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On Weighted Positivity of Ordinary Differential Operators

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Some elliptic differential operators possess a weighted positivity property, where the weight is a fundamental solution of the operator. This property has interesting applications to partial differential operators. The present paper is devoted to the property for ordinary differential operators.

It is shown that the operator $(1-\mathrm{d}^2/\mathrm{d}x^2)^m$ has the positivity property if and only if $m=0,1,2,3$, while there exist operators of arbitrary even order for which the positivity holds. Some necessary conditions for the property are given.

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1 INTRODUCTION

Let $p(D)$ be an elliptic differential operator with constant coefficients, having the fundamental solution Γ. The inequality

$$\operatorname{Re}\int_{\mathbf{R}^n} p(D)u\cdot\overline{u\Gamma}\,\mathrm{d}x\geq 0,\quad u\in C_0^\infty(\mathbf{R}^n)\tag{1}$$

was used by Maz'ya [1,2] to obtain a necessary condition for the Wiener regularity of a boundary point for the biharmonic operator in dimensions 4, 5, 6, 7. The inequality (1) fails for that operator if $n\geq 8$.

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This result was extended to the polyharmonic operator by Maz'ya–Donchev [3], and to the fractional Laplacian by Eilertsen [4].

Since it is not at all clear exactly which differential operators possess the weighted positivity property (1), we are motivated in the present paper to study the property in the one-dimensional case. We will be concerned with operators $p(D)$ in $\mathbf{R}^1$, where p is a positive polynomial and $Du(x) = -\mathrm{i}\,\mathrm{d}u/\mathrm{d}x$.

The structure of the paper is as follows. In Section 2 we prove that there exist operators of arbitrary even order satisfying (1) in the one-dimensional case. In fact, we prove that if the sequence (a_j) grows sufficiently fast then (1) holds for

$$p(D) = (a_1 + D^2)(a_2 + D^2)\cdots(a_m + D^2).$$

We also give explicit examples of such operators. In Section 3 we find some necessary conditions for operators to satisfy (1), and deduce examples of operators not having this property, for instance $1 + D^4$. Finally, in Section 4, we study the operators $(1 + D^2)^m$. We prove that they satisfy (1) if and only if $m = 0, 1, 2, 3$. The case $m = 3$ is more complicated than the others. For this case, an important step in the proof is the identity (24). In the cited papers it was essential to have a certain minorant (instead of 0) on the right of (1). We will also see in Section 4 that the operator $(1 + D^2)^3$ has a different behavior, with respect to this, than the operators $1 + D^2$ and $(1 + D^2)^2$.

By Parseval's formula, these results can also be interpreted as results for certain integral operators. For instance, it follows from Proposition 11 that if $m = 1, 2, 3$ then

$$\iint \frac{(1 + x^2)^m}{(1 + (x - y)^2)^m} f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y \geq 0, \quad f \text{ real in } C_0^\infty(\mathbf{R}),$$

with equality only for $f = 0$, while for $m \geq 4$, the double integral can take negative values.

Some notation: F denotes the Fourier transform,

$$(Fu)(\xi) = \hat{u}(\xi) = \int \mathrm{e}^{-\mathrm{i}\xi x} u(x)\,\mathrm{d}x.$$

We write $\int$ instead of $\int_{-\infty}^{\infty}$. Let S denote the Schwartz space of rapidly decreasing C^∞-functions on $\mathbf{R}^1$. We also write C_0^∞ instead of $C_0^\infty(\mathbf{R}^1)$.

The letter c denotes positive constants. The notation $a \sim b$ means that there exists c such that $c^{-1}a \leq b \leq ca$.

2 POSITIVITY

For a positive polynomial p we let Γ be defined by $\widehat{\Gamma} = 1/p$. Thus Γ is a fundamental solution of the operator $p(D)$. By Parseval's formula we have

$$\int p(D)u \cdot \overline{u\Gamma}\,\mathrm{d}x = (2\pi)^{-2} \int\int \frac{p(x)}{p(x-y)} \hat{u}(x)\overline{\hat{u}(y)}\,\mathrm{d}x\,\mathrm{d}y, \tag{2}$$

for $u \in \mathcal{S}$. We define $\mathcal{P}$ to be the class of those positive polynomials p for which the real part of (2) is nonnegative for all $u \in \mathcal{S}$.

LEMMA 1 *For any polynomial p of degree $2n$ or $2n+1$ there are polynomials q_j such that*

$$p(x) + p(-y) = \sum_{j=0}^{n} (xy)^j q_j(x-y). \tag{3}$$

In $q_j(t)$ and $p(x)$ the coefficients for t^m and x^{m+2j} are proportional and have the same sign.

Proof With the new variables $s = (x+y)/2$, $t = (x-y)/2$ and $u = xy$ we can write $x^m + (-y)^m$ as

$$\begin{aligned}(t+s)^m + (t-s)^m &= 2\sum_{\substack{k=0\\ k \text{ even}}}^{m} \binom{m}{k} s^k t^{m-k}\\ &= 2\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k} (t^2+u)^k t^{m-2k}\\ &= 2\sum_{k=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{k} \binom{m}{2k}\binom{k}{j} u^j t^{m-2j}.\end{aligned}$$

The statement follows.

Remark It can be shown that

$$q_j(x) = \frac{1}{j!}(E^j(p(x) + p(-y)))|_{y=0} = \sum_{m=0}^{N-2j} c_{m,j}\, b_{m+2j} x^m, \tag{4}$$

where the operator E is given by

$$E = (x+y)^{-1}(\partial/\partial x + \partial/\partial y),$$

N is the degree of p, b_m is the coefficient for x^m in $p(x)$, and the coefficients $c_{m,j}$ are given by

$$c_{m,j} = 2^{1-m} \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m+2j}{2(k+j)} \binom{k+j}{j}.$$

COROLLARY 2 *If $p > 0$ is an even polynomial of degree $2n$ and $F(q_j/p) \geq 0$ for $j = 0, \ldots, n$, then $p \in \mathcal{P}$ and for all $u \in \mathcal{S}$ it holds,*

$$2\,\mathrm{Re} \int P(D)u \cdot \bar{u}\Gamma \,\mathrm{d}x \geq |u(0)|^2 + \int (b_0|u|^2 + 2b_{2n}|u^{(n)}|^2)\Gamma \,\mathrm{d}x, \qquad (5)$$

where $b_0 = p(0)$ and b_{2n} is the leading coefficient of p. (The hypothesis imply that $\Gamma > 0$.)

Proof Using (2) and expanding $p(x) + p(y)$ according to the lemma we see that the left-hand side equals

$$(2\pi)^{-2} \iint \frac{p(x)+p(y)}{p(x-y)} \hat{u}(x)\overline{\hat{u}(y)} \,\mathrm{d}x\,\mathrm{d}y = (2\pi)^{-1} \sum_{j=0}^{n} \int F(q_j/p)|u^{(j)}|^2 \,\mathrm{d}\xi.$$

If we put $y = 0$ in (3) we get $q_0(x) = p(x) + p(0)$, so

$$F(q_0/p) = 2\pi\delta + p(0)F(1/p).$$

Similarly, if we let $x = y \to \infty$, we get $q_n = 2b_{2n}$. Since for p even $F(1/p) = 2\pi\Gamma$, this proves the assertion.

PROPOSITION 3 *For each integer $n \geq 1$ there is an $\varepsilon > 0$ such that if the positive constants $a_1, \ldots, a_n$ satisfy $a_j/a_{j+1} \leq \varepsilon$, then the polynomial*

$$p(x) = (a_1 + x^2)(a_2 + x^2) \cdots (a_n + x^2)$$

belongs to $\mathcal{P}$ and satisfies the inequality (5).

Proof Define the polynomials p_j and the constants b_k^j by

$$p_j(x) = (a_1 + x^2)(a_2 + x^2) \cdots (a_j + x^2) = x^{2j} + b_1^j x^{2(j-1)} + \cdots + b_j^j.$$

Thus $p = p_n$ and we have by Lemma 1 that the corresponding q_j's are of the form

$$q_{n-j}(x) = c_0^j x^{2j} + c_1^j b_1^n x^{2(j-1)} + \cdots + c_j^j b_j^n, \tag{6}$$

where c_k^j are positive constants.

Now, writing

$$q_{n-j}(x) = c_0^j p_j(x) + b_1^n (c_1^j + d_1^j) p_{j-1} + \cdots + b_j^n (c_j^j + d_j^j), \tag{7}$$

we claim that $d_l^j = O(\varepsilon)$, as $\varepsilon \to 0$. If we assume that this has been proved for $l = 1, 2 \ldots, k-1$, and identify the coefficients for $x^{2(j-k)}$ in (7) and (6), we get

$$c_0^j b_k^j + b_1^n (c_1^j + O(\varepsilon)) b_{k-1}^{j-1} + \cdots + b_{k-1}^n (c_{k-1}^j + O(\varepsilon)) b_1^{j-(k-1)} + b_k^n d_k^j = 0. \tag{8}$$

Observe that if $\varepsilon \leq 1$ (so that (a_j) is increasing) there is a number M such that we have the following estimation for b_k^j:

$$a_{j-k+1} a_{j-k+2} \cdots a_j \leq b_k^j \leq M a_{j-k+1} a_{j-k+2} \cdots a_j.$$

Therefore, if $0 \leq l \leq k-1$,

$$\frac{b_l^n b_{k-l}^{j-l}}{b_k^n} \leq M^2 \frac{a_{j-k+1} \cdots a_{j-l}}{a_{n-k+1} \cdots a_{n-l}} = O(\varepsilon).$$

Hence (8) shows that $d_k^j = O(\varepsilon)$. Since for $k = 1$ no assumptions were used, the claim is proved.

Since p_j/p has positive Fourier transform and the coefficients in (7) are positive for small ε, the proof is completed by Corollary 2.

Example 4 For any polynomial p of degree $2n$, we can easily compute (for instance by using (4)) the following:

$$\begin{aligned} q_0(x) &= p(x) + p(0), \\ q_1(x) &= (p'(x) - p'(0))/x \\ q_{n-1}(x) &= n^2 b_{2n} x^2 + (2n-1) b_{2n-1} x + 2b_{2n-2}, \\ q_n(x) &= 2b_{2n}, \end{aligned}$$

where b_m is the coefficient for x^m in $p(x)$. Now let p be as in the proposition. It follows immediately that q_0/p and q_n/p have positive Fourier transforms. The same is true for q_1/p since

$$\frac{p'(x)}{xp(x)} = 2\sum_{j=1}^{n} \frac{1}{a_j + x^2}.$$

As for q_{n-1} we now have

$$q_{n-1}(x) = n^2(a_1 + x^2) + 2\sum_{k=1}^{n} a_j - n^2 a_1,$$

so the condition $2\sum_1^n a_j \geq n^2 a_1$ is sufficient for the Fourier transform of q_{n-1}/p to be positive.

Taking $n = 1, 2$ and 3, we have proved that the polynomials

$$\begin{aligned} a + x^2, &\quad a > 0, \\ (a + x^2)(b + x^2), &\quad a, b > 0, \\ (a + x^2)(b + x^2)(c + x^2), &\quad a, b, c > 0,\ 7a \leq 2(b + c) \end{aligned} \tag{9}$$

are in $\mathcal{P}$ and satisfy the inequality (5).

3 NON-POSITIVITY

It is quite immediate that a necessary and sufficient condition for a positive polynomial p to belong to $\mathcal{P}$ is the Bochner type condition

$$\sum_{j,k=1}^{n} c_j c_k \frac{p(t_j)}{p(t_j - t_k)} \geq 0, \quad \text{for all } c_j,\ t_j \in \mathbf{R} \text{ and } n = 1, 2, \ldots. \tag{10}$$

We can consider (10) as the limit of the right side of (2) as the function $\hat{u}$ tends to the distribution $2\pi \sum_1^n c_j \delta_{t_j}$, where δ_{t_j} is the Dirac measure at t_j.

If we instead let $\hat{u}$ tend to the distribution $2\pi L(-iD)\delta_t$, where L is a polynomial with real coefficients, we obtain the necessary condition

$$L(\partial/\partial x)L(\partial/\partial y)\frac{p(x)}{p(x-y)}\bigg|_{x=y=t} \geq 0, \quad t \in \mathbf{R}, \tag{11}$$

in which only the two points 0 and t occur. The following proposition provides an equivalent form of this condition.

PROPOSITON 5 *Let L be any polynomial with real coefficients. The condition*

$$(L(-\mathrm{i}D)(p(t+\mathrm{i}D)L)(\mathrm{i}D))(1/p)(0) \geq 0, \quad t \in \mathbf{R} \tag{12}$$

is necessary for $p \in \mathcal{P}$.

Proof Let $\phi \in C_0^\infty$ have $\phi(0)=1$. Taking $u(x)=\mathrm{e}^{\mathrm{i}tx}L(\mathrm{i}x)\phi(\varepsilon x)$ and letting $\varepsilon \to 0$, the real part of (2) tends to

$$\int \overline{p(t+D)(L(\mathrm{i}x))}L(\mathrm{i}x)\Gamma(x)\,\mathrm{d}x = \int L(\mathrm{i}x)(p(t+\mathrm{i}D)L)(-\mathrm{i}x)\Gamma(x)\,\mathrm{d}x.$$

(When passing to the limit, we notice that $\Gamma(x)$ decreases exponentially as $|x| \to \infty$.) Since $\widehat{\Gamma} = 1/p$, the last integral equals the left side of (12).

COROLLARY 6 *The condition*

$$4p(t)^2(p(0)p''(0) - p'(0)^2) \geq (2p(t)p'(0) - p(0)p'(t))^2, \quad t \in \mathbf{R} \tag{13}$$

is necessary for $p \in \mathcal{P}$.

Proof If we take $L(x)=a+x$, the operator that acts on $1/p$ in (12) becomes

$$a^2p(t) + ap'(t) - (p(t)(\mathrm{i}D)^2 + p'(t)\mathrm{i}D).$$

Thus the left side of (12) becomes a quadratic form in a. This form being nonnegative for all real a is equivalent to

$$-4(p(t)/p(0))(p(t)(\mathrm{i}D)^2 + p'(t)\mathrm{i}D)(1/p)(0) \geq (p'(t)/p(0))^2.$$

The last inequality, multiplied by $p(0)^4$, can be written as (13).

Example 7 Condition (13) implies that if $p''(0) \leq 0$ then p is either constant or does not belong to $\mathcal{P}$.

Example 8 Let $p(x) = (1+x^2)^m$. For this polynomial (13) reads

$$2(1+t^2)^2 \geq mt^2,$$

which is equivalent to $m \leq 8$.

If we take $L(x) = ax + x^2$ and $t = 1$, Proposition 5 leads to

$$12m(m+1) - 2m(m^2 - ma - a^2) \geq 0.$$

This is equivalent to $24(m+1) \geq 5m^2$, that is, $m \leq 5$.

It is also possible to prove nonpositivity for $(1+x^2)^5$ and $(1+x^2)^4$ with the aid of Proposition 5, but then one has to use a polynomial L of degree 3 in the former and of degree 4 in the latter case. In Section 4 we give another proof of the nonpositivity when $m \geq 4$.

PROPOSITION 9 *If $p \in \mathcal{P}$ then the real part of $F(1/p)$ is nonnegative.*

Proof Assume that $\operatorname{Re} F(1/p) < 0$ at the point ξ_0 and hence also at the point $-\xi_0$. Let ϕ be a real, even function with $\phi(0) = 1$ and $\hat{\phi} \in C_0^\infty$. Put $f(x) = \cos(\xi_0 x)\phi(\varepsilon x)$, so that $\operatorname{supp} \hat{f} \to \{-\xi_0, \xi_0\}$ as $\varepsilon \to 0$.

Let q_j be as in Lemma 1. Thus q_n is a positive constant so, by the continuity of $F(1/p)$, there is an $a > 0$ such that $\operatorname{Re} F(q_n/p) \leq -a$ in $\operatorname{supp} \hat{f}$, if ε is small enough. Also, there is a number A such that $\operatorname{Re} F(q_j/p)(\xi) \leq A$, for $j = 0, \ldots, n-1$ and $\xi \neq 0$.

Now, using the inequality

$$\int g(K * g)\, \mathrm{d}x \leq \sup_{\operatorname{supp} \hat{g}} (\operatorname{Re} \hat{K}) \int g^2\, \mathrm{d}x, \quad g, K \text{ real},$$

we get, since f is even, that (2) with f in place of $\hat{u}$ can be estimated by a constant times

$$\iint \frac{p(x)+p(-y)}{p(x-y)} f(x) f(y)\, \mathrm{d}x\, \mathrm{d}y = \int \sum_{j=0}^{n} x^j f(x) \big((q_j/p) * y^j f(y)\big)(x)\, \mathrm{d}x$$
$$\leq \int \left(A \sum_{j=0}^{n-1} x^{2j} - a x^{2n} \right) f(x)^2\, \mathrm{d}x.$$

The last expression is clearly negative for small ε.

Example 10 Let $p > 0$ be a nonconstant even polynomial with $F(1/p) \geq 0$ (for instance, p can be any nonconstant even polynomial in $\mathcal{P}$). Since

$$\operatorname{Re} F(1/p(x-\varepsilon))(\xi) = F(1/p)(\xi)\cos(\varepsilon\xi),$$

Proposition 9 shows that if $\varepsilon \neq 0$ then $p(x-\varepsilon)$ does not belong to $\mathcal{P}$.

4 THE OPERATORS $(1+D^2)^m$

We introduce some notation. Let $p_m(x)=(1+x^2)^m$, $\Gamma_m=(2\pi)^{-1}F(1/p_m)$ and put $\lambda_s(x)=(1+|x|)^s e^{-|x|}$. We observe that $\Gamma_m \sim \lambda_{m-1}$, according to Lemma 12 below. We define the form

$$Q_m(u)=\mathrm{Re}\int p_m(D)u\cdot\bar{u}\Gamma_m\,\mathrm{d}x,$$

and the weighted Sobolev norms

$$\|u\|_{m,s}=\left(\sum_{j=0}^{m}\int|u^{(j)}|^2\lambda_s\,\mathrm{d}x\right)^{1/2}.$$

We remark that the subsequent inequality (20) shows that $\|u\|_{m,s}\geq c|u(0)|$, if $m\geq 1$ and $s\geq 0$.

The main result of this section is the following proposition. The five lemmas that follow it are needed for the proof.

PROPOSITION 11 *The polynomial* $(1+x^2)^m$ *belongs to* $\mathcal{P}$ *if and only if* $m=0,1,2,3$. *The following inequalities hold:*

$$\|u\|_{1,0}^2\sim Q_1(u), \tag{14}$$

$$\|u\|_{2,1}^2\sim Q_2(u), \tag{15}$$

$$c^{-1}\|u\|_{3,1}^2\leq Q_3(u)\leq c\|u\|_{3,2}^2. \tag{16}$$

The inequality (16) *cannot be improved by replacing any of the squared norms by another one of the type* $\|u\|_{3,s}^2$.

LEMMA 12 *The following identities hold:*

$$\Gamma_1(x)=\tfrac{1}{2}e^{-|x|},$$

$$\Gamma_2(x)=\tfrac{1}{4}(|x|+1)e^{-|x|},$$

$$\Gamma_3(x)=\tfrac{1}{16}(x^2+3|x|+3)e^{-|x|},$$

$$\Gamma_{m+2}(x)=\frac{2m+1}{2(m+1)}\Gamma_{m+1}(x)+\frac{x^2}{4m(m+1)}\Gamma_m(x).$$

Proof The recursion formula follows from the relation

$$(1/p)'' = -4m(m+1)/p_{m+2} + 2m(2m+1)/p_{m+1}.$$

The formulas for Γ_1 and Γ_2 can be calculated directly.

For the next lemma, which will be used for counter examples, we construct the functions u_t, $t \geq 1$. Let $\phi \in C_0^\infty((0,2))$ be real with $\phi = 1$ in a neighborhood of 1. Define $\phi_t \in C_0^\infty((0, t+1))$ so that $\phi_t = 1$ on [1,t] and

$$\phi_t(x) = \phi(x), \quad \phi_t(x+t) = \phi(x+1), \qquad x \in [0,1].$$

Let ω be a fixed real number and put

$$u_t(x) = \mathrm{e}^{|x|/2}\phi_t(|x|)\cos(\omega x/2).$$

Then $u_t \in C_0^\infty$ is real and even.

LEMMA 13 *Let u_t be as above. As $t \to \infty$ we have*

$$Q_m(u_t) = (2^{-3m}/m!)\mathrm{Re}\big(3 + \omega^2 - 2\omega\mathrm{i}\big)^m t^m + O(t^{m-1}). \tag{17}$$

Proof It follows from Lemma (12) that $\Gamma_m(x) = r(|x|)\mathrm{e}^{-|x|}$, where r is a polynomial of degree $m-1$ having leading coefficient $2^{-m}/(m-1)!$. Since u_t is real and even,

$$Q_m(u_t) = 2\int_0^\infty r(x)\mathrm{e}^{-x}u_t(x)p_m(D)u_t(x)\,\mathrm{d}x = 2\int_0^\infty r(x)\psi_t(x)\,\mathrm{d}x,$$

where ψ_t is introduced in the obvious way. For $x \in$ [1,t] we have

$$\begin{aligned}\psi_t(x) &= \cos(\omega x/2)p_m(D - \mathrm{i}/2)\cos(\omega x/2)\\ &= \cos(\omega x/2)\mathrm{Re}(\mathrm{e}^{\mathrm{i}\omega x/2}p_m((\omega - \mathrm{i})/2))\\ &= 2^{-(2m+1)}\mathrm{Re}((3 + \omega^2 - 2\omega\mathrm{i})^m(1 + \mathrm{e}^{\mathrm{i}\omega x}))\end{aligned}$$

and it follows that

$$\begin{aligned}Q_m(u_t) = 2\int_0^1 &(r(\xi)\psi_t(x) + r(x+t)\psi_t(x+t))\,\mathrm{d}x\\ &+ 2^{-2m}\,\mathrm{Re}\bigg((3+\omega^2 - 2\omega\mathrm{i})^m\int_1^t r(x)(1 + \mathrm{e}^{\mathrm{i}\omega x})\,\mathrm{d}x\bigg).\end{aligned}$$

Since r has degree $m-1$ the boundedness of $\{\psi_t\}$ implies that the first integral is $O(t^{m-1})$. After one integration by parts also the integral $\int_1^t r(x)e^{i\omega x}\,dx$ is seen to be $O(t^{m-1})$, so

$$Q_m(u_t) = 2^{-2m}\mathrm{Re}(3+\omega^2-2\omega i)^m \int_1^t r(x)\,dx + O(t^{m-1}), \quad \text{as } t \to \infty.$$

This gives (17).

LEMMA 14 *If $\varepsilon > 0$, $s \in (1,2)$ and k is a nonnegative integer then there exist $u, v \in C_0^\infty$ such that*

$$Q_3(u) \leq \varepsilon \int |u^{(j)}|^2 \lambda_s\,dx, \quad 0 \leq j, \tag{18}$$

$$Q_3(v) \geq \varepsilon^{-1} \int |v^{(j)}|^2 \lambda_s\,dx, \quad 0 \leq j \leq k. \tag{19}$$

Proof If we take $\omega = \sqrt{3}$ in the definition of u_t, Lemma 13 gives $Q_3(u_t) = O(t^2)$, as $t \to \infty$. On the other hand, for $x \in [1,t]$, a simple calculation shows that

$$|u_t^{(j)}(x)|^2 = e^x(1+\cos(\sqrt{3}x + j2\pi/3))/2,$$

so for large t, the integral in (18) majorizes

$$\int_1^t x^s(1+\cos(\sqrt{3}x+j2\pi/3))\,dx \geq t^{s+1}/3.$$

This proves (18).

To prove (19) we take $\omega = 0$. Lemma 13 then gives $Q_3(u_t) \geq ct^3$, for large t. But, similarly as in the proof of Lemma 13, we see that the right-hand side of (19), with u_t in place of v, is $O(t^{s+1})$.

The proof of the following simple lemma, which we use to establish equivalent norms in Proposition 11, also indicates the idea behind the more nontrivial Lemma 16.

LEMMA 15 *If $a > 0$ then for every $u \in \mathcal{S}$ we have*

$$0 \leq -2|u(0)|^2 + \int e^{-|x|}((1+a)|u|^2 + a^{-1}|u'|^2)\,dx, \tag{20}$$

$$0 \leq \int (1+|x|)e^{-|x|}((1+a)|u|^2 - 2|u'|^2 + a^{-1}|u''|^2)\,dx. \tag{21}$$

Proof We begin by proving the second inequality. Let $v \in \mathcal{S}$ be a real function that is either even or odd (thus $v(0)v'(0)=0$). By partial integration

$$\int_0^\infty (1+x)e^{-x}vv''\,dx = -\int_0^\infty (1+x)e^{-x}(v')^2\,dx - \frac{1}{2}\int_0^\infty (1-x)e^{-x}v^2\,dx$$
$$\leq \frac{1}{2}\int_0^\infty (1+x)e^{-x}(v^2 - 2(v')^2)\,dx,$$

so for any $a>0$,

$$0 \leq a\int_0^\infty (1+x)e^{-x}\left(v + a^{-1}v''\right)^2 dx$$
$$\leq \int_0^\infty (1+x)e^{-x}((1+a)v^2 - 2(v')^2 + a^{-1}(v'')^2)\,dx. \qquad (22)$$

Now, if u is real, (21) follows from (22) and the observation

$$\int \varphi(|x|)u^{(j)}(x)^2\,dx = 2\int_0^\infty \varphi(x)(u_0^{(j)}(x)^2 + u_1^{(j)}(x)^2)\,dx,$$

where $u = u_0 + u_1$ is the decomposition of u into even and odd functions. The complex case follows immediately. Similarly, the identity

$$\int_0^\infty e^{-x}(av + a^{-1}v')^2\,dx = -v(0)^2 + \int_0^\infty e^{-x}((a+1)v^2 + a^{-1}(v')^2)\,dx$$

leads to the first inequality.

LEMMA 16 *For every $u \in \mathcal{S}$ it holds,*

$$0 \leq 4|u(0)|^2 + \int e^{-|x|}(x^2|u|^2 + 6(1-|x|)|u'|^2$$
$$+ 3|x|(2-|x|)|u''|^2 + 2x^2|u'''|^2)\,dx. \qquad (23)$$

Proof This follows, as in the proof of the preceding lemma, from the identity

$$4v(0)^2 + 2\int_0^\infty e^{-x}(x^2v^2 + 6(1-x)(v')^2 + 3x(2-x)(v'')^2 + 2x^2(v''')^2)dx$$
$$= \int_0^\infty x^2e^{-x}(3(v''-v'+v)^2 + (2v'''-3v''+3v'-v)^2)\,dx, \qquad (24)$$

for a real $v \in \mathcal{S}$. To verify (24), one can expand the right-hand side and integrate by parts several times.

Proof of Proposition 11 The range of the argument for $3+\omega^2-2\omega\mathrm{i}$ is $[-\pi/6, \pi/6]$ (the endpoints are attained for $\omega = \pm\sqrt{3}$), so if $m \geq 4$ then $(3+\omega^2-2\omega\mathrm{i})^m$ assumes values with negative real part. By Lemma 13, p_m does not belong to $\mathcal{P}$ if $m \geq 4$.

We now turn to proving the inequalities. Using the decompositions

$$\frac{p_1(x)+p_1(y)}{p_1(x-y)} = 1 + \frac{1+2xy}{p_1(x-y)},$$
$$\frac{p_2(x)+p_2(y)}{p_2(x-y)} = 1 + \frac{4xy}{p_1(x-y)} + \frac{1+2x^2y^2}{p_2(x-y)},$$
$$\frac{p_3(x)+p_3(y)}{p_3(x-y)} = 1 + \frac{6xy}{p_1(x-y)} + \frac{9x^2y^2}{p_2(x-y)} + \frac{1-3x^2y^2+2x^3y^3}{p_3(x-y)},$$

along with Parseval's formula and the formulas for $F(1/p_m) = 2\pi\Gamma_m$ we obtain (similarly as in the proof of Corollary 2) the identities

$$2Q_1(u) = |u(0)|^2 + \frac{1}{2}\int \mathrm{e}^{-|x|}(|u|^2 + 2|u'|^2)\,\mathrm{d}x, \tag{25}$$

$$2Q_2(u) = |u(0)|^2 + \frac{1}{4}\int \mathrm{e}^{-|x|}(8|u'|^2 + (1+|x|)(|u|^2 + 2|u''|^2))\,\mathrm{d}x, \tag{26}$$

$$\begin{aligned}2Q_3(u) = |u(0)|^2 + \frac{1}{16}\int \mathrm{e}^{-|x|}((x^2+3|x|+3)|u|^2 + 48|u'|^2 \\ + 3(-x^2+9|x|+9)|u''|^2 + 2(x^2+3|x|+3)|u'''|^2)\,\mathrm{d}x.\end{aligned} \tag{27}$$

Now (25) immediately gives (14).

(15) follows from (26) and a combination of (26) and (21).

The right side of (27) minus the right side of

$$\begin{aligned}2Q_3(u) \geq \frac{3}{4}|u(0)|^2 + \frac{3}{16}\int \mathrm{e}^{-|x|}((1+|x|)|u|^2 + 2(7+|x|)|u'|^2 \\ + (9+7|x|)|u''|^2 + 2(1+|x|)|u'''|^2)\,\mathrm{d}x,\end{aligned} \tag{28}$$

multiplied by 16 equals the nonnegative expression in (23). Thus (28) holds. Now (16) follows from (28) and (27).

The last statement in the proposition is a consequence of Lemma 14.

Remark From the viewpoint of (27), the negative term on the right makes it nontrivial that $Q_3(u) \geq 0$. The proof of Lemma 14 shows that (27) minus the integral $\int |x|^s e^{-|x|}|u^{(j)}(x)|^2 dx$, multiplied by any positive number, can be negative if $s > 1$ and $j \geq 0$.

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