n; n \geq 1\}$ is in the domain of attraction of a stable law $\mathcal{L}$ with index $\alpha \in (1, 2]$, there exists a numerical sequence A_n (for $\alpha = 2$, it can be taken as $\sigma\sqrt{n}$) such that

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{\mu/A_n} \xrightarrow{d} e^{(\Gamma(\alpha+1))^{1/\alpha})\mathcal{L}}, \tag{1.4}$$

as $n \to \infty$, where $\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x} dx$. Furthermore, Zhang and Huang [4] extended Theorem A to the invariance principle.

In this paper, we aim to study the weak invariance principle for self-normalized products of trimmed sums of i.i.d. sequences. Before stating our main results, we need to introduce some necessary notions. Let $\{X, X_n;\ n \geq 1\}$ be a sequence of i.i.d. random variables with a continuous distribution function F. Assume that the right extremity of F satisfies

$$\gamma_F = \sup\{x : F(x) < 1\} = \infty, \tag{1.5}$$

and the limiting tail quotient

$$\lim_{x \to \infty} \frac{\overline{F}(x+a)}{\overline{F}(x)}, \tag{1.6}$$

exists, where $\overline{F}(x) = 1 - F(x)$. Then, the above limit is e^{-ca} for some $c \in [0, \infty)$, and F or X is said to have a thick tail if $c = 0$, a medium tail if $0 < c < \infty$, and a thin tail if $c = \infty$. Denote $M_n = \max_{1 \leq j \leq n} X_j$. For a fixed constant $a > 0$, we say X_j is a near-maximum if and only if $X_j \in (M_n - a, M_n]$, and the number of near-maxima is

$$K_n(a) := \operatorname{Card}\{j \leq n;\ X_j \in (M_n - a, M_n]\}. \tag{1.7}$$

These concepts were first introduced by Pakes and Steutel [5], and their limit properties have been widely studied by Pakes and Steutel [5], Pakes and Li [6], Li [7], Pakes [8], and Hu and Su [9]. Now, set

$$S_n(a) := \sum_{i=1}^{n} X_i I\{M_n - a < X_i \leq M_n\}, \tag{1.8}$$

where

$$I\{A\} = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A, \end{cases}$$
$$T_n(a) := S_n - S_n(a), \tag{1.9}$$

which are the sum of near-maxima and the trimmed sum, respectively. From Remark 1 of Hu and Su [9], we have that if F has a medium tail and $\mathrm{E}X \neq 0$, then $T_n(a)/n \xrightarrow{\text{a.s.}} \mathrm{E}X$, which implies that with probability one $\mathrm{Card}\{k : T_k(a) = 0, k \geq 1\}$ is finite at most. Thus, we can redefine $T_k(a) = 1$ if $T_k(a) = 0$.

2. Main Result

Now we are ready to state our main results.

Theorem 2.1. *Let $\{X, X_n;\ n \geq 1\}$ be a sequence of positive i.i.d. random variables with a continuous distribution function F, and $\mathrm{E}X = \mu, \mathrm{Var}\, X = \sigma^2$. Assume that F has a medium tail. Then, one has*

$$\left(\prod_{k=1}^{[nt]} \frac{T_k(a)}{\mu k}\right)^{\mu/V_n} \xrightarrow{d} \exp\left\{\int_0^t \frac{W(x)}{x} dx\right\} \quad \text{in } D[0,1], \text{ as } n \longrightarrow \infty, \tag{2.1}$$

where $\{W(t); t \geq 0\}$ is a standard Wiener process.

In particular, when we take $t = 1$, it yields the following corollary.

Corollary 2.2. *Under the assumptions of Theorem 2.1, one has*

$$\left(\prod_{k=1}^{n} \frac{T_k(a)}{\mu k}\right)^{\mu/V_n} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}}, \tag{2.2}$$

as $n \rightarrow \infty$, where $\mathcal{N}$ is a standard normal random variable.

Remark 2.3. Since $\int_0^1 (W(x)/x) dx$ is a normal random variable with

$$\mathrm{E}\int_0^1 \frac{W(x)}{x} dx = \int_0^1 \frac{EW(x)}{x} dx = 0,$$
$$\mathrm{E}\left(\int_0^1 \frac{W(x)}{x} dx\right)^2 = \iint_0^1 \frac{EW(x)W(y)}{xy} dx\, dy = \iint_0^1 \frac{\min(x,y)}{xy} dx\, dy = 2. \tag{2.3}$$

Corollary 2.2 follows from Theorem 2.1 immediately.

3. Proof of Theorem 2.1

In this section, we will give the proof of Theorem 2.1. In the sequel, let C denote a positive constant which may take different values in different appearances and $[x]$ mean the largest integer $\le x$.

Note that via Remark 1 of Hu and Su [9], we have $C_k := T_k(a)/\mu k \overset{\text{a.s.}}{\to} 1$. It follows that for any $\delta > 0$, there exists a positive integer R such that

$$\mathsf{P}\left(\sup_{k\ge R}|C_k - 1| > \delta\right) < \delta. \tag{3.1}$$

Consequently, there exist two sequences $\delta_m \downarrow 0(\delta_1 = 1/2)$ and $R^*_m \uparrow \infty$ such that

$$\mathsf{P}\left(\sup_{k\ge R^*_m}|C_k - 1| > \delta_m\right) < \delta_m. \tag{3.2}$$

The strong law of large numbers also implies that there exists a sequence $R'_m \uparrow \infty$ such that

$$\sup_{k\ge R'_m}|C_k - 1| \overset{\text{a.s.}}{\le} \frac{1}{m}. \tag{3.3}$$

Here and in the sequel, we take $R_m = \max\{R^*_m, R'_m\}$, and it yields

$$\begin{gathered}\mathsf{P}\left(\sup_{k\ge R_m}|C_k - 1| > \delta_m\right) < \delta_m \\ \sup_{k\ge R_m}|C_k - 1| \overset{\text{a.s.}}{\le} \frac{1}{m}.\end{gathered} \tag{3.4}$$

Then, it leads to

$$\begin{aligned}\mathsf{P}\left(\frac{\mu}{V_n}\sum_{k=1}^{[nt]}\log(C_k) \le x\right) &= \mathsf{P}\left(\frac{\mu}{V_n}\sum_{k=1}^{[nt]}\log(C_k) \le x, \sup_{k\ge R_m}|C_k - 1| > \delta_m\right) \\ &\quad + \mathsf{P}\left(\frac{\mu}{V_n}\sum_{k=1}^{[nt]}\log(C_k) \le x, \sup_{k\ge R_m}|C_k - 1| \le \delta_m\right) \\ &=: A_{m,n} + B_{m,n},\end{aligned} \tag{3.5}$$

and $A_{m,n} < \delta_m$. By using the expansion of the logarithm $\log(1+x) = x - x^2/2(1+\theta x)^2$, where $\theta \in (0,1)$ depends on $|x| < 1$, we have that

$$\begin{aligned}
B_{m,n} &= \mathsf{P}\left(\frac{\mu}{V_n}\sum_{k=1}^{[nt]}\log(C_k) \le x, \sup_{k\ge R_m}|C_k - 1| \le \delta_m\right) \\
&= \mathsf{P}\left(\frac{\mu}{V_n}\sum_{k=1}^{R_m\wedge([nt]-1)}\log(C_k) + \frac{\mu}{V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}\log(1 + C_k - 1) \le x, \sup_{k\ge R_m}|C_k - 1| \le \delta_m\right) \\
&= \mathsf{P}\left(\frac{\mu}{V_n}\sum_{k=1}^{R_m\wedge([nt]-1)}\log(C_k) + \frac{\mu}{V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}(C_k - 1)\right. \\
&\qquad \left. - \frac{\mu}{V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}\frac{(C_k-1)^2}{2(1+\theta_k(C_k-1))^2} \le x, \sup_{k\ge R_m}|C_k - 1| \le \delta_m\right) \\
&= \mathsf{P}\left(\frac{\mu}{V_n}\sum_{k=1}^{R_m\wedge([nt]-1)}\log(C_k) + \frac{\mu}{V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}(C_k - 1)\right. \\
&\qquad \left. - \frac{\mu}{V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}\frac{(C_k-1)^2}{2(1+\theta_k(C_k-1))^2} I\left(\sup_{k\ge R_m}|C_k - 1| \le \delta_m\right) \le x\right) \\
&\quad - \mathsf{P}\left(\frac{\mu}{V_n}\sum_{k=1}^{R_m\wedge([nt]-1)}\log(C_k) + \frac{\mu}{V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}(C_k - 1) \le x, \sup_{k\ge R_m}|C_k - 1| > \delta_m\right) \\
&=: D_{m,n} - E_{m,n},
\end{aligned} \tag{3.6}$$

where θ_k $(k = 1, \ldots, [nt])$ are (0-1)-valued and $E_{m,n} < \delta_m$.

Also, we can rewrite $D_{m,n}$ as

$$\begin{aligned}
D_{m,n} = \mathsf{P}\Bigg(&\frac{\mu}{V_n}\sum_{k=1}^{R_m\wedge([nt]-1)}\left(\log(C_k) - C_k + 1\right) + \frac{\mu}{V_n}\sum_{k=1}^{[nt]}(C_k - 1) \\
&- \frac{\mu}{V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}\frac{(C_k-1)^2}{2(1+\theta_k(C_k-1))^2} I\left(\sup_{k\ge R_m}|C_k - 1| \le \delta_m\right) \le x\Bigg).
\end{aligned} \tag{3.7}$$

Observe that, for any fixed m, it is easy to obtain

$$\frac{\mu}{V_n}\sum_{k=1}^{R_m\wedge([nt]-1)}\left(\log(C_k) - C_k + 1\right) \xrightarrow{p} 0 \quad \text{as } n \longrightarrow \infty, \tag{3.8}$$

by noting that $V_n^2 \xrightarrow{p} \infty$.

And if $R_m \geq [nt]-1$, then we have

$$\frac{\mu}{V_n}\frac{\left(C_{[nt]}-1\right)^2}{2\left(1+\left(C_{[nt]}-1\right)\theta_{[nt]}\right)^2} \overset{\text{a.s.}}{\leq} \frac{C}{V_n} \xrightarrow{p} 0, \tag{3.9}$$

as $n \longrightarrow \infty$. If $R_m < [nt]-1$, then $R_m+1 < [nt]$. Denote

$$F_{m,n} := \left(\frac{\mu}{V_n}\sum_{k=R_m+1}^{[nt]}\frac{(C_k-1)^2}{2(1+\theta_k(C_k-1))^2}\right) I\left(\sup_{k\geq R_m}|C_k-1| \leq \delta_m\right), \tag{3.10}$$

and, by observing that $x^2/(1+\theta x)^2 \leq 4x^2$, then we can obtain

$$\begin{aligned} F_{m,n} &\leq \frac{C}{V_n}\sum_{k=R_m+1}^{[nt]}(C_k-1)^2 = \frac{C}{V_n}\sum_{k=R_m+1}^{[nt]}\left(\frac{S_k-S_k(a)}{\mu k}-1\right)^2 \\ &\leq \frac{C}{V_n}\sum_{k=R_m+1}^{[nt]}\left(\frac{S_k}{\mu k}-1\right)^2 + \frac{C}{V_n}\sum_{k=R_m+1}^{[nt]}\left(\frac{S_k(a)}{\mu k}\right)^2 \\ &=: H_{m,n} + L_{m,n}. \end{aligned} \tag{3.11}$$

For any $\varepsilon > 0$, by the Markov's inequality, we have

$$\begin{aligned} \mathsf{P}\left(\frac{1}{\sqrt{n}}\sum_{k=R_m+1}^{[nt]}\left(\frac{S_k}{\mu k}-1\right)^2 > \varepsilon\right) &\leq \frac{C}{\varepsilon\sqrt{n}}\mathsf{E}\left(\sum_{k=R_m+1}^{[nt]}\left(\frac{S_k}{\mu k}-1\right)^2\right) \\ &= \frac{C}{\varepsilon\sqrt{n}}\sum_{k=R_m+1}^{[nt]}\operatorname{Var}\left(\frac{S_k}{\mu k}\right) = \frac{C\sigma^2}{\varepsilon\mu^2\sqrt{n}}\sum_{k=R_m+1}^{[nt]}\frac{1}{k} \xrightarrow{\text{a.s.}} 0. \end{aligned} \tag{3.12}$$

Then, $H_{m,n} \xrightarrow{p} 0$. To obtain this result, we need the following fact:

$$\frac{V_n^2}{n} \xrightarrow{\text{a.s.}} \sigma^2, \qquad \frac{\sum_{i=1}^n\left(X_i-\overline{X}\right)^2}{\sum_{i=1}^n\left(X_i-\mu\right)^2} \xrightarrow{\text{a.s.}} 1, \quad \text{as } n \longrightarrow \infty. \tag{3.13}$$

Indeed,

$$\begin{aligned} \frac{\sum_{i=1}^n\left(X_i-\overline{X}\right)^2}{\sum_{i=1}^n\left(X_i-\mu\right)^2} &= \frac{\sum_{i=1}^n\left(X_i-\mu\right)^2 - n\left(\mu-\overline{X}\right)^2}{\sum_{i=1}^n\left(X_i-\mu\right)^2} \\ &= 1 - \frac{\left(\mu-\overline{X}\right)^2}{\left(\sum_{i=1}^n\left(X_i-\mu\right)^2\right)/n}. \end{aligned} \tag{3.14}$$

Now, we choose two constants $N > 0$ and $0 < \delta < 1$ such that $P(|X - \mu| > N) < \delta$. Hence, in view of the strong law of large numbers, we have for n large enough

$$\begin{aligned}\frac{\left(\mu-\overline{X}\right)^2}{\left(\sum_{i=1}^n (X_i-\mu)^2\right)/n} &\le \frac{\left(\mu-\overline{X}\right)^2}{\left(\sum_{i=1}^n (X_i-\mu)^2 I\left(\left|X_i-\mu\right|>N\right)\right)/n}\\ &\le \frac{\left(\mu-\overline{X}\right)^2}{N^2\left(\sum_{i=1}^n I\left(\left|X_i-\mu\right|>N\right)\right)/n}\\ &= \frac{o(1)}{N^2\left(P\left(\left|X-\mu\right|>N\right)+o(1)\right)} \overset{\text{a.s.}}{=} o(1),\end{aligned} \tag{3.15}$$

which together with (3.14) implies that

$$\frac{\sum_{i=1}^n \left(X_i-\overline{X}\right)^2}{\sum_{i=1}^n (X_i-\mu)^2} = \frac{V_n^2}{\sum_{i=1}^n (X_i-\mu)^2} \xrightarrow{\text{a.s.}} 1, \tag{3.16}$$

as $n \to \infty$. Furthermore, in view of the strong law of large numbers again, we obtain

$$\frac{V_n^2}{n} = \frac{\sum_{i=1}^n \left(X_i-\overline{X}\right)^2}{\sum_{i=1}^n (X_i-\mu)^2} \cdot \frac{\sum_{i=1}^n (X_i-\mu)^2}{n} \xrightarrow{\text{a.s.}} \sigma^2, \tag{3.17}$$

as $n \to \infty$, where $\sigma^2 = \operatorname{Var}(X) > 0$. For $L_{m,n}$, by noting that $S_n(a)/S_n \overset{\text{a.s.}}{\to} 0$, as $n \to \infty$ (see Hu and Su [9]), thus we can easily get

$$\frac{S_n(a)}{n} = \frac{S_n(a)}{S_n} \cdot \frac{S_n}{n} \xrightarrow{\text{a.s.}} 0, \tag{3.18}$$

as $n \to \infty$. Then, for any $0 < \delta' < 1$, there exists a positive integer R' such that

$$\mathsf{P}\left(\sup_{k\ge R'} \frac{S_k(a)}{k} \ge \delta'\right) < \delta'. \tag{3.19}$$

Consequently, coupled with (3.18), we have

$$\begin{aligned}\mathsf{P}\left(L_{m,n} > \delta'\right) &\le \mathsf{P}\left(\frac{C}{V_n}\sum_{k=1}^n\left(\frac{S_k(a)}{\mu k}\right)^2 > \delta', \sup_{k\ge R'}\frac{S_k(a)}{k} < \delta'\right) + \mathsf{P}\left(\sup_{k\ge R'}\frac{S_k(a)}{k} \ge \delta'\right)\\ &\le \mathsf{P}\left(\frac{C}{V_n}\sum_{k=1}^n \frac{S_k(a)}{k} > \delta'\right) + \delta'.\end{aligned} \tag{3.20}$$

Clearly, to show $L_{m,n} \xrightarrow{p} 0$, as $n \to \infty$, it is sufficient to prove

$$\frac{1}{V_n}\sum_{k=1}^{n}\frac{S_k(a)}{k} \xrightarrow{p} 0. \tag{3.21}$$

Indeed, combined with (3.17), we only need to show

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{S_k(a)}{k} \xrightarrow{p} 0. \tag{3.22}$$

As a matter of fact, by the definitions of $S_n(a)$ and $K_n(a)$, we have

$$(M_n - a)K_n(a) < S_n(a) \leq M_n K_n(a). \tag{3.23}$$

In view of the fact $M_n \uparrow \infty$(a.s.), we can get from Hu and Su [9] that

$$\frac{S_n(a)}{M_n} \overset{\text{a.s.}}{\sim} K_n(a), \tag{3.24}$$

and thus it suffices to prove

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{M_k K_k(a)}{k} \xrightarrow{p} 0. \tag{3.25}$$

Actually, for all $\varepsilon, \delta > 0$, and N_1 large enough, we can have that

$$\begin{aligned}\mathsf{P}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{K_k(a)M_k}{k} > \varepsilon\right) &= \mathsf{P}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{K_k(a)}{\sqrt{k}}\cdot\frac{M_k}{\sqrt{k}} > \varepsilon\right)\\ &\leq \mathsf{P}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{K_k(a)}{\sqrt{k}}\cdot\delta > \varepsilon, \sup_{k\geq N_1}\frac{M_k}{\sqrt{k}} < \delta\right) + \mathsf{P}\left(\sup_{k\geq N_1}\frac{M_k}{\sqrt{k}} \geq \delta\right).\end{aligned} \tag{3.26}$$

Observe that if F has a medium tail, then we have $M_n/\sqrt{n} = (M_n/\log n)(\log n/\sqrt{n}) \xrightarrow{\text{a.s.}} 0$ by noting that $M_n/\log n \xrightarrow{\text{a.s.}} 1/c$ [9], where c is the limit defined in Section 1. Thus it follows

$$\mathsf{P}\left(\sup_{k\geq N_1}\frac{M_k}{\sqrt{k}} \geq \delta\right) \longrightarrow 0, \tag{3.27}$$

as $N_1 \to \infty$. Further, by the Markov's inequality and the bounded property of $EK_k(a)$ from Hu and Su [9], we have

$$\begin{aligned}
\mathrm{P}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{K_k(a)}{\sqrt{k}}\cdot\delta>\varepsilon,\sup_{k\geq N_1}\frac{M_k}{\sqrt{k}}<\delta\right) &\leq \mathrm{P}\left(\frac{\delta}{\sqrt{n}}\sum_{k=1}^{n}\frac{K_k(a)}{\sqrt{k}}>\varepsilon\right)\\
&\leq C\frac{\delta}{\varepsilon\sqrt{n}}\sum_{k=1}^{n}\frac{EK_k(a)}{\sqrt{k}}\\
&\leq C\frac{\delta}{\varepsilon\sqrt{n}}\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\leq C\frac{\delta}{\varepsilon},
\end{aligned}\tag{3.28}$$

and, hence, the proof of (3.22) is terminated. Thus $L_{m,n} \xrightarrow{p} 0$ follows. Finally, in order to complete the proof, it is sufficient to show that

$$\Upsilon_n(t) := \frac{\mu}{V_n}\sum_{k=1}^{[nt]}(C_k-1) \xrightarrow{d} \int_0^t \frac{W(x)}{x}dx,\tag{3.29}$$

and, coupled with (3.21), we only need to prove

$$\Upsilon_n(t) := \frac{\mu}{V_n}\sum_{k=1}^{[nt]}\left(\frac{S_k}{\mu k}-1\right) \xrightarrow{d} \int_0^t \frac{W(x)}{x}dx.\tag{3.30}$$

Let

$$\begin{gathered}
H_\epsilon(f)(t) = \begin{cases}\displaystyle\int_\epsilon^t \frac{f(x)}{x}dx, & t>\epsilon,\\ 0, & 0\leq t\leq\epsilon,\end{cases}\\
\Upsilon_{n,\epsilon}(t) = \begin{cases}\displaystyle\frac{1}{V_n}\sum_{k=[n\epsilon]+1}^{[nt]}\frac{S_k-\mu k}{k}, & t>\epsilon,\\ 0, & 0\leq t\leq\epsilon.\end{cases}
\end{gathered}\tag{3.31}$$

It is obvious that

$$\max_{0\leq t\leq 1}\left|\int_0^t\frac{W(x)}{x}dx - H_\epsilon(W)(t)\right| = \sup_{0\leq t\leq\epsilon}\left|\int_0^t\frac{W(x)}{x}dx\right| \xrightarrow{\text{a.s.}} 0, \quad \text{as } \epsilon \longrightarrow 0.\tag{3.32}$$

Note that

$$\max_{0\leq t\leq\epsilon}\left|\Upsilon_n(t)-\Upsilon_{n,\epsilon}(t)\right| = \max_{0\leq t\leq\epsilon}\frac{1}{V_n}\sum_{k=1}^{[nt]}\frac{\left|S_k-\mu k\right|}{k} \leq \frac{1}{V_n}\sum_{k=1}^{[n\epsilon]}\frac{\left|S_k-\mu k\right|}{k},\tag{3.33}$$

and then, for any $\epsilon_1 > 0$, by the Cauchy-Schwarz inequality and (3.17), it follows that

$$\begin{aligned}
\lim_{\epsilon\to 0}\limsup_{n\to\infty} \mathsf{P}\left(\max_{0\le t\le\epsilon}|Y_n(t)-Y_{n,\epsilon}(t)|\ge\epsilon_1\right) &\le \lim_{\epsilon\to 0}\limsup_{n\to\infty} \mathsf{P}\left(\frac{1}{V_n}\sum_{k=1}^{[n\epsilon]}\frac{|S_k-\mu k|}{k}\ge\epsilon_1\right)\\
&\le \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{C}{\sqrt{n}}\sum_{k=1}^{[n\epsilon]}\frac{E|S_k-\mu k|}{k}\\
&\le \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{C}{\sqrt{n}}\sum_{k=1}^{[n\epsilon]}\frac{1}{\sqrt{k}}\left(\mathrm{Var}\left(\frac{S_k-\mu k}{\sqrt{k}}\right)\right)^{1/2}\\
&= \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{C}{\sqrt{n}}\sum_{k=1}^{[n\epsilon]}\frac{1}{\sqrt{k}}\\
&\le \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{C}{\sqrt{n}}\sqrt{[n\epsilon]}.
\end{aligned}\tag{3.34}$$

Furthermore, we can obtain

$$\begin{aligned}
&\sup_{\epsilon\le t\le 1}\frac{1}{V_n}\left|\sum_{k=[n\epsilon]+1}^{[nt]}\frac{S_k-\mu k}{k}-\int_{n\epsilon}^{nt}\frac{S_{[x]}-[x]\mu}{x}dx\right|\\
&\quad\le \sup_{\epsilon\le t\le 1}\frac{1}{V_n}\left|\int_{[n\epsilon]+1}^{[nt]+1}\frac{S_{[x]}-[x]\mu}{[x]}dx-\int_{n\epsilon}^{nt}\frac{S_{[x]}-[x]\mu}{x}dx\right|\\
&\quad\le \frac{1}{V_n}\left|\int_{n\epsilon}^{[n\epsilon]+1}\frac{S_{[x]}-[x]\mu}{x}dx\right|+\sup_{\epsilon\le t\le 1}\frac{1}{V_n}\left|\int_{nt}^{[nt]+1}\frac{S_{[x]}-[x]\mu}{x}dx\right|\\
&\qquad+\sup_{\epsilon\le t\le 1}\frac{1}{V_n}\left|\int_{[n\epsilon]+1}^{[nt]+1}\left(S_{[x]}-[x]\mu\right)\left(\frac{1}{x}-\frac{1}{[x]}\right)dx\right|\\
&\quad\le \frac{\max_{k\le n}|S_k-\mu k|}{V_n}\sup_{\epsilon\le t\le 1}\left(\frac{2}{n\epsilon}+\frac{2}{nt}+\frac{1}{n\epsilon}\right)\\
&\quad\le C\frac{\max_{k\le n}|S_k-\mu k|}{nV_n}\le C\frac{\max_{k\le n}\sum_{i=1}^{k}|X_i-\mu|}{nV_n}\\
&\quad=\frac{C}{V_n}\frac{\sum_{i=1}^{n}|X_i-\mu|}{n}\xrightarrow{\text{a.s.}}0.
\end{aligned}\tag{3.35}$$

Therefore, uniformly for $t\in[\epsilon,1]$, we have

$$\frac{1}{V_n}\sum_{k=[n\epsilon]+1}^{[nt]}\frac{S_k-\mu k}{k}=\frac{1}{V_n}\int_{n\epsilon}^{nt}\frac{S_{[x]}-[x]\mu}{x}dx+o_P(1)=\int_{\epsilon}^{t}\frac{W_n(t)}{x}dx+o_P(1),\tag{3.36}$$

where $W_n(t) := (S_{[nt]} - [nt]\mu)/V_n$. Notice that $H_\epsilon(\cdot)$ is a continuous mapping on the space $D[0,1]$. Thus, using the continuous mapping theorem (c.f., Theorem 2.7 of Billingsley [10]), it follows that

$$Y_{n,\epsilon}(t) = H_\epsilon(W_n)\ (t) + o_P(1) \xrightarrow{d} H_\epsilon(W)(t), \quad \text{in } D[0,1], \text{ as } n \longrightarrow \infty. \tag{3.37}$$

Hence, (3.32), (3.34), and (3.37) coupled with Theorem 3.2 of Billingsley [10] lead to (3.30). The proof is now completed.

4. Application to U-Statistics

A useful notion of a U-statistic has been introduced by Hoeffding [11]. Let a U-statistic be defined as

$$U_n = \binom{n}{m}^{-1} \sum_{1 \le i_1 < \cdots < i_m \le n} h(X_{i_1}, \dots, X_{i_m}), \tag{4.1}$$

where h is a symmetric real function of m arguments and $\{X_i; i \ge 1\}$ is a sequence of i.i.d. random variables. If we take $m = 1$ and $h(x) = x$, then U_n reduces to S_n/n. Assume that $Eh(X_1, \dots, X_m)^2 < \infty$, and let

$$\begin{aligned} h_1(x) &= Eh(x, X_2, \dots, X_m), \\ \widehat{U}_n &= \frac{m}{n} \sum_{i=1}^{n} (h_1(X_i) - Eh) + Eh. \end{aligned} \tag{4.2}$$

Thus, we may write

$$U_n = \widehat{U}_n + R_n, \tag{4.3}$$

where

$$\begin{aligned} R_n &= \binom{n}{m}^{-1} \sum_{1 \le i_1 < \cdots < i_m \le n} H(X_{i_1}, \dots, X_{i_m}), \\ H(x_1, \dots, x_m) &= h(x_1, \dots, x_m) - \sum_{i=1}^{m} (h_1(x_i) - Eh) - Eh. \end{aligned} \tag{4.4}$$

It is well known (cf. Resnick [12]) that

$$\begin{aligned} \operatorname{Cov}\left(\widehat{U}_n, R_n\right) &= 0, \\ n \operatorname{Var}\left(\binom{n}{m}^{-1} R_n\right) &\longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{4.5}$$

Theorem 2.1 now is extended to U-statistics as follows.

Theorem 4.1. *Let U_n be a U-statistic defined as above. Assume that* $\mathsf{E}h^2 < \infty$ *and* $\mathsf{P}(h(X_1, \ldots, X_m) > 0) = 1$. *Denote* $\mu = \mathsf{E}h > 0$ *and* $\sigma^2 = \mathrm{Var}(h_1(X_1)) \neq 0$. *Then,*

$$\left(\prod_{k=m}^{[nt]} \frac{U_k}{\mu\binom{n}{m}}\right)^{\mu/mV_n} \xrightarrow{d} \exp\left\{\int_0^t \frac{W(x)}{x}dx\right\}, \quad \text{in } D[0,1], \text{ as } n \longrightarrow \infty, \tag{4.6}$$

where $W(x)$ is a standard wiener process, and $V_n^2 = \sum_{i=1}^n (X_i - \overline{X})^2$.

In order to prove this theorem, by (3.17), we only need to prove

$$\frac{\mu}{m\sqrt{n}} \sum_{k=m}^{[nt]} \left(\frac{U_k}{\mu\binom{n}{m}} - 1\right) \xrightarrow{d} \sigma \int_0^t \frac{W(x)}{x}dx, \quad \text{in } D[0,1], \text{ as } n \longrightarrow \infty. \tag{4.7}$$

If this result is true, then with the fact that $\binom{n}{m}^{-1}U_n \overset{\text{a.s.}}{\to} Eh = \mu$ deduced from $E|h| < \infty$ (see Resnick [12]) and (4.3), Theorem 4.1 follows immediately from the method used in the proof of Theorem 2.1 with S_k/k replaced by $\binom{n}{m}^{-1}U_n$. Now, we begin to show (4.7). By (4.3), we have

$$\frac{\mu}{m\sqrt{n}} \sum_{k=m}^{[nt]} \left(\frac{U_k}{\mu\binom{n}{m}} - 1\right) = \frac{\mu}{m\sqrt{n}} \sum_{k=m}^{[nt]} \left(\frac{\widehat{U}_k}{\mu\binom{n}{m}} - 1\right) + \frac{\mu}{m\sqrt{n}} \sum_{k=m}^{[nt]} \frac{R_k}{\mu\binom{n}{m}}. \tag{4.8}$$

By applying (3.30) to random variables $mh_1(X_i)$ for $i \geq 1$, we have

$$\begin{aligned} \frac{\mu}{m\sqrt{n}} \sum_{k=m}^{[nt]} \left(\frac{\widehat{U}_k}{\mu\binom{n}{m}} - 1\right) &= \frac{\mu}{\sqrt{n}} \left(\sum_{k=1}^{n} \left(\frac{\sum_{i=1}^k h_1(x_i)}{\mu k} - 1\right) - \sum_{k=1}^{m-1} \left(\frac{\sum_{i=1}^k h_1(x_i)}{\mu k} - 1\right)\right) \\ &\xrightarrow{d} \sigma \int_0^t \frac{W(x)}{x}dx, \end{aligned} \tag{4.9}$$

in $D[0,1]$, as $n \to \infty$, since the second expression converges to zero a.s. as $n \to \infty$. Therefore, for proving (4.7), we only need to prove

$$\frac{\mu}{m\sqrt{n}} \sum_{k=m}^{[nt]} \frac{R_k}{\mu\binom{n}{m}} \xrightarrow{p} 0, \quad \text{as } n \longrightarrow \infty, \tag{4.10}$$

and it is sufficient to demonstrate

$$\widetilde{R}_n := \frac{\mu}{m\sqrt{n}} \sum_{k=m}^{n} \frac{R_k}{\mu\binom{n}{m}} \xrightarrow{p} 0, \quad \text{as } n \longrightarrow \infty. \tag{4.11}$$

Indeed, we can easily obtain $E\widetilde{R}_n^2 \to 0$ as $n \to \infty$ from Hoeffding [11]. Thus, we complete the proof of (4.7), and, hence, Theorem 3.1 holds.

Acknowledgment

The author thanks the referees for valuable comments that have led to improvements in this work.

References

[1] B. C. Arnold and J. A. Villaseñor, "The asymptotic distributions of sums of records," *Extremes*, vol. 1, no. 3, pp. 351–363, 1999.

[2] G. Rempała and J. Wesołowski, "Asymptotics for products of sums and U-statistics," *Electronic Communications in Probability*, vol. 7, pp. 47–54, 2002.

[3] Y. Qi, "Limit distributions for products of sums," *Statistics & Probability Letters*, vol. 62, no. 1, pp. 93–100, 2003.

[4] L.-X. Zhang and W. Huang, "A note on the invariance principle of the product of sums of random variables," *Electronic Communications in Probability*, vol. 12, pp. 51–56, 2007.

[5] A. G. Pakes and F. W. Steutel, "On the number of records near the maximum," *The Australian Journal of Statistics*, vol. 39, no. 2, pp. 179–192, 1997.

[6] A. G. Pakes and Y. Li, "Limit laws for the number of near maxima via the Poisson approximation," *Statistics & Probability Letters*, vol. 40, no. 4, pp. 395–401, 1998.

[7] Y. Li, "A note on the number of records near the maximum," *Statistics & Probability Letters*, vol. 43, no. 2, pp. 153–158, 1999.

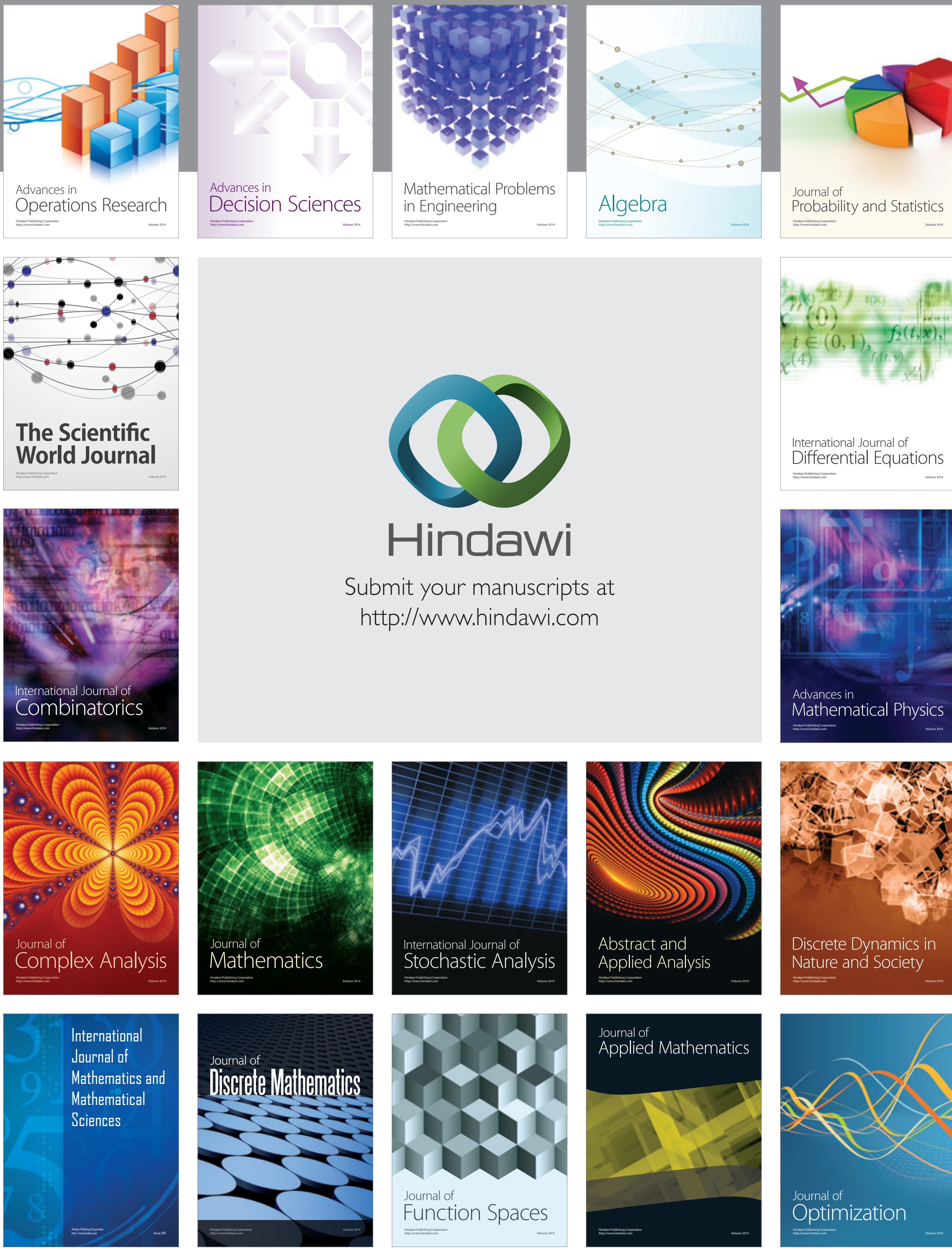
[8] A. G. Pakes, "The number and sum of near-maxima for thin-tailed populations," *Advances in Applied Probability*, vol. 32, no. 4, pp. 1100–1116, 2000.

[9] Z. Hu and C. Su, "Limit theorems for the number and sum of near-maxima for medium tails," *Statistics & Probability Letters*, vol. 63, no. 3, pp. 229–237, 2003.

[10] P. Billingsley, *Convergence of Probability Measures*, Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons Inc., New York, NY, USA, 2nd edition, 1999.

[11] W. Hoeffding, "A class of statistics with asymptotically normal distribution," *Annals of Mathematical Statistics*, vol. 19, pp. 293–325, 1948.

[12] S. I. Resnick, "Limit laws for record values," *Stochastic Processes and their Applications*, vol. 1, pp. 67–82, 1973.



Advances in Operations Research
Advances in Decision Sciences
Mathematical Problems in Engineering
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
International Journal of Stochastic Analysis
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
Journal of Applied Mathematics
Journal of Optimization