Research Article

Estimating the Conditional Tail Expectation in the Case of Heavy-Tailed Losses

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The conditional tail expectation (CTE) is an important actuarial risk measure and a useful tool in financial risk assessment. Under the classical assumption that the second moment of the loss variable is finite, the asymptotic normality of the nonparametric CTE estimator has already been established in the literature. The noted result, however, is not applicable when the loss variable follows any distribution with infinite second moment, which is a frequent situation in practice. With a help of extreme-value methodology, in this paper, we offer a solution to the problem by suggesting a new CTE estimator, which is applicable when losses have finite means but infinite variances.

1. Introduction

One of the most important actuarial risk measures is the conditional tail expectation (CTE) (see, e.g., [1]), which is the average amount of loss given that the loss exceeds a specified quantile. Hence, the CTE provides a measure of the capital needed due to the exposure to the loss, and thus serves as a risk measure. Not surprisingly, therefore, the CTE continues to receive increased attention in the actuarial and financial literature, where we also find its numerous extensions and generalizations (see, e.g., [2–8], and references therein). We next present basic notation and definitions.

Let X be a loss random variable with cumulative distribution function (cdf) F. Usually, the cdf F is assumed to be continuous and defined on the entire real line, with negative loss interpreted as gain. We also assume the continuity of F throughout the present paper. The CTE of the risk or loss X is then defined, for every $t \in (0,1)$,

by

$$\text{CTE}_F(t) = \mathbf{E}[X \mid X > Q(t)], \tag{1.1}$$

where $Q(t) = \inf\{x : F(x) \geq t\}$ is the quantile function corresponding to the cdf F. Since the cdf F is continuous, we easily check that

$$\text{CTE}_F(t) = \frac{1}{1-t} \int_t^1 Q(s)ds. \tag{1.2}$$

Naturally, the CTE is unknown since the cdf F is unknown. Hence, it is desirable to establish statistical inferential results such as confidence intervals for $\text{CTE}_F(t)$ with specified confidence levels and margins of error. We shall next show how to accomplish this task, initially assuming the classical moment assumption $\mathbf{E}[X^2] < \infty$. Namely, suppose that we have independent random variables $X_1, X_2, \ldots$, each with the cdf F, and let $X_{1:n} < \cdots < X_{n:n}$ denote the order statistics of $X_1, \ldots, X_n$. It is natural to define an empirical estimator of $\text{CTE}_F(t)$ by the formula

$$\widehat{\text{CTE}}_n(t) = \frac{1}{1-t} \int_t^1 Q_n(s)ds, \tag{1.3}$$

where $Q_n(s)$ is the empirical quantile function, which is equal to the ith order statistic $X_{i:n}$ for all $s \in ((i-1)/n, i/n]$, and for all $i = 1, \ldots, n$. The asymptotic behavior of the estimator $\widehat{\text{CTE}}_n(t)$ has been studied by Brazauskas et al. [9], and we next formulate their most relevant result for our paper as a theorem.

Theorem 1.1. *Assume that $\mathbf{E}[X^2] < \infty$. Then for every $t \in (0,1)$, we have the asymptotic normality statement*

$$\sqrt{n}\left(\widehat{\text{CTE}}_n(t) - \text{CTE}_F(t)\right)(1-t) \longrightarrow_d \mathcal{N}\left(0, \sigma^2(t)\right), \tag{1.4}$$

when $n \to \infty$, where the asymptotic variance $\sigma^2(t)$ is given by the formula

$$\sigma^2(t) = \int_t^1 \int_t^1 (\min(x,y) - xy)dQ(x)dQ(y). \tag{1.5}$$

The assumption $\mathbf{E}[X^2] < \infty$ is, however, quite restrictive as the following example shows. Suppose that F is the Pareto cdf with index $\gamma > 0$, that is, $1 - F(x) = x^{-1/\gamma}$ for all $x \geq 1$. Let us focus on the case $\gamma < 1$, because when $\gamma \geq 1$, then $\text{CTE}_F(t) = +\infty$ for every $t \in (0,1)$. Theorem 1.1 covers only the values $\gamma \in (0, 1/2)$ in view of the assumption $\mathbf{E}[X^2] < \infty$. When $\gamma \in [1/2, 1)$, we have $\mathbf{E}[X^2] = \infty$ but, nevertheless, $\text{CTE}_F(t)$ is well defined and finite since $\mathbf{E}[X] < \infty$. Analogous remarks hold for other distributions with Pareto-like tails, an we shall indeed work with such general distributions in this paper.

Namely, recall that the cdf F is regularly varying at infinity with index $(-1/\gamma) < 0$ if

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-1/\gamma} \tag{1.6}$$

for every $x > 0$. This class includes a number of popular distributions such as Pareto, generalized Pareto, Burr, Fréchet, Student, and so forth, which are known to be appropriate models for fitting large insurance claims, fluctuations of prices, log-returns, and so forth (see, e.g., [10]). In the remainder of this paper, therefore, we restrict ourselves to this class of distributions. For more information on the topic and, generally, on extreme value models and their manifold applications, we refer to the monographs by Beirlant et al. [11], Castillo et al. [12], de Haan and Ferreira [13], Resnick [14].

The rest of the paper is organized as follows. In Section 2 we construct an alternative, called "new", CTE estimator by utilizing an extreme value approach. In Section 3 we establish the asymptotic normality of the new CTE estimator and illustrate its performance with a little simulation study. The main result, which is Theorem 3.1 stated in Section 3, is proved in Section 4.

2. Construction of a New CTE Estimator

We have already noted that the "old" estimator $\widehat{\text{CTE}}_n(t)$ does not yield the asymptotic normality (in the classical sense) beyond the condition $\mathbf{E}[X^2] < \infty$. Indeed, this follows by setting $t = 0$, in which case $\widehat{\text{CTE}}_n(t)$ becomes the sample mean of $X_1, \ldots, X_n$, and thus the asymptotic normality of $\widehat{\text{CTE}}_n(0)$ is equivalent to the classical Central Limit Theorem (CLT). Similar arguments show that the finite second moment is necessary for having the asymptotic normality (in the classical sense) of $\widehat{\text{CTE}}_n(t)$ at any fixed "level" $t \in (0, 1)$. Indeed, note that the asymptotic variance $\sigma^2(t)$ in Theorem 1.1 is finite only if $\mathbf{E}[X^2] < \infty$.

For this reason, we next construct an alternative CTE estimator, which takes into account different asymptotic properties of moderate and high quantiles in the case of heavy-tailed distributions. Hence, from now on we assume that $\gamma \in (1/2, 1)$. Before indulging ourselves into construction details, we first formulate the new CTE estimator:

$$\widetilde{\text{CTE}}_n(t) = \frac{1}{1-t} \int_t^{1-k/n} Q_n(s)ds + \frac{kX_{n-k:n}}{n(1-t)(1-\widehat{\gamma})}, \tag{2.1}$$

where we use the simplest yet useful and powerful Hill's [15] estimator

$$\widehat{\gamma}_n = \frac{1}{k} \sum_{i=1}^{k} \log X_{n-i+1:n} - \log X_{n-k:n} \tag{2.2}$$

of the tail index $\gamma \in (1/2, 1)$. Integers $k = k_n \in \{1, \ldots, n\}$ are such that $k \to \infty$ and $k/n \to 0$ when $n \to \infty$, and we note at the outset that their choices present a challenging task. In Figures 1 and 2, we illustrate the performance of the new estimator $\widetilde{\text{CTE}}_n(t)$ with respect to the sample size $n \geq 1$, with the integers $k = k_n$ chosen according to the method proposed by Cheng and Peng [16]. Note that when t increases through the values 0.25, 0.50, 0.75, and 0.90
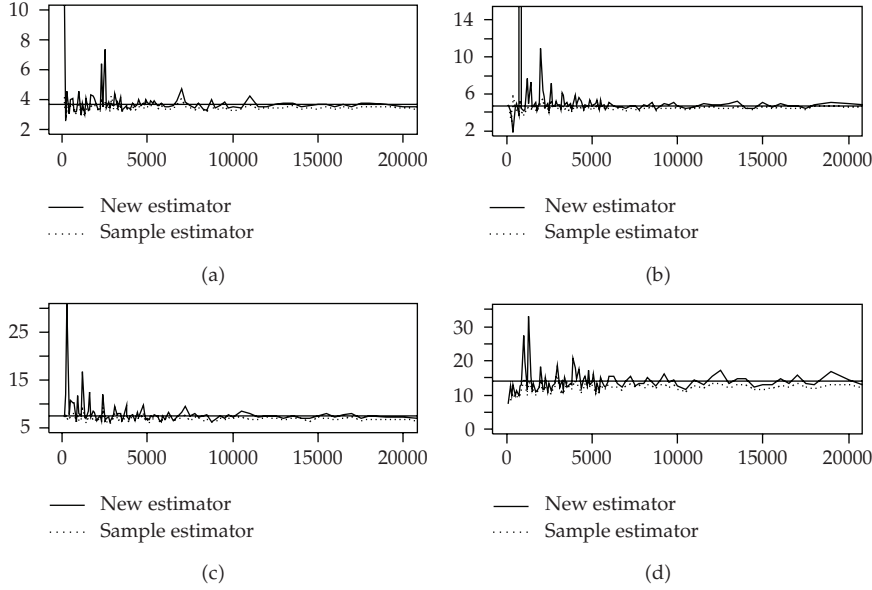
Figure 1: Values of the CTE estimator $\widetilde{\text{CTE}}_n(t)$ (vertical axis) versus sample sizes n (horizontal axis) evaluated at the levels $t = 0.25$, $t = 0.50$, $t = 0.75$, and $t = 0.90$ (panels (a)–(d), resp.) in the Pareto case with the tail index $\gamma = 2/3$.
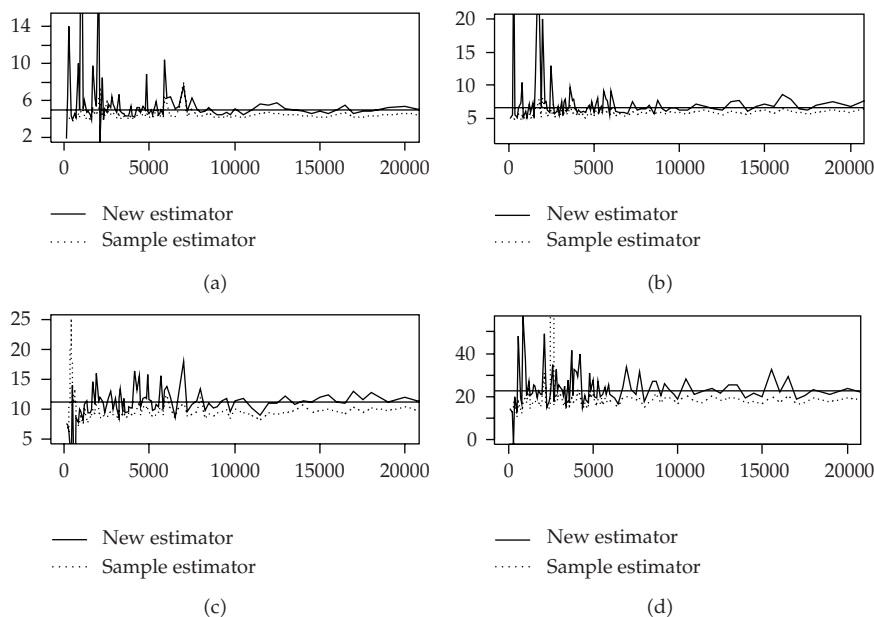
(panels (a)–(d), resp.), the vertical axes of the panels also increase, which reflects the fact that the larger the t gets, the more erratic the "new" and "old" estimators become. Note also that the empirical (i.e., "old") estimator underestimates the theoretical $\text{CTE}_F(t)$, which is a well known phenomenon (see [17]).

We have based the construction of $\widetilde{\text{CTE}}_n(t)$ on the recognition that one should estimate moderate and high quantiles differently when the underlying distribution is heavy-tailed. For this, we first recall that the high quantile q_s is, by definition, equal to $Q(1 - s)$ for sufficiently small s. For an estimation theory of high quantiles in the case of heavy-tailed distributions we refer to, for example, Weissman [18], Dekkers and de Haan [19], Matthys and Beirlant [20], Gomes et al. [21], and references therein. We shall use the Weissman estimator

$$\tilde{q}_s = \left(\frac{k}{n}\right)^{\widehat{\gamma}} X_{n-k:n} s^{-\widehat{\gamma}}, \quad 0 < s < \frac{k}{n}, \tag{2.3}$$

of the high quantile q_s. Then we write $\text{CTE}_F(t)$ as the sum $\text{CTE}_{1,n}(t) + \text{CTE}_{2,n}(t)$ with the two summands defined together with their respective empirical estimators $\widetilde{\text{CTE}}_{1,n}(t)$ and $\widetilde{\text{CTE}}_{2,n}(t)$ as follows:

$$\text{CTE}_{1,n}(t) = \frac{1}{1-t} \int_t^{1-k/n} Q(s)ds \approx \frac{1}{1-t} \int_t^{1-k/n} Q_n(s)ds = \widetilde{\text{CTE}}_{1,n}(t),$$

$$\text{CTE}_{2,n}(t) = \frac{1}{1-t} \int_{1-k/n}^1 Q(s)ds \approx \frac{1}{1-t} \int_{1-k/n}^1 \tilde{q}_{1-s}ds = \widetilde{\text{CTE}}_{2,n}(t). \tag{2.4}$$

Figure 2: Values of the CTE estimator $\widetilde{\text{CTE}}_n(t)$ (vertical axis) versus sample sizes n (horizontal axis) evaluated at the levels $t = 0.25$, $t = 0.50$, $t = 0.75$, and $t = 0.90$ (panels (a)–(d), resp.) in the Pareto case with the tail index $\gamma = 3/4$.

Simple integration gives the formula

$$\widetilde{\text{CTE}}_{2,n}(t) = \frac{kX_{n-k,n}}{n(1-\widehat{\gamma})(1-t)}. \tag{2.5}$$

Consequently, the sum $\widetilde{\text{CTE}}_{1,n}(t) + \widetilde{\text{CTE}}_{2,n}(t)$ is an estimator of $\text{CTE}_F(t)$, and this is exactly the estimator $\widetilde{\text{CTE}}_n(t)$ introduced above. We shall investigate asymptotic normality of the new estimator in the next section, accompanied with an illustrative simulation study.

3. Main Theorem and Its Practical Implementation

We start this section by noting that Hill's estimator $\widehat{\gamma}_n$ has been thoroughly studied, improved, and generalized in the literature. For example, weak consistency of $\widehat{\gamma}_n$ has been established by Mason [22] assuming only that the underlying distribution is regularly varying at infinity. Asymptotic normality of $\widehat{\gamma}$ has been investigated under various conditions by a number of researchers, including Csörgő and Mason [23], Beirlant and Teugels [24], Dekkers et al. [25], see also references therein.

The main theoretical result of this paper, which is Theorem 3.1 below, establishes asymptotic normality of the new CTE estimator $\widetilde{\text{CTE}}_n(t)$. To formulate the theorem, we need to introduce an assumption that ensures the asymptotic normality of Hill's estimator $\widehat{\gamma}_n$. Namely, the cdf F satisfies the generalized second-order regular variation condition with

second-order parameter $\rho \leq 0$ (see [26, 27]) if there exists a function $a(t)$ which does not change its sign in a neighbourhood of infinity and is such that, for every $x > 0$,

$$\lim_{t \to \infty} \frac{1}{a(t)} \left(\frac{1 - F(tx)}{1 - F(t)} - x^{-1/\gamma} \right) = x^{-1/\gamma} \frac{x^{\rho/\gamma} - 1}{\rho/\gamma}. \tag{3.1}$$

When $\rho = 0$, then the ratio on the right-hand side of (3.1) is interpreted as $\log x$. For statistical inference concerning the second-order parameter ρ, we refer, for example, to Peng and Qi [28], Gomes et al. [21], Gomes and Pestana [29]. Furthermore, in the formulation of Theorem 3.1, we shall also use the function $A(z) = \gamma^2 a(\mathbb{U}(z))$, where $\mathbb{U}(z) = Q(1 - 1/z)$.

Theorem 3.1. *Assume that the cdf F satisfies condition* (3.1) *with $\gamma \in (1/2, 1)$. Then for any sequence of integers $k = k_n \to \infty$ such that $k/n \to 0$ and $k^{1/2} A(n/k) \to 0$ when $n \to \infty$, we have that for any fixed $t \in (0, 1)$,*

$$\frac{\sqrt{n} \left(\widetilde{\mathrm{CTE}}_n(t) - \mathrm{CTE}_F(t) \right)(1 - t)}{(k/n)^{1/2} X_{n-k:n}} \longrightarrow_d \mathcal{N}\left(0, \sigma_\gamma^2\right), \tag{3.2}$$

where the asymptotic variance σ_γ^2 is given by the formula

$$\sigma_\gamma^2 = \frac{\gamma^4}{(1 - \gamma)^4 (2\gamma - 1)}. \tag{3.3}$$

The asymptotic variance σ_γ^2 does not depend on t, unlike the variance in **Theorem 1.1**. This is not surprising because the heaviness of the right-most tail of F makes the asymptotic behaviour of $\int_t^1 (Q_n(s) - Q(s)) ds$ "heavier" than the classical CLT-type behaviour of $\int_0^t (Q_n(s) - Q(s)) ds$, for any fixed t. This in turn implies that under the conditions of **Theorem 3.1**, statement (3.2) is equivalent to the same statement in the case $t = 0$. The latter statement concerns estimating the mean $\mathbf{E}[X]$ of a heavy-tailed distribution. Therefore, we can view **Theorem 3.1** as a consequence of Peng [30], and at the same time we can view results of Peng [30] as a consequence of **Theorem 3.1** by setting $t = 0$ in it. Despite this equivalence, in Section 4 we give a proof of **Theorem 3.1** for the sake of completeness. Our proof, however, is crucially based on a powerful technique called the Vervaat process (see [31–33], for details and references).

To discuss practical implementation of **Theorem 3.1**, we first fix a significance level $\varsigma \in (0, 1)$ and use the classical notation $z_{\varsigma/2}$ for the $(1 - \varsigma/2)$-quantile of the standard normal distribution $\mathcal{N}(0, 1)$. Given a realization of the random variables $X_1, \ldots, X_n$ (e.g., claim amounts), which follow a cdf F satisfying the conditions of **Theorem 3.1**, we construct a level $1 - \varsigma$ confidence interval for $\mathrm{CTE}_F(t)$ as follows. First, we choose an appropriate number k of extreme values. Since Hill's estimator has in general a substantial variance for small k and a considerable bias for large k, we search for a k that balances between the two shortcomings, which is indeed a well-known hurdle when estimating the tail index. To resolve this issue, several procedures have been suggested in the literature, and we refer to, for example, Dekkers and de Haan [34], Drees and Kaufmann [35], Danielsson et al. [36], Cheng and Peng [16], Neves and Fraga Alves [37], Gomes et al. [38], and references therein.

Table 1: Point estimates $\widetilde{\text{CTE}}_n(t)$ and 95% confidence intervals for $\text{CTE}_F(t)$ when $\gamma = 2/3$.

$t = 0.75$				$\text{CTE}_F(t) = 7.005$				
n	k^*	$\widetilde{\text{CTE}}_n(t)$	error	rmse	lower	upper	cover	length
1000	054	6.876	0.045	0.303	6.356	7.397	0.839	1.041
2000	100	6.831	0.025	0.231	6.463	7.199	0.882	0.736
5000	219	7.119	0.016	0.194	6.881	7.357	0.895	0.476
$t = 0.90$				$\text{CTE}_F(t) = 12.533$				
n	k^*	$\widetilde{\text{CTE}}_n(t)$	error	rmse	lower	upper	cover	length
1000	054	12.753	0.017	0.534	12.241	13.269	0.847	1.028
2000	100	12.487	0.003	0.294	12.137	12.838	0.841	0.701
5000	219	12.461	0.005	0.236	12.246	12.676	0.887	0.430

Table 2: Point estimates $\widetilde{\text{CTE}}_n(t)$ and 95% confidence intervals for $\text{CTE}_F(t)$ when $\gamma = 3/4$.

$t = 0.75$				$\text{CTE}_F(t) = 9.719$				
n	k^*	$\widetilde{\text{CTE}}_n(t)$	error	rmse	lower	upper	cover	length
1000	051	9.543	0.018	0.582	8.589	9.543	0.854	0.954
2000	104	9.808	0.009	0.466	9.150	10.466	0.888	1.316
5000	222	9.789	0.007	0.410	9.363	10.215	0.915	0.852
$t = 0.90$				$\text{CTE}_F(t) = 18.494$				
n	k^*	$\widetilde{\text{CTE}}_n(t)$	error	rmse	lower	upper	cover	length
1000	051	18.199	0.015	0.989	17.437	18.960	0.874	1.523
2000	104	18.696	0.011	0.858	18.052	19.340	0.895	1.288
5000	222	18.541	0.002	0.798	18.092	18.990	0.925	0.898

In our current study, we employ the method of Cheng and Peng [16] for an appropriate value k^* of the "parameter" k. Having computed Hill's estimator and consequently determined $X_{n-k^*:n}$, we then compute the corresponding values of $\widetilde{\text{CTE}}_n(t)$ and $\sigma^2_{\hat{\gamma}_n}$, and denote them by $\widetilde{\text{CTE}}^*_n(t)$ and $\sigma^{2*}_{\hat{\gamma}_n}$, respectively. Finally, using Theorem 3.1 we arrive at the following $(1 - \varsigma)$-confidence interval for $\text{CTE}_F(t)$:

$$\widetilde{\text{CTE}}^*_n(t) \pm z_{\varsigma/2} \frac{(k^*/n)^{1/2} X_{n-k^*:n} \, \sigma^{2*}_{\hat{\gamma}_n}}{(1 - t)\sqrt{n}}. \tag{3.4}$$

To illustrate the performance of this confidence interval, we have carried out a small-scale simulation study based on the Pareto cdf $F(x) = 1 - x^{-1/\gamma}$, $x \geq 1$, with the tail index γ set to $2/3$ and $3/4$, and the level t set to 0.75 and 0.90. We have generated 200 independent replicates of three samples of sizes $n = 1000$, 2000, and 5000. For every simulated sample, we have obtained estimates $\widetilde{\text{CTE}}_n(t)$. Then we have calculated the arithmetic averages over the values from the 200 repetitions, with the absolute error (error) and root mean squared error (rmse) of the new estimator $\widetilde{C}_n(t)$ reported in Table 1 ($\gamma = 2/3$) and Table 2 ($\gamma = 3/4$). In the tables, we have also reported 95%-confidence intervals (3.4) with their lower and upper bounds, coverage probabilities, and lengths.

We note emphatically that the above coverage probabilities and lengths of confidence intervals can be improved by employing more precise but, naturally, considerably more

complex estimators of the tail index. Such estimators are described in the monographs by Beirlant et al. [11], Castillo et al. [12], de Haan and Ferreira [13], and Resnick [14]. Since the publication of these monographs, numerous journal articles have appeared on the topic. Our aim in this paper, however, is to present a simple yet useful result that highlights how much Actuarial Science and developments in Mathematical Statistics, Probability, and Stochastic Processes are interrelated, and thus benefit from each other.

4. Proof of Theorem 3.1

We start the proof of Theorem 3.1 with the decomposition

$$\left(\widetilde{\mathrm{CTE}}_n(t) - \mathrm{CTE}_F(t)\right)(1 - t) = A_{n,1}(t) + A_{n,2}, \tag{4.1}$$

where

$$A_{n,1}(t) = \int_t^{1-k/n} (Q_n(s) - Q(s))ds,$$

$$A_{n,2} = \frac{k/n}{1 - \widehat{\gamma}_n} X_{n-k:n} - \int_{1-k/n}^1 Q(s)ds. \tag{4.2}$$

We shall show below that there are Brownian bridges B_n such that

$$\frac{\sqrt{n}A_{n,1}(t)}{(k/n)^{1/2}Q(1 - k/n)} = -\frac{\int_0^{1-k/n} B_n(s)dQ(s)}{(k/n)^{1/2}Q(1 - k/n)} + o_{\mathbf{P}}(1), \tag{4.3}$$

$$\frac{\sqrt{n}A_{n,2}}{(k/n)^{1/2}Q(1 - k/n)} = \frac{\gamma^2}{(1 - \gamma)^2}\sqrt{\frac{n}{k}}\,B_n\left(1 - \frac{k}{n}\right)$$

$$- \frac{\gamma}{(1 - \gamma)^2}\sqrt{\frac{n}{k}}\int_{1-k/n}^1 \frac{B_n(s)}{1 - s}ds + o_{\mathbf{P}}(1). \tag{4.4}$$

Assuming for the time being that statements (4.3) and (4.4) hold, we next complete the proof of Theorem 3.1. To simplify the presentation, we use the following notation:

$$W_{1,n} = -\frac{\int_0^{1-k/n} B_n(s)dQ(s)}{(k/n)^{1/2}Q(1 - k/n)},$$

$$W_{2,n} = \frac{\gamma^2}{(1 - \gamma)^2}\sqrt{\frac{n}{k}}B_n\left(1 - \frac{k}{n}\right), \tag{4.5}$$

$$W_{3,n} = -\frac{\gamma}{(1 - \gamma)^2}\sqrt{\frac{n}{k}}\int_{1-k/n}^1 \frac{B_n(s)}{1 - s}ds.$$

Hence, we have the asymptotic representation

$$\frac{\sqrt{n}\left(\widetilde{\mathrm{CTE}}_n(t) - \mathrm{CTE}_F(t)\right)(1-t)}{(k/n)^{1/2}Q(1-k/n)} = W_{1,n} + W_{2,n} + W_{3,n} + o_{\mathbf{P}}(1). \tag{4.6}$$

The sum $W_{1,n} + W_{2,n} + W_{3,n}$ is a centered Gaussian random variable. To calculate its asymptotic variance, we establish the following limits:

$$\mathbf{E}\left[W_{1,n}^2\right] \longrightarrow \frac{2\gamma}{2\gamma - 1}, \qquad \mathbf{E}\left[W_{2,n}^2\right] \longrightarrow \frac{\gamma^4}{(1-\gamma)^4}, \qquad \mathbf{E}\left[W_{3,n}^2\right] \longrightarrow \frac{2\gamma^2}{(1-\gamma)^4},$$

$$2\mathbf{E}[W_{1,n}W_{2,n}] \longrightarrow \frac{-2\gamma^2}{(1-\gamma)^2}, \qquad 2\mathbf{E}[W_{1,n}W_{3,n}] \longrightarrow \frac{2\gamma}{(1-\gamma)^2}, \tag{4.7}$$

$$2\mathbf{E}[W_{2,n}W_{3,n}] \longrightarrow \frac{-2\gamma^3}{(1-\gamma)^4}.$$

Summing up the right-hand sides of the above six limits, we obtain σ_γ^2, whose expression in terms of the parameter γ is given in Theorem 3.1. Finally, since $X_{n-k:n}/Q(1-k/n)$ converges in probability to 1 (see, e.g., the proof of Corollary in [39]), the classical Sultsky's lemma completes the proof of Theorem 3.1. Of course, we are still left to verify statements (4.3) and (4.4), which make the contents of the following two subsections.

4.1. Proof of Statement (4.3)

If Q were continuously differentiable, then statement (4.3) would follow easily from the proof of Theorem 2 in [39]. We do not assume differentiability of Q and thus a new proof is required, which is crucially based on the Vervaat process (see [31–33], and references therein)

$$V_n(t) = \int_0^t (Q_n(s) - Q(s))ds + \int_{-\infty}^{Q(t)} (F_n(x) - F(x))dx. \tag{4.8}$$

Hence, for every t such that $0 < t < 1 - k/n$, which is satisfied for all sufficiently large n since t is fixed, we have that

$$A_{n,1}(t) = \int_0^{1-k/n} (Q_n(s) - Q(s))ds - \int_0^t (Q_n(s) - Q(s))ds$$

$$= -\int_{Q(t)}^{Q(1-k/n)} (F_n(x) - F(x))dx + V_n\left(1 - \frac{k}{n}\right) - V_n(t). \tag{4.9}$$

It is well known (see [31–33]) that $V_n(t)$ is nonnegative and does not exceed $-(F_n(Q(t)) - t)(Q_n(t) - Q(t))$. Since the cdf F is continuous by assumption, we therefore have that

$$\sqrt{n}V_n(t) \le |e_n(t)||Q_n(t) - Q(t)|, \tag{4.10}$$

where $e_n(t)$ is the uniform empirical process $\sqrt{n}(F_n(Q(t)) - F(Q(t)))$, which for large n looks like the Brownian bridge $B_n(t)$. Note also that with the just introduced notation e_n, the integral on the right-hand side of (4.9) is equal to $\int_{Q(t)}^{Q(1-k/n)} e_n(F(x))dx$. Hence,

$$
\begin{aligned}
\frac{\sqrt{n}A_{n,1}(t)}{(k/n)^{1/2}Q(1-k/n)} = {} & -\frac{\int_{Q(t)}^{Q(1-k/n)} e_n(F(x))dx}{(k/n)^{1/2}Q(1-k/n)} \\
& + O_{\mathbf{P}}(1)\frac{|e_n(1-k/n)||Q_n(1-k/n) - Q(1-k/n)|}{(k/n)^{1/2}Q(1-k/n)} \\
& + O_{\mathbf{P}}(1)\frac{|e_n(t)||Q_n(t) - Q(t)|}{(k/n)^{1/2}Q(1-k/n)}.
\end{aligned}
\tag{4.11}
$$

We shall next replace the empirical process e_n by an appropriate Brownian bridge B_n in the first integral on the right-hand side of (4.11) with an error term of magnitude $o_{\mathbf{P}}(1)$, and we shall also show that the second and third summands on the right-hand side of (4.11) are of the order $o_{\mathbf{P}}(1)$. The replacement of e_n by B_n can be accomplished using, for example, Corollary 2.1 on page 48 of Csörgő et al. [40], which states that on an appropriately constructed probability space and for any $0 \le \nu < 1/4$, we have that

$$
\sup_{1/n \le s \le 1-1/n} \frac{|e_n(s) - B_n(s)|}{s^{1/2-\nu}(1-s)^{1/2-\nu}} = O_{\mathbf{P}}(n^{-\nu}).
\tag{4.12}
$$

This result is applicable in the current situation since we can always place our original problem into the required probability space, because our main results are "in probability". Furthermore, since $Q(t) \le x \le Q(1-k/n)$, we have that $t \le F(x) \le 1-k/n$. Hence, statement (4.12) implies that

$$
-\frac{\int_{Q(t)}^{Q(1-k/n)} e_n(F(x))dx}{(k/n)^{1/2}Q(1-k/n)} = -\frac{\int_{Q(t)}^{Q(1-k/n)} B_n(F(x))dx}{(k/n)^{1/2}Q(1-k/n)} + O_{\mathbf{P}}(1)\frac{\int_{Q(t)}^{Q(1-k/n)} (1-F(x))^{1/2-\nu}dx}{n^{\nu}(k/n)^{1/2}Q(1-k/n)}.
\tag{4.13}
$$

Changing the variables of integration and using the property $(k/n)^{1/2}Q(1-k/n) \to \infty$ when $n \to \infty$, we obtain that

$$
-\frac{\int_{Q(t)}^{Q(1-k/n)} B_n(F(x))dx}{(k/n)^{1/2}Q(1-k/n)} = -\frac{\int_0^{1-k/n} B_n(s)dQ(s)}{(k/n)^{1/2}Q(1-k/n)} + o_{\mathbf{P}}(1).
\tag{4.14}
$$

The main term on the right-hand side of (4.14) is $W_{1,n}$. We shall next show that the right-most summand of (4.13) converges to 0 when $n \to \infty$.

Changing the variable of integration and then integrating by parts, we obtain the bound

$$\frac{\int_{Q(t)}^{Q(1-k/n)} (1-F(x))^{1/2-\nu} dx}{n^\nu (k/n)^{1/2} Q(1-k/n)} \leq \frac{(1-s)^{1/2-\nu} Q(s)\Big|_t^{1-k/n}}{n^\nu (k/n)^{1/2} Q(1-k/n)} + O(1)\frac{\int_t^{1-k/n} (1-s)^{-1/2-\nu} Q(s) ds}{n^\nu (k/n)^{1/2} Q(1-k/n)}.$$

(4.15)

We want to show that the right-hand side of bound (4.15) converges to 0 when $n \to \infty$. For this, we first note that

$$\frac{(1-s)^{1/2-\nu} Q(s)\Big|_t^{1-k/n}}{n^\nu (k/n)^{1/2} Q(1-k/n)} = \frac{1}{k^\nu} - \frac{(1-t)^{1/2-\nu} Q(t)}{n^\nu (k/n)^{1/2} Q(1-k/n)} \longrightarrow 0.$$

(4.16)

Next, with the notation $\phi(u) = Q(1-u)/u^{1/2+\nu}$, we have that

$$\frac{\int_t^{1-k/n} (1-s)^{-1/2-\nu} Q(s) ds}{n^\nu (k/n)^{1/2} Q(1-k/n)} = \frac{1}{k^\nu} \frac{\int_{k/n}^{1-t} \phi(s) ds}{(k/n)\phi(k/n)} \longrightarrow 0$$

(4.17)

when $n \to \infty$, where the convergence to 0 follows from Result 1 in the Appendix of Necir and Meraghni [39]. Taking statements (4.15)–(4.17) together, we have that the right-most summand of (4.13) converges to 0 when $n \to \infty$.

Consequently, in order to complete the proof of statement (4.3), we are left to show that the second and third summands on the right-hand side of (4.11) are of the order $o_P(1)$. The third summand is of the order $o_P(1)$ because $|e_n(t)\|Q_n(t) - Q(t)| = O_P(1)$ and $(k/n)^{1/2} Q(1-k/n) \to \infty$. Hence, we are only left to show that the second summand on the right-hand side of equation (4.11) is of the order $o_P(1)$, for which we shall show that

$$\frac{|e_n(1-k/n)|}{(k/n)^{1/2}}\left|\frac{Q_n(1-k/n)}{Q(1-k/n)} - 1\right| = o_P(1).$$

(4.18)

To prove statement (4.18), we first note that

$$\frac{|e_n(1-k/n)|}{(k/n)^{1/2}} \leq \frac{|e_n(1-k/n) - B_n(1-k/n)|}{(k/n)^{1/2}} + \frac{|B_n(1-k/n)|}{(k/n)^{1/2}}.$$

(4.19)

The first summand on the right-hand side of bound (4.19) is of the order $O_P(1)$ due to statement (4.12) with $\nu = 0$. The second summand on the right-hand side of bound (4.19) is of the order $O_P(1)$ due to a statement on page 49 of Csörgő et al. [40] (see the displayed bound just below statement (2.39) therein). Hence, to complete the proof of statement (4.18), we need to check that

$$\frac{Q_n(1-k/n)}{Q(1-k/n)} = 1 + o_P(1).$$

(4.20)

Observe that, for each n, the distribution of $Q_n(1-k/n)$ is the same as that of $Q(E_n^{-1}(1-k/n))$, where E_n^{-1} is the uniform empirical quantile function. Furthermore, the processes $\{1 - E_n^{-1}(1-s),\ 0 \le s \le 1\}$ and $\{E_n^{-1}(s),\ 0 \le s \le 1\}$ are equal in distribution. Hence, statement (4.20) is equivalent to

$$\frac{Q\left(1 - E_n^{-1}(k/n)\right)}{Q(1 - k/n)} = 1 + o_{\mathbf{P}}(1). \tag{4.21}$$

From the Glivenko-Cantelli theorem we have that $E_n^{-1}(k/n) - k/n \to 0$ almost surely, which also implies that $E_n^{-1}(k/n) \to 0$ since $k/n \to 0$ by our choice of k. Moreover, we know from Theorem 0 and Remark 1 of Wellner [41] that

$$\sup_{1/n \le s \le 1} s^{-1} \left| E_n^{-1}(s) - s \right| = o_{\mathbf{P}}(1), \tag{4.22}$$

from which we conclude that

$$\frac{n E_n^{-1}(k/n)}{k} = 1 + o_{\mathbf{P}}(1). \tag{4.23}$$

Since the function $s \mapsto Q(1 - s)$ is slowly varying at zero, using Potter's inequality (see the 5th assertion of Proposition B.1.9 on page 367 of de Haan and Ferreira [13],we obtain that

$$\frac{Q\left(1 - E_n^{-1}(k/n)\right)}{Q(1 - k/n)} = (1 + o_{\mathbf{P}}(1)) \left(\frac{n E_n^{-1}(k/n)}{k}\right)^{-\gamma \pm \theta} \tag{4.24}$$

for any $\theta \in (0, \gamma)$. In view of (4.23), the right-hand side of (4.24) is equal to $1 + o_{\mathbf{P}}(1)$, which implies statement (4.21) and thus finishes the proof of statement (4.3).

4.2. Proof of Statement (4.4)

The proof of statement (4.4) is similar to that of Theorem 2 in Necir et al. [42], though some adjustments are needed since we are now concerned with the CTE risk measure. We therefore present main blocks of the proof together with pinpointed references to Necir et al. [42] for specific technical details.

We start the proof with the function $\mathbb{U}(z) = Q(1 - 1/z)$ that was already used in the formulation of Theorem 3.1. Hence, if Y is a random variable with the distribution function $G(z) = 1 - 1/z$, $z \ge 1$, then $\mathbb{U}(Y) = Q(G(Y)) =_d X$ because $G(Y)$ is a uniform on the interval $[0, 1]$ random variable. Hence,

$$A_{n,2} = \frac{k/n}{1 - \widehat{\gamma}_n} \mathbb{U}(Y_{n-k:n}) - \int_0^{k/n} \mathbb{U}\left(\frac{1}{s}\right) ds, \tag{4.25}$$

and so we have

$$\frac{\sqrt{n} A_{n,2}}{(k/n)^{1/2} Q(1 - k/n)} = \sqrt{k} \left(\frac{1}{1 - \widehat{\gamma}_n} \frac{\mathbb{U}(Y_{n-k:n})}{\mathbb{U}(n/k)} - \frac{(n/k) \int_0^{k/n} \mathbb{U}(1/s) ds}{\mathbb{U}(n/k)} \right)$$

$$= \sqrt{k} \left(\frac{1}{1 - \widehat{\gamma}_n} \frac{\mathbb{U}(Y_{n-k:n})}{\mathbb{U}(n/k)} - \frac{1}{1 - \gamma} \right) \qquad (4.26)$$

$$+ \sqrt{k} \left(\frac{1}{1 - \gamma} - \frac{\int_1^\infty s^{-2} \mathbb{U}(ns/k) ds}{\mathbb{U}(n/k)} \right).$$

We next show that the right-most term in (4.26) converges to 0 when $n \to \infty$. For this reason, we first rewrite the term as follows:

$$\sqrt{k} \left(\frac{1}{1 - \gamma} - \frac{\int_1^\infty s^{-2} \mathbb{U}(ns/k) ds}{\mathbb{U}(n/k)} \right) = -\sqrt{k} \int_1^\infty \frac{1}{s^2} \left(\frac{\mathbb{U}(ns/k)}{\mathbb{U}(n/k)} - s^\gamma \right) ds. \qquad (4.27)$$

The right-hand side of (4.27) converges to 0 (see notes on page 149 of Necir et al. [42]) due to the second-order condition (3.1), which can equivalently be rewritten as

$$\lim_{z \to \infty} \frac{1}{A(z)} \left(\frac{\mathbb{U}(zs)}{\mathbb{U}(z)} - s^\gamma \right) = s^\gamma \frac{s^\rho - 1}{\rho} \qquad (4.28)$$

for every $s > 0$, where $A(z) = \gamma^2 a(\mathbb{U}(z))$. Note that $\sqrt{k} A(n/k) \to 0$ when $n \to \infty$. Hence, in order to complete the proof of statement (4.4), we need to check that

$$\sqrt{k} \left(\frac{1}{1 - \widehat{\gamma}_n} \frac{\mathbb{U}(Y_{n-k:n})}{\mathbb{U}(n/k)} - \frac{1}{1 - \gamma} \right) = \frac{\gamma^2}{(1 - \gamma)^2} \sqrt{\frac{n}{k}} B_n \left(1 - \frac{k}{n} \right)$$

$$- \frac{\gamma}{(1 - \gamma)^2} \sqrt{\frac{n}{k}} \int_{1-k/n}^1 \frac{B_n(s)}{1 - s} ds + o_{\mathbf{P}}(1). \qquad (4.29)$$

With Hill's estimator written in the form

$$\widehat{\gamma}_n = \frac{1}{k} \sum_{i=1}^k \log \left(\frac{\mathbb{U}(Y_{n-i+1:n})}{\mathbb{U}(Y_{n-k:n})} \right), \qquad (4.30)$$

we proceed with the proof of statement (4.29) as follows:

$$\sqrt{k} \left(\frac{1 - \gamma}{1 - \widehat{\gamma}_n} \frac{\mathbb{U}(Y_{n-k:n})}{\mathbb{U}(n/k)} - 1 \right)$$

$$= \sqrt{k} \frac{1 - \gamma}{1 - \widehat{\gamma}_n} \left(\frac{\mathbb{U}(Y_{n-k:n})}{\mathbb{U}(n/k)} - \left(\frac{Y_{n-k:n}}{n/k} \right)^\gamma \right) + \sqrt{k} \frac{1 - \gamma}{1 - \widehat{\gamma}_n} \left(\left(\frac{Y_{n-k:n}}{n/k} \right)^\gamma - 1 \right) + \sqrt{k} \frac{\widehat{\gamma}_n - \gamma}{1 - \widehat{\gamma}_n}. \qquad (4.31)$$

Furthermore, we have that

$$
\sqrt{k}\,\frac{\widehat{\gamma}_n - \gamma}{1 - \widehat{\gamma}_n} = \frac{1}{1 - \widehat{\gamma}_n}\frac{1}{\sqrt{k}}\sum_{i=1}^{k}\left(\log\left(\frac{\mathbb{U}(Y_{n-i+1:n})}{\mathbb{U}(Y_{n-k:n})}\right) - \gamma\log\left(\frac{Y_{n-i+1:n}}{Y_{n-k:n}}\right)\right)
$$

$$
+ \frac{\gamma}{1 - \widehat{\gamma}_n}\frac{1}{\sqrt{k}}\sum_{i=1}^{k}\left(\log\left(\frac{Y_{n-i+1:n}}{Y_{n-k:n}}\right) - 1\right). \tag{4.32}
$$

Arguments on page 156 of Necir et al. [42] imply that the first term on the right-hand side of (4.32) is of the order $O_{\mathbf{P}}(\sqrt{k}A(Y_{n-k:n}))$, and a note on page 157 of Necir et al. [42] says that $\sqrt{k}\,A(Y_{n-k:n}) = o_{\mathbf{P}}(1)$. Hence, the first term on the right-hand side of (4.32) is of the order $o_{\mathbf{P}}(1)$. Analogous considerations using bound (2.5) instead of (2.4) on page 156 of Necir et al. [42] imply that the first term on the right-hand side of (4.31) is of the order $o_{\mathbf{P}}(1)$. Hence, in summary, we have that

$$
\sqrt{k}\left(\frac{1 - \gamma}{1 - \widehat{\gamma}_n}\frac{\mathbb{U}(Y_{n-k:n})}{\mathbb{U}(n/k)} - 1\right) = \frac{1 - \gamma}{1 - \widehat{\gamma}_n}\sqrt{k}\left(\left(\frac{Y_{n-k:n}}{n/k}\right)^{\gamma} - 1\right)
$$

$$
+ \frac{\gamma}{1 - \widehat{\gamma}_n}\frac{1}{\sqrt{k}}\sum_{i=1}^{k}\left(\log\left(\frac{Y_{n-i+1:n}}{Y_{n-k:n}}\right) - 1\right) + o_{\mathbf{P}}(1). \tag{4.33}
$$

We now need to connect the right-hand side of (4.33) with Brownian bridges B_n. To this end, we first convert the Y-based order statistics into U-based (i.e., uniform on $[0,1]$) order statistics. For this we recall that the cdf of Y is G, and thus Y is equal in distribution to $G^{-1}(U)$, which is $1/(1 - U)$. Consequently,

$$
\sqrt{k}\left(\frac{1 - \gamma}{1 - \widehat{\gamma}_n}\frac{\mathbb{U}(Y_{n-k:n})}{\mathbb{U}(n/k)} - 1\right) = \frac{1 - \gamma}{1 - \widehat{\gamma}_n}\sqrt{k}\left(\left(\frac{1}{(n/k)(1 - U_{n-k:n})}\right)^{\gamma} - 1\right)
$$

$$
+ \frac{\gamma}{1 - \widehat{\gamma}_n}\frac{1}{\sqrt{k}}\sum_{i=1}^{k}\left(\log\left(\frac{(1 - U_{n-k:n})}{(1 - U_{n-i+1:n})}\right) - 1\right) + o_{\mathbf{P}}(1). \tag{4.34}
$$

Next we choose a sequence of Brownian bridges B_n (see pages 158-159 in [42] and references therein) such that the following two asymptotic representations hold:

$$
\sqrt{k}\left(\left(\frac{1}{(n/k)(1 - U_{n-k:n})}\right)^{\gamma} - 1\right) = -\gamma\,\sqrt{\frac{n}{k}}B_n\left(1 - \frac{k}{n}\right) + o_{\mathbf{P}}(1),
$$

$$
\frac{1}{\sqrt{k}}\sum_{i=1}^{k}\left(\log\left(\frac{(1 - U_{n-k:n})}{(1 - U_{n-i+1:n})}\right) - 1\right) = \sqrt{\frac{n}{k}}B_n\left(1 - \frac{k}{n}\right) \tag{4.35}
$$

$$
- \sqrt{\frac{n}{k}}\int_{1-k/n}^{1}\frac{B_n(s)}{1 - s}ds + o_{\mathbf{P}}(1).
$$

Using these two statements on the right-hand side of (4.34) and also keeping in mind that $\hat{\gamma}_n$ is a consistent estimator of γ (see [22]), we have that

$$
\sqrt{k}\left(\frac{1-\gamma}{1-\hat{\gamma}_n}\frac{\mathbb{U}(Y_{n-k:n})}{\mathbb{U}(n/k)}-1\right) = \frac{\gamma^2}{1-\gamma}\sqrt{\frac{n}{k}}\,B_n\left(1-\frac{k}{n}\right)
$$
$$
-\frac{\gamma}{1-\gamma}\sqrt{\frac{n}{k}}\int_{1-k/n}^{1}\frac{B_n(s)}{1-s}ds + o_{\mathbf{P}}(1).
$$

(4.36)

Dividing both sides of equation (4.36) by $1-\gamma$, we arrive at (4.29). This completes the proof of statement (4.4) and of Theorem 3.1 as well.

Acknowledgments

References

[1] M. Denuit, J. Dhaene, M. Goovaerts, and R. Kaas, *Actuarial Theory for Dependent Risks: Measures, Orders and Models*, Wiley, Chichester, UK, 2005.

[2] Z. M. Landsman and E. A. Valdez, "Tail conditional expectations for elliptical distributions," *North American Actuarial Journal*, vol. 7, no. 4, pp. 55–71, 2003.

[3] M. R. Hardy and J. L. Wirch, "The iterated CTE: a dynamic risk measure," *North American Actuarial Journal*, vol. 8, no. 4, pp. 62–75, 2004.

[4] J. Cai and H. Li, "Conditional tail expectations for multivariate phase-type distributions," *Journal of Applied Probability*, vol. 42, no. 3, pp. 810–825, 2005.

[5] B. J. Manistre and G. H. Hancock, "Variance of the CTE estimator," *North American Actuarial Journal*, vol. 9, no. 2, pp. 129–156, 2005.

[6] E. Furman and Z. Landsman, "Tail variance premium with applications for elliptical portfolio of risks," *Astin Bulletin*, vol. 36, no. 2, pp. 433–462, 2006.

[7] E. Furman and R. Zitikis, "Weighted risk capital allocations," *Insurance: Mathematics & Economics*, vol. 43, no. 2, pp. 263–269, 2008.

[8] E. Furman and R. Zitikis, "Weighted pricing functionals with applications to insurance: an overview," *North American Actuarial Journal*, vol. 13, pp. 483–496, 2009.

[9] V. Brazauskas, B. L. Jones, M. L. Puri, and R. Zitikis, "Estimating conditional tail expectation with actuarial applications in view," *Journal of Statistical Planning and Inference*, vol. 138, no. 11, pp. 3590–3604, 2008.

[10] J. Beirlant, G. Matthys, and G. Dierckx, "Heavy-tailed distributions and rating," *Astin Bulletin*, vol. 31, no. 1, pp. 37–58, 2001.

[11] J. Beirlant, Y. Goegebeur, J. Teugels, and J. Segers, *Statistics of Extremes*, Wiley Series in Probability and Statistics, John Wiley & Sons, Chichester, UK, 2004.

[12] E. Castillo, A. S. Hadi, N. Balakrishnan, and J. M. Sarabia, *Extreme Value and Related Models with Applications in Engineering and Science*, Wiley Series in Probability and Statistics, Wiley-Interscience, Hoboken, NJ, USA, 2005.

[13] L. de Haan and A. Ferreira, *Extreme Value Theory: An Introduction*, Springer Series in Operations Research and Financial Engineering, Springer, New York, NY, USA, 2006.

[14] S. I. Resnick, *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling*, Springer Series in Operations Research and Financial Engineering, Springer, New York, NY, USA, 2007.

[15] B. M. Hill, "A simple general approach to inference about the tail of a distribution," *The Annals of Statistics*, vol. 3, no. 5, pp. 1163–1174, 1975.

[16] S. Cheng and L. Peng, "Confidence intervals for the tail index," *Bernoulli*, vol. 7, no. 5, pp. 751–760, 2001.

[17] J. T. Kim and M. R. Hardy, "Quantifying and correcting the bias in estimated risk measures," *Astin Bulletin*, vol. 37, no. 2, pp. 365–386, 2007.

[18] I. Weissman, "Estimation of parameters and large quantiles based on the k largest observations," *Journal of the American Statistical Association*, vol. 73, no. 364, pp. 812–815, 1978.

[19] A. L. M. Dekkers and L. de Haan, "On the estimation of the extreme-value index and large quantile estimation," *The Annals of Statistics*, vol. 17, no. 4, pp. 1795–1832, 1989.

[20] G. Matthys and J. Beirlant, "Estimating the extreme value index and high quantiles with exponential regression models," *Statistica Sinica*, vol. 13, no. 3, pp. 853–880, 2003.

[21] M. I. Gomes, F. Figueiredo, and S. Mendonça, "Asymptotically best linear unbiased tail estimators under a second-order regular variation condition," *Journal of Statistical Planning and Inference*, vol. 134, no. 2, pp. 409–433, 2005.

[22] D. M. Mason, "Laws of large numbers for sums of extreme values," *The Annals of Probability*, vol. 10, no. 3, pp. 754–764, 1982.

[23] S. Csörgő and D. M. Mason, "Central limit theorems for sums of extreme values," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 98, no. 3, pp. 547–558, 1985.

[24] J. Beirlant and J. L. Teugels, "Asymptotic normality of Hill's estimator," in *Extreme Value Theory (Oberwolfach, 1987)*, J. Hüsler and R.-D. Reiss, Eds., vol. 51 of *Lecture Notes in Statistics*, pp. 148–155, Springer, New York, NY, USA, 1989.

[25] A. L. M. Dekkers, J. H. J. Einmahl, and L. de Haan, "A moment estimator for the index of an extreme-value distribution," *The Annals of Statistics*, vol. 17, no. 4, pp. 1833–1855, 1989.

[26] L. de Haan and U. Stadtmüller, "Generalized regular variation of second order," *Australian Mathematical Society Journal Series A*, vol. 61, no. 3, pp. 381–395, 1996.

[27] J. Geluk, L. de Haan, S. Resnick, and C. Stărică, "Second-order regular variation, convolution and the central limit theorem," *Stochastic Processes and Their Applications*, vol. 69, no. 2, pp. 139–159, 1997.

[28] L. Peng and Y. Qi, "Estimating the first- and second-order parameters of a heavy-tailed distribution," *Australian & New Zealand Journal of Statistics*, vol. 46, no. 2, pp. 305–312, 2004.

[29] M. I. Gomes and D. Pestana, "A simple second-order reduced bias' tail index estimator," *Journal of Statistical Computation and Simulation*, vol. 77, no. 5-6, pp. 487–504, 2007.

[30] L. Peng, "Estimating the mean of a heavy tailed distribution," *Statistics & Probability Letters*, vol. 52, no. 3, pp. 255–264, 2001.

[31] R. Zitikis, "The Vervaat process," in *Asymptotic Methods in Probability and Statistics*, B. Szyszkowicz, Ed., pp. 667–694, North-Holland, Amsterdam, The Netherlands, 1998.

[32] Y. Davydov and R. Zitikis, "Generalized Lorenz curves and convexifications of stochastic processes," *Journal of Applied Probability*, vol. 40, no. 4, pp. 906–925, 2003.

[33] Y. Davydov and R. Zitikis, "Convex rearrangements of random elements," in *Asymptotic Methods in Stochastics*, B. Szyszkowicz, Ed., vol. 44 of *Fields Institute Communications*, pp. 141–171, American Mathematical Society, Providence, RI, USA, 2004.

[34] A. L. M. Dekkers and L. de Haan, "Optimal choice of sample fraction in extreme-value estimation," *Journal of Multivariate Analysis*, vol. 47, no. 2, pp. 173–195, 1993.

[35] H. Drees and E. Kaufmann, "Selecting the optimal sample fraction in univariate extreme value estimation," *Stochastic Processes and Their Applications*, vol. 75, no. 2, pp. 149–172, 1998.

[36] J. Danielsson, L. de Haan, L. Peng, and C. G. de Vries, "Using a bootstrap method to choose the sample fraction in tail index estimation," *Journal of Multivariate Analysis*, vol. 76, no. 2, pp. 226–248, 2001.

[37] C. Neves and M. I. Fraga Alves, "Reiss and Thomas' automatic selection of the number of extremes," *Computational Statistics & Data Analysis*, vol. 47, no. 4, pp. 689–704, 2004.

[38] M. I. Gomes, D. Pestana, and F. Caeiro, "A note on the asymptotic variance at optimal levels of a bias-corrected Hill estimator," *Statistics & Probability Letters*, vol. 79, no. 3, pp. 295–303, 2009.

[39] A. Necir and D. Meraghni, "Empirical estimation of the proportional hazard premium for heavy-tailed claim amounts," *Insurance: Mathematics & Economics*, vol. 45, no. 1, pp. 49–58, 2009.

[40] M. Csörgő, S. Csörgő, L. Horváth, and D. M. Mason, "Weighted empirical and quantile processes," *The Annals of Probability*, vol. 14, no. 1, pp. 31–85, 1986.

[41] J. A. Wellner, "Limit theorems for the ratio of the empirical distribution function to the true distribution function," *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, vol. 45, no. 1, pp. 73–88, 1978.

[42] A. Necir, D. Meraghni, and F. Meddi, "Statistical estimate of the proportional hazard premium of loss," *Scandinavian Actuarial Journal*, no. 3, pp. 147–161, 2007.