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# A Volterra Inequality with the Power Type Nonlinear Kernel

W. MYDLARCZYK*

*Institute of Mathematics, Technical University of Wrocław, Wybrzeze Wyspianskiego 27, 50-367 Wrocław, Poland*

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In the paper, we characterize nonnegative, locally integrable functions $k$, for which the nonlinear convolution integral inequality $u(s) \leq k * g(u(s))$, with the power type nonlinearity $g$ has nontrivial solutions.

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## 1. INTRODUCTION

We study the integral inequality

$$u(x) \leq \int_0^x k(x-s)[u(s)]^\beta ds \quad (0 < x,\ 0 < \beta), \tag{1.1}$$

where $k > 0$ is a given locally integrable function. It is clear that $u(x) \equiv 0$ is a trivial solution of (1.1). Therefore, we are interested further in nontrivial continuous, nonnegative solutions $u$ of (1.1).

*e-mail: mydlar@im.pwr.wroc.pl

This inequality arises in the study of uniqueness problem for a more general integral equation

$$y(t) = \int_0^t h(t, s, y(s))ds + f(t), \quad t \geq 0$$

in some Banach space. For example, if one considers two solutions $y_1$ and $y_2$, takes $x(t) = \|y_1(t) - y_2(t)\|$ and assumes that

$$\|h(t, s, y_1(s)) - h(t, s, y_2(s))\| \leq k(t - s)\|y_1(s) - y_2(s)\|^\beta,$$

then one obtains inequality (1.1) for $x(t)$.

First, we note that if $1 \leq \beta$, then (1.1) has no nontrivial solutions. This due the fact that the integral operator

$$Tu(x) = \int_0^x k(x - s)[u(s)]^\beta ds \quad (\beta \geq 1)$$

is Lipschitz continuous in the class of nonnegative, continuous functions. Therefore, throughout the paper, we assume that $0 < \beta < 1$. It is also important to note that the existence of a nontrivial solution to (1.1) is equivalent to the existence of such a nontrivial solution to the corresponding equation

$$u(x) = \int_0^x k(x - s)[u(s)]^\beta ds \quad (0 < x,\ 0 < \beta < 1). \tag{1.2}$$

To see this, we consider any nontrivial solution $v(x)$ of (1.1). To deal with nondecreasing functions, we define

$$\bar{v}(x) = \sup v(s), \quad 0 \leq s \leq x.$$

Since, the integral operator $T$ has the following monotonicity properties:

$$Tw_1(x) \leq Tw_2(x) \quad \text{for any } 0 \leq w_1(x) \leq w_2(x)$$

and

$Tw(x)$ is nondecreasing for any nondecreasing function $0 \leq w(x)$, we easily see that $\bar{v}(x)$ is also a nontrivial solution to (1.1). Furthermore, it follows from the inequality

$$\bar{v}(x) \leq \int_0^x k(x-s)[\bar{v}(s)]^\beta ds \leq K(x)[\bar{v}(x)]^\beta,$$

where $K(x) = \int_0^x k(s)ds$ that

$$\bar{v}(x) \leq K(x)^{1/(1-\beta)}.$$

Now, we construct a function sequence

$$v_0(x) = K(x)^{1/(1-\beta)}, \quad v_{n+1}(x) = Tv_n(x), \quad n = 1, 2, \ldots .$$

We verify directly that $Tv_0(x) \leq v_0(x)$ and as a consequence of this, we obtain

$$v_{n+1}(x) = Tv_n(x) \leq v_n(x) \quad \text{for } n = 1, 2, \ldots .$$

Thus $\{v_n(x)\}$ is a nonincreasing sequence of continuous functions. Since

$$\bar{v}(x) \leq v_0(x) \quad \text{and} \quad \bar{v}(x) \leq T\bar{v}(x) \leq Tv_0(x) \leq v_0(x),$$

we obtain $\bar{v}(x) \leq v_n(x)$ for $n = 1, 2, \ldots$ . Now, we consider the limit function

$$u(x) = \lim_{n\to\infty} v_n(x) = \lim_{n\to\infty} Tv_n(x) \geq \bar{v}(x).$$

Such a $u(x)$ is a nontrivial solution of (1.2).

Equation (1.2) is a very special case of the equation

$$u(x) = \int_0^x k(x-s)g(u(s))ds, \tag{1.3}$$

where $g$ is a continuous and nondecreasing function.

There is a wide literature, where the problem of the existence of nontrivial solutions for (1.3) was studied and some necessary and sufficient conditions were given, see [1, 4, 7]. They were formulated in the form of so called the generalized Osgood conditions. One of the most strength results was obtained for the logarithmicly concave kernels $k$. For example, it is known that for such kernels the following condition

$$\int_0^\delta (K^{-1})'\left(\frac{s}{g(s)}\right)\frac{ds}{g(s)} < \infty,$$

where $K^{-1}$ is inverse to $K$ and $\delta > 0$ is sufficiently small, is necessary for the existence of nontrivial solutions to (1.3). Moreover, in the case $k(x) = x^{\alpha-1}$ or $\exp(-x^{-\alpha})$, $\alpha > 0$ this condition is also sufficient, see [2, 3, 5]. Unfortunately, if $g(u) = u^\beta$, $0 < \beta < 1$ this condition is satisfied for any $k$. On the other hand, it is known that if $k(x) = \exp(-\exp(x^{-\alpha}))$, then Eq. (1.3) has a nontrivial solution if and only if $0 < \alpha < 1$, see [6, 8]. Our aim is to characterize those kernels $k$, for which the inequality (1.1) or equivalently Eq. (1.2) has nontrivial solutions. Our main result is established in the following theorem.

THEOREM *The inequality* (1.1) *has a nontrivial solution if and only if* $0 < \beta < 1$ *and*

$$\int_0^\delta K^{-1}(s)\frac{ds}{s(-ln\ s)} < \infty,$$

*where* $\delta > 0$ *is a sufficiently small number.*

*Remark 1* We directly verify that for the kernels $k(x) = \exp(-\exp(x^{-\alpha}))$ mentioned above the following inequalities $k(0.5x) \leq K(x) \leq k(2x)$ hold at the vicinity of zero. Now, we easily see that the condition in Theorem is satisfied in this case, if and only if $0 < \alpha < 1$.

*Remark 2* A substitution $s = \tau^\alpha$ $(0 < \alpha < 1)$ into the integral above changes the condition in Theorem to the following

$$\int_0^\delta K^{-1}(\tau^\alpha)\frac{d\tau}{\tau(-\ln\tau)} < \infty.$$

## 2. MAIN STEPS OF THE PROOF OF THEOREM

The necessity part of the theorem. Consider the nontrivial solution $u$ of (1.2) constructed above. We note that Eq. (1.2) has also other nontrivial solutions. For example, the functions $u_c(x) = 0$ for $0 \leq x < c$

and $u_c(x) = u(x-c)$ for $x \geq c$ $(c > 0)$ are such solutions. Manipulating with $c$, if necessary we can choose $u$ such that $u(0) = 0$ and $u(x) > 0$ for $x > 0$. It follows from the construction described above that $u$ is nondecreasing. Furthemore, the integration by parts gives

$$u(x) = \int_0^x K(x-\tau)d[u(\tau)^\beta], \tag{2.1}$$

from which we infer that $u$ is absolutely continuous and increasing. Finally, the substitution $s = u(\tau)$ into integral (2.1) gives

$$x = \int_0^x K(u^{-1}(x) - u^{-1}(s))d(s^\beta),$$

where $u^{-1}$ is inverse to $u$. Let $\phi(x) = x^{1/\beta} < x < 1$. Splitting the integral above into two parts we obtain

$$x \leq K(u^{-1}(x))\phi(x)^\beta + K(u^{-1}(x) - u^{-1}(\phi(x)))x^\beta. \tag{2.2}$$

Since $K(u^{-1}(x)) \to 0$ as $x \to 0$, it follows from (2.2) that

$$\frac{1}{2}x^{1-\beta} \leq K(u^{-1}(x) - u^{-1}(\phi(x))),$$

or

$$K^{-1}\left(\frac{1}{2}x^{1-\beta}\right) \leq u^{-1}(x) - u^{-1}(\phi(x)) \tag{2.3}$$

for $0 < x < \delta$, where $\delta > 0$ is sufficiently small.

Now, we note that for any $0 < x < \delta$ the sequence

$$x_0 = x, \quad x_{n+1} = \phi(x_n), \quad n = 1, 2, \ldots$$

is decreasing and convergent to zero.

Since

$$\int_{x_{n+1}}^{x_n} K^{-1}\left(\frac{1}{2}s^{1-\beta}\right)\frac{ds}{s(-\ln s)} \leq (-\ln\beta)K^{-1}\left(\frac{1}{2}x_n^{1-\beta}\right),$$

it follows from (2.3) that

$$\int_0^x K^{-1}\left(\frac{1}{2}s^{1-\beta}\right)\frac{ds}{s(-\ln s)} < \infty$$

for $0 < x < \delta$, which gives easily our assertion.

The sufficient part of the theorem. We are going to construct one of the solutions to (1.1). Let $\psi(x) = x^{2/(1+\beta)} < x < 1$. We expect that the function $F$ given by its inverse

$$F^{-1}(x) = \gamma \int_0^x K^{-1}(s^{(1-\beta)/2})\frac{ds}{s(-\ln s)}, \quad \gamma = 1/\ln(2/(1+\beta))$$

is such a solution.

First, we note that

$$\begin{aligned}
&\int_0^x K\big(F^{-1}(x) - F^{-1}(s)\big)d(s^\beta) \\
&\geq \int_0^{\psi(x)} K\big(F^{-1}(x) - F^{-1}(s)\big)d(s^\beta) \\
&\geq K\big(F^{-1}(x) - F^{-1}(\psi(x))\big)\psi(x)^\beta.
\end{aligned}$$

We observe also that

$$\begin{aligned}
F^{-1}(x) - F^{-1}(\psi(x)) &= \gamma \int_{\psi(x)}^x K^{-1}\big(s^{(1-\beta)/2}\big)\frac{ds}{s(-\ln s)} \\
&\geq \gamma K^{-1}\big(\psi(x)^{(1-\beta)/2}\big)\int_{\psi(x)}^x \frac{ds}{s(-\ln s)} \\
&= K^{-1}\big(\psi(x)^{(1-\beta)/2}\big).
\end{aligned}$$

It follows from two inequalities above that

$$\int_0^x K\big(F^{-1}(x) - F^{-1}(s)\big)d(s^\beta) \geq \psi(x)^{(1+\beta)/2} = x,$$

for $0 < x < 1$. Now the substitution $\tau = F(s)$ into the integral above shows that

$$\int_0^x K(x - s)d(F(\tau)^\beta) \geq F(x).$$

Finally, the integration by parts shows that $F(x)$ satisfies (1.1), which ends the proof.

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