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Existence Criteria for Integral Equations in Banach Spaces

DONAL O'REGAN [a,*] and RADU PRECUP [b]

[a] *Department of Mathematics, National University of Ireland, Galway, Ireland;*
[b] *Department of Applied Mathematics, "Babeş-Bolyai" University, Cluj, Romania*

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A Carathéodory existence theory for nonlinear Volterra and Urysohn integral equations in Banach spaces is presented using a Mönch type approach.

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1 INTRODUCTION

This paper studies Volterra and Urysohn integral equations in a ball of a Banach space E when the nonlinear kernel satisfies Carathéodory type conditions. Such integral equations were studied recently by the first author [8,9] assuming that the kernel $f(t,s,x)$ satisfies a global set-Lipschitz condition of the form

$$\alpha(f([0,T]\times[0,T]\times M))\leq c\alpha(M) \qquad (1.1)$$

for each bounded set $M\subset E$, with a suitable small positive constant c, α being the Kuratowski measure of noncompactness. In the present paper

* Corresponding author.

the global condition (1.1) is replaced by a local one, namely

$$\alpha(f(t,s,M)) \leq \omega(t,s,\alpha(M)) \tag{1.2}$$

for all $t \in [0, T]$, a.e. $s \in [0, T]$ and any bounded set $M \subset E$.

Our existence principles do not require uniform continuity of f and are based upon the continuation theorem of Mönch [6] and a result by Heinz [5] concerning the interchanging of α and integral for countable sets of Bochner integrable functions. More applicable existence results are derived from the general principles by means of differential and integral inequalities.

The results in this paper improve and complement those in [4,8,9,12]. In particular, our criteria yield old and new existence results for the abstract Cauchy problem and boundary value problems for differential equations in infinite dimensions; see [1,2,6,7,10,11,13].

Throughout this paper E will be a real Banach space with norm $|\cdot|$. For every $x \in E$ and $R > 0$, let $B_R(x)$ and $\bar{B}_R(x)$ be the open and closed balls

$$B_R(x) = \{y \in E;\ |x-y| < R\}, \bar{B}_R(x) = \{y \in E;\ |x-y| \leq R\}.$$

We denote by $C([a,b];E)$ the space of continuous functions $u:[a,b] \to E$ and by $|\cdot|_\infty$ its max-norm $|u|_\infty = \max_{t\in[a,b]}|u(t)|$. For any subset $M \subset E$, we denote by $C([a,b];M)$ the set of all functions in $C([a,b];E)$ which take values in M.

A function $u:[a,b] \to E$ is said to be *finitely-valued* if it is constant $\neq 0$ on each of a finite number of disjoint measurable sets I_j and $u(t) = 0$ on $[a,b]\backslash\bigcup_j I_j$. Let the value of u on I_j be x_j and let $\chi(I_j)$ be the characteristic function of I_j, $\chi(I_j)(t) = 1$ if $t \in I_j$, $\chi(I_j)(t) = 0$ if $t \in [a,b]\backslash I_j$. Then u can be represented as a finite sum

$$u = \sum_j x_j \chi(I_j)$$

and the element

$$\sum_j x_j \operatorname{mes}(I_j) \in E$$

is defined as the *Bochner integral* of u over $[a,b]$ and is denoted by $\int_a^b u(s)\,\mathrm{d}s$. More generally, a function $u:[a,b] \to E$ is said to be *Bochner* *integrable* on $[a, b]$ if there exists a sequence of finitely-valued functions u_n with

$$u_n(t) \to u(t) \quad \text{as } n \to \infty, \text{ a.e. } t \in [a, b] \tag{1.3}$$

(i.e. u is *strongly measurable*) and

$$\int_a^b |u_n(s) - u(s)| \, \mathrm{d}s \to 0 \quad \text{as } n \to \infty. \tag{1.4}$$

In this case the *Bochner integral* of u is defined by

$$\int_a^b u(s) \, \mathrm{d}s = \lim_{n \to \infty} \int_a^b u_n(s) \, \mathrm{d}s.$$

Recall that a strongly measurable function u is Bochner integrable if and only if $|u|$ is Lebesgue integrable (see [14, Theorem 5.5.1]).

For any real $p \in [1, \infty]$, we consider the space $L^p([a, b]; E)$ of all strongly measurable functions $u : [a, b] \to E$ such that $|u|^p$ is Lebesgue integrable on $[a, b]$. $L^p([a, b]; E)$ is a Banach space under the norm

$$|u|_p = \left(\int_a^b |u(s)|^p \, \mathrm{d}s \right)^{1/p}$$

for $p < \infty$ and

$$|u|_\infty = \operatorname*{ess\,sup}_{t \in [a,b]} |u(t)| = \inf\{c \geq 0; \ |u(t)| \leq c \text{ a.e. } t \in [a, b]\}.$$

When this will be important, we shall denote $|u|_p$ also by $|u|_{L^p([a,b];E)}$. In particular, $L^1([a, b]; E)$ is the space of Bochner integrable functions on $[a, b]$. When $E = \mathbf{R}$, the space $L^p([a, b]; \mathbf{R})$ is simply denoted by $L^p[a, b]$.

Recall that a function $\phi : [a, b] \times D \to E$, $D \subset E$, is said to be L^p-*Carathéodory* $(1 \leq p \leq \infty)$ if $\phi(\cdot, x)$ is strongly measurable for each $x \in D$, $\phi(t, \cdot)$ is continuous for a.e. $t \in [a, b]$ and for each $r > 0$ there exists $h_r \in L^p[a, b]$ with $|\phi(t, x)| \leq h_r(t)$ for all $x \in D$ satisfying $|x| \leq r$ and a.e. $t \in [a, b]$.

Now we recall the definition of the *Kuratowski measure of noncompactness* and the *Hausdorff ball measure of noncompactness.*

Let $M \subset E$ be bounded. Then

$$\alpha(M) = \inf\left\{\varepsilon > 0;\ M \subset \bigcup_{j=1}^{m} M_j \text{ and } \operatorname{diam}(M_j) \le \varepsilon\right\}$$

and

$$\beta(M) = \inf\left\{\varepsilon > 0;\ M \subset \bigcup_{j=1}^{m} B_\varepsilon(x_j) \text{ where } x_j \in E\right\}.$$

If F is a linear subspace of E and $M \subset F$ is bounded, then we define

$$\beta_F(M) = \inf\left\{\varepsilon > 0;\ M \subset \bigcup_{j=1}^{m} B_\varepsilon(x_j) \text{ where } x_j \in F\right\}.$$

We have

$$\beta(M) \le \alpha(M) \le 2\beta(M) \quad \text{for } M \subset E \text{ bounded} \tag{1.5}$$

and

$$\beta(M) \le \beta_F(M) \le \alpha(M) \quad \text{for } M \subset F \text{ bounded.} \tag{1.6}$$

For a separable Banach space E, the ball measure of noncompactness β has the following representation on countable sets.

PROPOSITION 1.1 [6] *Let E be a separable Banach space and (E_n) an increasing sequence of finite dimensional subspaces with $E = \overline{\cup_{n \in \mathbf{N}} E_n}$. Then for every bounded countable set $M = \{x_m;\ m \in \mathbf{N}\} \subset E$, we have*

$$\beta(M) = \lim_{n\to\infty} \overline{\lim_{m\to\infty}}\, d(x_m, E_n)$$

(here $d(x, E_n) = \inf_{y \in E_n} |x - y|$).

Let γ be α or β. The next proposition gives the representation of γ on bounded equicontinuous sets of $C([a,b];E)$ and its property of interchanging with integral.

PROPOSITION 1.2 [2] *Let E be a Banach space and $M \subset C([a,b];E)$ bounded and equicontinuous. Then the function $\mu : [a,b] \to \mathbf{R}$ given by*

$\mu(t) = \gamma(M(t))$ is continuous on $[a, b]$,

$$\gamma(M) = \max_{t\in[a,b]} \gamma(M(t))$$

and

$$\gamma\left(\int_a^b M(s)\,\mathrm{d}s\right) \le \int_a^b \gamma(M(s))\,\mathrm{d}s \tag{1.7}$$

(*here $\int_a^b M(s)\,\mathrm{d}s$ stands for the set $\{\int_a^b u(s)\,\mathrm{d}s;\ u \in M\} \subset E$*).

A result of type (1.7) holds without assuming the equicontinuity of M.

PROPOSITION 1.3 (Mönch–von Harten [7]) *Let E be a separable Banach space and $M \subset C([a, b]; E)$ countable with $|u(t)| \le h(t)$ on $[a, b]$ for every $u \in M$, where $h \in L^1[a, b]$. Then the function $\psi : [a, b] \to \mathbf{R}$ given by $\psi(t) = \beta(M(t))$ belongs to $L^1[a, b]$ and*

$$\beta\left(\int_a^b M(s)\,\mathrm{d}s\right) \le \int_a^b \beta(M(s))\,\mathrm{d}s. \tag{1.8}$$

Now we state the extension of Proposition 1.3 for countable sets of Bochner integrable functions.

PROPOSITION 1.4 (Heinz [5]) (a) *If E is a separable Banach space and $M \subset L^1([a, b]; E)$ countable with $|u(t)| \le h(t)$ for a.e. $t \in [a, b]$ and every $u \in M$, where $h \in L^1[a, b]$, then the function $\psi(t) = \beta(M(t))$ belongs to $L^1[a, b]$ and satisfies* (1.8).

(b) *If E is a Banach space (not necessarily separable) and $M \subset L^1([a, b]; E)$ countable with $|u(t)| \le h(t)$ for a.e. $t \in [a, b]$ and every $u \in M$, where $h \in L^1[a, b]$, then the function $\varphi(t) = \alpha(M(t))$ belongs to $L^1[a, b]$ and satisfies*

$$\alpha\left(\int_a^b M(s)\,\mathrm{d}s\right) \le 2\int_a^b \alpha(M(s))\,\mathrm{d}s. \tag{1.9}$$

Remark A proof of Proposition 1.4 can be found in Heinz [5] in a more general setting. However in our setting an easier proof can be presented for (a) and we include it here for the convenience of the reader.

Proof Let $M = \{u_m; m \in \mathbf{N}\}$.

(a) The fact that $\psi \in L^1[a, b]$ is a direct consequence of Proposition 1.1. Now, let (E_n) be any increasing sequence of finite dimensional subspaces of E with $E = \overline{\cup_{n\in\mathbf{N}} E_n}$. Let us fix $n, m \in \mathbf{N}$ and take any $\varepsilon > 0$. Since u_m is Bochner integrable, there is a finitely-valued function $\tilde{u}_m$, say

$$\tilde{u}_m = \sum_{j=1}^{k_m} x_{mj}\chi(I_{mj}),$$

where $\chi(I_{mj})$ is the characteristic function of a measurable set $I_{mj} \subset [a, b]$, such that

$$\int_a^b |\tilde{u}_m(s) - u_m(s)|\, ds \leq \varepsilon. \tag{1.10}$$

The sublinearity of $d(\cdot, E_n)$ gives

$$d\left(\sum_{j=1}^{k_m} x_{mj} \operatorname{mes}(I_{mj}), E_n\right) \leq \sum_{j-1}^{k_m} d(x_{mj}, E_n) \operatorname{mes}(I_{mj}). \tag{1.11}$$

It is clear that

$$\sum_{j=1}^{k_m} x_{mj} \operatorname{mes}(I_{mj}) = \int_a^b \tilde{u}_m(s)\, ds$$

and

$$\sum_{j=1}^{k_m} d(x_{mj}, E_n) \operatorname{mes}(I_{mj}) = \int_a^b d(\tilde{u}_m(s), E_n)\, ds.$$

Thus (1.11) can be rewritten as

$$d\left(\int_a^b \tilde{u}_m(s)\, ds, E_n\right) \leq \int_a^b d(\tilde{u}_m(s), E_n)\, ds. \tag{1.12}$$

On the other hand, using (1.10), we have

$$\begin{aligned} d\left(\int_a^b \tilde{u}_m(s)\,\mathrm{d}s, E_n\right) &\geq d\left(\int_a^b u_m(s)\,\mathrm{d}s, E_n\right) - d\left(\int_a^b u_m(s)\,\mathrm{d}s, \int_a^b \tilde{u}_m(s)\,\mathrm{d}s\right) \\ &\geq d\left(\int_a^b u_m(s)\,\mathrm{d}s, E_n\right) - \varepsilon \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} \int_a^b d(\tilde{u}_m(s), E_n)\,\mathrm{d}s &\leq \int_a^b d(u_m(s), E_n)\,\mathrm{d}s + \int_a^b d(u_m(s), \tilde{u}_m(s))\,\mathrm{d}s \\ &\leq \int_a^b d(u_m(s), E_n)\,\mathrm{d}s + \varepsilon. \end{aligned} \tag{1.14}$$

Now (1.12)–(1.14) imply

$$d\left(\int_a^b u_m(s)\,\mathrm{d}s, E_n\right) \leq \int_a^b d(u_m(s), E_n)\,\mathrm{d}s + 2\varepsilon.$$

Letting $\varepsilon \searrow 0$, we obtain

$$d\left(\int_a^b u_m(s)\,\mathrm{d}s, E_n\right) \leq \int_a^b d(u_m(s), E_n)\,\mathrm{d}s.$$

The rest of the proof is identical with that in the proof of Proposition 3 in [7]: By means of Fatou's lemma, we have

$$\overline{\lim_{m\to\infty}}\, d\left(\int_a^b u_m(s)\,\mathrm{d}s, E_n\right) \leq \int_a^b \overline{\lim_{m\to\infty}}\, d(u_m(s), E_n)\,\mathrm{d}s.$$

Finally, since

$$\overline{\lim_{m\to\infty}}\, d(u_m(s), E_n) \leq h(t),$$

using the Lebesgue dominated convergence theorem, we find

$$\lim_{n\to\infty} \overline{\lim_{m\to\infty}}\, d\left(\int_a^b u_m(s)\,\mathrm{d}s, E_n\right) \leq \int_a^b \lim_{n\to\infty} \overline{\lim_{m\to\infty}}\, d(u_m(s), E_n)\,\mathrm{d}s$$

which is exactly (1.8).

(b) (see the proof of part (b) of Corollary 3.1 in [5]): Let F be a separable closed linear subspace of E such that $u_m(t) \in F$ for every $m \in \mathbf{N}$ and for every $t \in [a, b]$ outside a fixed set of measure zero (the subspace F can be obtained as follows: for any m, u_m being Bochner integrable can be approximated in the sense of (1.3) and (1.4) by a sequence $(u_{mn})_{n\in\mathbf{N}}$ of finitely-valued functions,

$$u_{mn} = \sum_{j=1}^{k_{mn}} x_{mnj}\,\chi(I_{mnj}).$$

Then we take as F the closure in E of the subspace generated by the countable set of elements x_{mnj}, $1 \le j \le k_{mn}$, $m \in \mathbf{N}, n \in \mathbf{N}$). From (a), (1.5) and (1.6) we have

$$\alpha(M) \le 2\beta_F(M) \le 2\int_a^b \beta_F(M(t))\,\mathrm{d}t \le 2\int_a^b \alpha(M(t))\,\mathrm{d}t.$$

We finish this section with two fixed point results due to Mönch [6] (see also [3, Chapter 5.18]). The first one contains as particular cases the fixed point theorems of Schauder, Darbo and Sadovskii.

THEOREM 1.5 (Mönch [6]) *Let K be a closed convex subset of a Banach space X and $N : K \to K$ continuous with the further property that for some $x_0 \in K$, we have*

$$M \subset K \text{ countable}, \quad \bar{M} = \overline{co}(\{x_0\} \cup N(M)) \implies \bar{M} \text{ compact}. \quad (1.15)$$

Then N has a fixed point.

The second result is the continuation analogue of Theorem 1.5.

THEOREM 1.6 (Mönch [6]) *Let K be a closed convex subset of a Banach space X, U a relatively open subset of K and $N : \bar{U} \to K$ continuous with the further property that for some $x_0 \in U$, we have*

$$M \subset \bar{U} \text{ countable}, \quad M \subset \overline{co}(\{x_0\} \cup N(M)) \implies \bar{M} \text{ compact}. \quad (1.16)$$

In addition, assume

$$x \neq (1-\lambda)x_0 + \lambda N(x) \quad \text{for all } x \in \bar{U}\backslash U \text{ and } \lambda \in (0,1). \quad (1.17)$$

Then N has a fixed point in $\bar{U}$.

2 EXISTENCE CRITERIA FOR VOLTERRA INTEGRAL EQUATIONS

In this section we establish existence criteria for the Volterra integral equation

$$u(t) = \int_0^t f(t, s, u(s))\,\mathrm{d}s, \quad t \in [0, T] \tag{2.1}$$

in a ball of the Banach space $(E, |\cdot|)$, under Carathéodory conditions on f.

Let $R > 0$ and $T > 0$. We denote by B the closed ball $\bar{B}_R(0)$ of E and we consider

$$\Lambda = \{(t, s);\ t \in [0, T],\ s \in [0, T]\}$$

and $D = \Lambda \times B$. Hence

$$D = \{(t, s, x);\ t \in [0, T],\ s \in [0, t],\ x \in E \text{ and } |x| \le R\}.$$

We assume that $f\colon D \to E$ and we look for solutions u in $C([0, T]; E)$ with $|u(t)| \le R$ for all $t \in [0, T]$.

For a fixed $t \in [0, T]$, let $f_t\colon [0, t] \times B \to E$ be the map given by

$$f_t(s, x) = f(t, s, x).$$

THEOREM 2.1 *Suppose*

(A) *for each* $t \in [0, T]$, *the map* f_t is L^1-*Carathéodory uniformly in* t, *in the sense that there exists a bounded function* $\eta\colon \Lambda \to \mathbf{R}_+$ *with*

$$\eta(t, t') \to 0 \quad \text{as } t - t' \searrow 0$$

and

$$\int_{t'}^{t} \sup_{|x| \le R} |f_t(s, x)|\,\mathrm{d}s \le \eta(t, t')$$

for $0 \le t' < t \le T$;

(B) *for each* $t \in [0, T]$,

$$\int_0^{t_0} \sup_{|x| \le R} |f_t(s, x) - f_{t'}(s, x)| \, \mathrm{d}s \to 0 \quad \textit{as } t' \to t,$$

where $t_0 = \min\{t, t'\}$;

(C) *there exists* $\omega : \Lambda \times [0, 2R]$ *such that for each* $t \in [0, T]$, $\omega_t = \omega(t, \cdot, \cdot)$ *is* L^1*-Carathéodory*,

$$\alpha(f(t, s, M)) \le \omega(t, s, \alpha(M)) \tag{2.2}$$

for a.e. $s \in [0, t]$ *and every* $M \subset B$, *and the unique solution* $\varphi \in C([0, T];$ $[0, 2R])$ *of the inequality*

$$\varphi(t) \le 2 \int_0^t \omega(t, s, \varphi(s)) \, \mathrm{d}s, \quad t \in [0, T]$$

is $\varphi \equiv 0$;

(D) $|u|_\infty < R$ *for any solution* $u \in C([0, T]; B)$ *to*

$$u(t) = \lambda \int_0^t f(t, s, u(s)) \, \mathrm{d}s, \quad t \in [0, T] \tag{2.3}$$

for each $\lambda \in (0, 1)$.

Then (2.1) *has a solution in* $C([0, T]; B)$.

Proof We shall apply Theorem 1.6 to $K = X = C([0, T]; E)$ with norm $|\cdot|_\infty$, $U = \{u \in C([0, T]; E);\ |u|_\infty < R\}$, x_0 the null function and $N : \bar{U} \to C([0, T]; E)$ given by

$$N(u)(t) = \int_0^t f(t, s, u(s)) \, \mathrm{d}s.$$

Since f_t is L^1-Carathéodory, standard arguments yield the conclusion that for each $u \in \bar{U}$, the function $f_t(\cdot, u(\cdot))$ is Bochner integrable on $[0, t]$. In addition

$$|N(u)(t)| \le \int_0^t |f(t, s, u(s))| \, \mathrm{d}s \le \int_0^t \sup_{|x| \le R} |f_t(s, x)| \, \mathrm{d}s$$
$$\le \eta(t, 0) \le \sup_\Lambda \eta < \infty. \tag{2.4}$$

Also, for $u \in \bar{U}$ and every $t, t' \in [0, T]$, we have: if $t < t'$, then

$$
\begin{aligned}
|N(u)(t) - N(u)(t')| &\leq \int_0^t |f(t,s,u(s)) - f(t',s,u(s))| \, \mathrm{d}s \\
&\quad + \int_t^{t'} |f(t',s,u(s))| \, \mathrm{d}s \\
&\leq \int_0^t \sup_{|x| \leq R} |f_t(s,x) - f_{t'}(s,x)| \, \mathrm{d}s + \eta(t',t),
\end{aligned}
$$

while if $t' < t$, then

$$
|N(u)(t) - N(u)(t')| \leq \int_0^{t'} \sup_{|x| \leq R} |f_t(s,x) - f_{t'}(s,x)| \, \mathrm{d}s + \eta(t,t').
$$

Hence, in both cases

$$
\begin{aligned}
&|N(u)(t) - N(u)(t')| \\
&\leq \int_0^{t_0} \sup_{|x| \leq R} |f_t(s,x) - f_{t'}(s,x)| \, \mathrm{d}s + \eta(t + t' - t_0, t_0), \qquad (2.5)
\end{aligned}
$$

where $t_0 = \min\{t, t'\}$. Now (2.5) shows that $N(u) \in C([0,T]; E)$ and $N(\bar{U})$ is equicontinuous on $[0, T]$. In addition, (2.4) shows that $N(\bar{U})$ is bounded.

Next we show that N is continuous. To see this, let $u_n \to u$ in $C([0,T]; E)$, where $u_n, u \in \bar{U}$. Since f_t is L^1-Carathéodory, $f_t(s, \cdot)$ is continuous for a.e. $s \in [0,t]$ and there exists $h_t \in L^1[0,t]$ with $|f_t(s,x)| \leq h_t(s)$ for a.e. $s \in [0,t]$ and all $x \in B$. It follows that

$$
f(t,s,u_n(s)) \to f(t,s,u(s)) \quad \text{as } n \to \infty
$$

and

$$
|f(t,s,u_n(s))| \leq h_t(s)
$$

for a.e. $s \in [0,t]$ and all $t \in [0,T]$. These together with the Lebesgue dominated convergence theorem yield

$$
N(u_n)(t) \to N(u)(t) \quad \text{as } n \to \infty
$$

for any $t \in [0, T]$. Now (2.5) guarantees that the convergence is uniform in t. Hence $N(u_n) \to N(u)$ in $C([0,T]; E)$ as desired.

To check (1.16), let $M \subset \bar{U}$ be countable with

$$M \subset \overline{\mathrm{co}}(\{0\} \cup N(M)). \tag{2.6}$$

Since $N(M) \subset N(\bar{U})$ and $N(\bar{U})$ is bounded and equicontinuous, from (2.6) we have that M is bounded and equicontinuous. To deduce that $\bar{M}$ is compact, that is $\alpha(M)=0$ (in $C([0,T];E)$), by Proposition 1.2 we have to prove that $\alpha(M(t))=0$ (in E) for any $t \in [0,T]$. For this, let $\varphi : [0,T] \to \mathbf{R}$ be given by $\varphi(t) = \alpha(M(t))$. Clearly $\varphi \in C([0,T];[0,2R])$. Now, using Proposition 1.4(b), (2.2) and (2.6), we obtain

$$\begin{aligned}\varphi(t) &= \alpha(M(t)) \leq \alpha(N(M)(t)) = \alpha\left(\int_0^t f_t(s, M(s))\,\mathrm{d}s\right)\\ &\leq 2\int_0^t \alpha(f_t(s, M(s)))\,\mathrm{d}s \leq 2\int_0^t \omega(t, s, \alpha(M(s)))\,\mathrm{d}s\\ &= 2\int_0^t \omega(t, s, \varphi(s))\,\mathrm{d}s.\end{aligned}$$

Now (C) guarantees $\varphi \equiv 0$ as desired.

Finally (D) guarantees (1.17). Thus Theorem 1.6 applies.

A special case of (2.1) is

$$u(t) = \int_0^t k(t,s) g(s, u(s))\,\mathrm{d}s, \quad t \in [0,T] \tag{2.7}$$

where $k : \Lambda \to \mathbf{R}$.

THEOREM 2.2 *Let $k : \Lambda \to \mathbf{R}$ and $g : [0,T] \times B \to E$. Suppose*

(a) *g is L^q-Carathéodory for some $q \geq 1$ and $k_t = k(t,\cdot) \in L^p[0,t]$ for any $t \in [0,T]$, where $1/p + 1/q = 1$;*

(b) *for each $t \in [0,T]$, we have*

$$|k_t - k_{t'}|_{L^p[0,t_0]} \to 0 \quad \text{as } t' \to t$$

where $t_0 = \min\{t, t'\}$;

(c) *there exists $\omega_0 : [0,T] \times [0,2R] \to \mathbf{R}$ L^q-Carathéodory with*

$$\alpha(g(s,M)) \leq \omega_0(s, \alpha(M)) \tag{2.8}$$

for a.e. $s\in[0,T]$ *and* $M\subset B$, *such that the unique solution* $\varphi\in C([0,T];[0,2R])$ *of the inequality*

$$\varphi(t)\le 2\int_0^t |k(t,s)|\omega_0(s,\varphi(s))\,\mathrm{d}s,\quad t\in[0,T]$$

is $\varphi\equiv 0$;

(d) $|u|_\infty < R$ *for any solution* $u\in C([0,T];B)$ *to*

$$u(t)=\lambda\int_0^t k(t,s)g(s,u(s))\,\mathrm{d}s,\quad t\in[0,T] \tag{2.9}$$

for each $\lambda\in(0,1)$.

Then (2.7) *has a solution in* $C([0,T];B)$.

Proof The result follows from Theorem 2.1. Here $f(t,s,x)=k(t,s)\,g(s,x)$ and

$$\eta(t,t')=\left(\int_{t'}^t h(s)^q\,\mathrm{d}s\right)^{1/q}\sup_{\tau\in[0,T]}|k_\tau|_{L^p[0,\tau]},\quad 0\le t'\le t\le T, \tag{2.10}$$

where $h\in L^q[0,T]$ is such that $|g(s,x)|\le h(s)$ for all $x\in B$ and a.e. $s\in[0,T]$. Also $\omega(t,s,\tau)=|k(t,s)|\omega_0(s,\tau)$. Note that the supremum in (2.10) is finite because of (b).

The next result contains a sufficient condition for (d).

THEOREM 2.3 *Let* $k:\Lambda\to\mathbf{R}$ *and* $g:[0,T]\times B\to E$. *Assume* (*a*)–(*c*) *hold. Also suppose that*

(d′) *there exists* $\delta\in L^1[0,T]$ *and* $w:(0,R]\to(0,\infty)$ *continuous and nondecreasing such that*

$$|k(t,s)g(s,x)|\le\delta(s)w(|x|)$$

for a.e. $s\in[0,t]$ *and all* $t\in[0,T]$, $x\in B\backslash\{0\}$, *and*

$$\int_0^T\delta(s)\,\mathrm{d}s\le\int_0^R\frac{\mathrm{d}r}{w(r)}. \tag{2.11}$$

Then (2.7) *has a solution in* $C([0,T];B)$.

Proof The result follows from Theorem 2.2 once we show (d) is true. Let $u \in C([0,T];B)$ be any solution to (2.9) for some $\lambda \in (0,1)$. Then

$$|u(t)| \leq \lambda \int_0^t |k(t,s)g(s,u(s))|\,\mathrm{d}s \leq \lambda \int_0^t \delta(s)w(|u(s)|)\,\mathrm{d}s$$

for all $t \in [0,T]$ (we put $w(0) = \lim_{t\searrow 0} w(t)$). Let

$$c(t) = \min\left\{R, \lambda \int_0^t \delta(s)w(|u(s)|)\,\mathrm{d}s\right\}.$$

Clearly c is nondecreasing. We claim that $c(T) < R$. Suppose the contrary. Then since $c(0) = 0$, there exists a subinterval $[a,b] \subset [0,T]$ with

$$c(a) = 0, \quad c(b) = R \qquad \text{and} \qquad c(t) \in (0,R) \text{ for } t \in (a,b).$$

Since $|u(t)| \leq c(t) \leq R$ on $[a,b]$ and w is nondecreasing on $[0,R]$, we have

$$c'(s) = \lambda\delta(s)w(|u(s)|) \leq \lambda\delta(s)w(c(s)) \quad \text{a.e. } s \in [a,b].$$

Now integration from a to b yields

$$\int_a^b \frac{c'(s)}{w(c(s))}\,\mathrm{d}s = \int_0^R \frac{\mathrm{d}r}{w(r)} \leq \lambda \int_a^b \delta(s)\,\mathrm{d}s$$
$$\leq \lambda \int_0^T \delta(s)\,\mathrm{d}s < \int_0^T \delta(s)\,\mathrm{d}s,$$

a contradiction. Notice we may assume $|\delta|_{L^1[0,T]} > 0$ since otherwise we have nothing to prove.

Observe that Theorem 2.3 can be derived directly from Theorem 1.5 if we take

$$K = \{u \in C([0,T];E);\ |u(t)| \leq b(t) \text{ for } t \in [0,T]\},$$

where

$$b(t) = I^{-1}\left(\int_0^t \delta(s)\,\mathrm{d}s\right) \quad \text{and} \quad I(\tau) = \int_0^\tau \frac{\mathrm{d}r}{w(r)}$$

(see the proof of Theorem 2.3 in [9]). Notice $b(t) \le R$ for all $t \in [0, T]$ because of (2.11).

COROLLARY 2.4 *Let $k: \Lambda \to \mathbf{R}$ and $g: [0, T] \times B \to E$ with $g = g_1 + g_2$, where $g_1(\cdot, 0) = 0$ and g_2 is completely continuous. Assume (a) and (b) hold with $q = 1, p = \infty$ and*

$$|k_t|_{L^\infty[0,t]} \le 1 \quad \textit{for all } t \in [0, T]. \tag{2.12}$$

Also suppose that

(c*) *there exists $\delta \in L^1[0, T]$ and $w_1 : (0, 2R] \to (0, \infty)$ continuous and nondecreasing with*

$$\int_0^{2R} \frac{\mathrm{d}r}{w_1(r)} = \infty \tag{2.13}$$

and

$$|g_1(s, x) - g_1(s, y)| \le \delta(s) w_1(|x - y|) \tag{2.14}$$

for a.e. $s \in [0, T]$ and all $x, y \in B$, $x \ne y$;

(d*) *there exists $w_2 : [0, R] \to \mathbf{R}_+$ continuous and nondecreasing such that*

$$|g_2(s, x)| \le \delta(s) w_2(|x|) \quad \textit{for a.e. } s \in [0, T] \textit{ and all } x \in B \tag{2.15}$$

and (2.11) holds with $w = w_1 + w_2$.

Then (2.7) has a solution in $C([0, T]; B)$.

Proof First we check (*c*). Using (2.14) we see that (2.8) holds for $\omega_0(s, r) = \delta(s) w_1(r)$. Now let $\varphi \in C([0, T]; [0, 2R])$ satisfies

$$\varphi(t) \le 2 \int_0^t |k(t, s)| \delta(s) w_1(\varphi(s)) \, \mathrm{d}s.$$

Then, by (2.12), we have

$$\varphi(t) \le 2 \int_0^t \delta(s) w_1(\varphi(s)) \, \mathrm{d}s.$$

Let

$$c(t) = 2 \int_0^t \delta(s) w_1(\varphi(s)) \, \mathrm{d}s.$$

It is clear that c is nondecreasing. Then $\varphi \equiv 0$ once we show $c(T) = 0$. Suppose the contrary, i.e. $c(T) > 0$. Then, since $c(0) = 0$, for each $\varepsilon \in (0, A)$, where $A = \min\{c(T), 2R\}$, there is a subinterval $[a, b] \subset [0, T]$ with

$$c(a) = \varepsilon, \ \ c(b) = A \quad \text{and} \quad c(t) \in (\varepsilon, A) \ \text{ for all } t \in (a, b).$$

Now $\varphi(t) \leq c(t) \leq 2R$ on $[a, b]$ and w_1 nondecreasing on $[0, 2R]$ guarantee that

$$c'(s) = 2\delta(s)w_1(\varphi(s)) \leq 2\delta(s)w_1(c(s))$$

a.e. $s \in [a, b]$. Consequently

$$\int_\varepsilon^A \frac{\mathrm{d}r}{w_1(r)} \leq 2\int_a^b \delta(s)\,\mathrm{d}s \leq 2|\delta|_{L^1[0,T]}.$$

This, for $\varepsilon \searrow 0$, yields a contradiction to (2.13). Thus $c(T) = 0$ and so $\varphi \equiv 0$.

Finally (d′) follows from (2.12), (2.14), $g_1(\cdot, 0) = 0$ and (2.15).

Example 2.1 We give an example of a function g_1 which satisfies (c^*). Let $E = C(I; \mathbf{R})$, $I \subset \mathbf{R}$ compact interval, and let $g_1 : [0, T] \times B \to E$ given by

$$g_1(s, x)(\tau) = \delta(s)w_1(|x(\tau)|), \quad \tau \in I \ \text{ and } \ x \in B,$$

where $\delta \in L^1([0, T]; \mathbf{R}_+)$ and $w_1 : [0, 2R] \to \mathbf{R}_+$ is continuous, nondecreasing and satisfies the following conditions:

$$w_1(0) = 0, \quad w_1(r) > 0 \qquad \text{on } (0, 2R]$$

$$|w_1(r) - w_1(r')| \leq w_1(|r - r'|) \quad \text{for } r, r' \in [0, 2R] \tag{2.16}$$

and

$$\int_0^{2R} \frac{\mathrm{d}r}{w_1(r)} = \infty. \tag{2.17}$$

Notice (2.14) can easily be deduced from (2.16). Examples of w_1 with the above properties are: $w_1(r) = r$ for any $R > 0$ and $w_1(r) = -r\ln r$ for $R = 1/(2e)$ (see [13]).

3 EXISTENCE CRITERIA FOR URYSOHN INTEGRAL EQUATIONS

In this section we discuss the Urysohn integral equation

$$u(t) = \int_0^T f(t,s,u(s))\,\mathrm{d}s, \quad t \in [0,T] \tag{3.1}$$

in a ball $B = \{x \in E; |x| \leq R\}$ of the Banach space $(E, |\cdot|)$.

Essentially the same reasoning as in Section 2 establish the following existence principles for (3.1).

THEOREM 3.1 *Let* $f : [0,T]^2 \times B \to E$. *Suppose*

(A) *for each* $t \in [0,T]$, f_t *is* L^1-*Carathéodory and*

$$\sup_{t\in[0,T]} \int_0^T \sup_{|x|\leq R} |f_t(s,x)|\,\mathrm{d}s < \infty;$$

(B) *one has*

$$\int_0^T \sup_{|x|\leq R} |f_t(s,x) - f_{t'}(s,x)|\,\mathrm{d}s \to 0 \quad \textit{as } t' \to t;$$

(C) *there exists* $\omega : [0,T]^2 \times [0,2R] \to \mathbf{R}$ *such that for each* $t \in [0,T]$, ω_t *is* L^1-*Carathéodory,*

$$\alpha(f(t,s,M)) \leq \omega(t,s,\alpha(M))$$

for a.e. $s \in [0,T]$, $M \subset B$, *and the unique* $\varphi \in C([0,T];[0,2R])$ *satisfying*

$$\varphi(t) \leq 2\int_0^T \omega(t,s,\varphi(s))\,\mathrm{d}s, \quad t \in [0,T]$$

is $\varphi \equiv 0$;

(D) $|u|_\infty < R$ *for any solution* $u \in C([0,T];B)$ *to*

$$u(t) = \lambda \int_0^T f(t,s,u(s))\,\mathrm{d}s, \quad t \in [0,T]$$

for each $\lambda \in (0,1)$.

Then (3.1) *has a solution in* $C([0,T];B)$.

An immediate consequence of Theorem 3.1 is the following result for the Hammerstein integral equation

$$u(t) = \int_0^T k(t,s)g(s,u(s))\,\mathrm{d}s, \quad t \in [0,T]. \tag{3.2}$$

THEOREM 3.2 *Let* $k:[0,T]^2 \to \mathbf{R}$ *and* $g:[0,T]\times B \to E$. *Suppose*

(a) g *is* L^q*-Carathéodory for some* $q \geq 1$ *and for each* $t \in [0,T]$, $k_t \in L^p[0,T]$ *where* $1/p + 1/q = 1$;
(b) *the map* $t \longmapsto k_t$ *is continuous from* $[0,T]$ *to* $L^p[0,T]$;
(c) *there exists* $\omega_0 : [0,T]\times[0,2R] \to \mathbf{R}$ L^q*-Carathéodory with*

$$\alpha(g(s,M)) \leq \omega_0(s,\alpha(M)) \tag{3.3}$$

for a.e. $s \in [0,T]$ *and* $M \subset B$, *such that the unique solution* $\varphi \in C([0,T];[0,2R])$ *of the inequality*

$$\varphi(t) \leq 2\int_0^T |k(t,s)|\omega_0(s,\varphi(s))\,\mathrm{d}s, \quad t \in [0,T] \tag{3.4}$$

is $\varphi \equiv 0$;
(d) $|u|_\infty < R$ *for any solution* $u \in C([0,T];B)$ *to*

$$u(t) = \lambda \int_0^T k(t,s)g(s,u(s))\,\mathrm{d}s, \quad t \in [0,T] \tag{3.5}$$

for each $\lambda \in (0,1)$.

Then (3.2) *has a solution in* $C([0,T];B)$.

Theorem 3.2 is now used to obtain an applicable result for (3.2).

COROLLARY 3.3 *Let* $k:[0,T]^2 \to \mathbf{R}$ *and* $g:[0,T]\times B \to E$ *with* $g = g_1 + g_2$, *where* $g_1(\cdot,0)=0$ *and* g_2 *is completely continuous. Assume* (*a*) *and* (*b*) *hold with* $q=1, p=\infty$ *and*

$$|k_t|_{L^\infty[0,T]} \le 1 \quad \textit{for all } t \in [0,T].$$

In addition suppose

(c*) *there exists* $\delta \in L^1[0,T]$ *and* $w_1:(0,2R] \to (0,\infty)$ *continuous and nondecreasing with*

$$\inf_{r\in(0,2R]} \frac{r}{w_1(r)} > 2|\delta|_{L^1[0,T]} \tag{3.6}$$

and

$$|g_1(s,x) - g_1(s,y)| \le \delta(s) w_1(|x-y|) \tag{3.7}$$

for a.e. $s\in[0,T]$ *and all* $x,y\in B$, $x \ne y$;

(d*) *there exists* $w_2:[0,R]\to\mathbf{R}_+$ *continuous and nondecreasing such that*

$$|g_2(s,x)| \le \delta(s) w_2(|x|)$$

for a.e. $s\in[0,T]$, *all* $x\in B$, *and*

$$\frac{R}{w(R)} \ge |\delta|_{L^1[0,T]} \tag{3.8}$$

where $w = w_1 + w_2$.

Then (3.2) *has a solution in* $C([0,T];B)$.

Proof First we claim that (c) holds with $\omega_0(s,r) = \delta(s)w_1(r)$. It is clear that (3.3) follows from (3.7) and the complete continuity of g_2. Now let $\varphi\in C([0,T];[0,2R])$ be any solution to (3.4) and suppose that $\varphi \not\equiv 0$. Then $r_0 = \max_{t\in[0,T]}\varphi(t) \in (0,2R]$. Let $t_0\in[0,T]$ be such that $\varphi(t_0)=r_0$. From (3.4), since $|k_t|_\infty \le 1$ and w_1 is nondecreasing, we deduce that

$$r_0 = \varphi(t_0) \le 2\int_0^T |k(t_0,s)|\delta(s) w_1(\varphi(s))\,\mathrm{d}s \le 2w_1(r_0)|\delta|_{L^1[0,T]}\,.$$

Hence $r_0/w_1(r_0) \le 2|\delta|_{L^1[0,T]}$, which contradicts (3.6). Thus $r_0 = 0$.

Next we check (d). Suppose $|u|_\infty = R$ for some $u \in C([0,T];B)$ solution to (3.5). Let $t_1 \in [0,T]$ with $|u(t_1)| = R$. From (3.5) we obtain

$$R = |u(t_1)| \leq \lambda \int_0^T \delta(s) w(|u(s)|)\, \mathrm{d}s \leq \lambda w(R) |\delta|_{L^1[0,T]}.$$

Since $\lambda \in (0,1)$, $w(R) > 0$ and we may assume $|\delta|_{L^1[0,T]} > 0$, we find $R < w(R)|\delta|_{L^1[0,T]}$, a contradiction to (3.8). Thus (d) holds and Theorem 3.2 applies.

Example 3.1 Let $E = C(I; \mathbf{R})$, $I \subset \mathbf{R}$ compact interval. Then an example of a function g_1 satisfying (c*) is given by Example 2.1, where (2.17) is now replaced by (3.6). For instance, we may take

$$w_1(r) = Cr \quad \text{for an arbitrary } R > 0 \text{ and } C < 1/(2|\delta|_1)$$

or

$$w_1(r) = C \sin r \quad \text{for } 0 < R \leq \pi/4 \text{ and } C < 1/(2|\delta|_1).$$

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