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Landau and Kolmogoroff Type Polynomial Inequalities

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Let $0<j<m\leq n$ be integers. Denote by $\|\cdot\|$ the norm $\|f\|^2=\int_{-\infty}^{\infty}f^2(x)\exp(-x^2)\,\mathrm{d}x$. For various positive values of A and B we establish Kolmogoroff type inequalities

$$\|f^{(j)}\|^2\leq\frac{A\|f^{(m)}\|+B\|f\|}{A\theta_k+B\mu_k},$$

with certain constants θ_k e μ_k, which hold for every $f\in\pi_n$ (π_n denotes the space of real algebraic polynomials of degree not exceeding n).

For the particular case $j=1$ and $m=2$, we provide a complete characterisation of the positive constants A and B, for which the corresponding Landau type polynomial inequalities

$$\|f'\|\leq\frac{A\|f''\|+B\|f\|}{A\theta_k+B\mu_k}$$

hold. In each case we determine the corresponding extremal polynomials for which equalities are attained.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\|f\|_{\infty,[a,b]} = \sup_{a\le x\le b}|f(x)|$. In 1913, Landau [4] proved that $\|f'\|_{\infty,[0,1]} \le 4$ for every $f \in C^2[0,1]$, for which $\|f\|_{\infty,[0,1]} = 1$ and $\|f''\|_{\infty,[0,1]} = 4$. Kolmogoroff [3] proved, for sufficiently smooth functions, inequalities of the form

$$\|f^{(j)}\|_{\infty,[0,1]} \le K(m,j)\|f^{(m)}\|_{\infty,[0,1]}^{j/m}\|f\|_{\infty,[0,1]}^{1-(j/m)}, \tag{1}$$

with the best constant $K(m,j)$ and determined the functions for which inequality in (1) is attained. These extremal functions are perfect splines and in none of the cases algebraic polynomials. On the other hand, the classical A. Markov's inequality [5]

$$\|p'\|_{\infty,[-1,1]} \le n^2\|p\|_{\infty,[-1,1]}, \quad p \in \pi_n,$$

and its extension,

$$\|p^{(k)}\|_{\infty,[-1,1]} \le \frac{1}{(2k-1)!!}\prod_{i=0}^{k-1}(n^2 - i^2)\|p\|_{\infty,[-1,1]}, \quad 1 \le k \le n, \;\; p \in \pi_n,$$

given by V. Markov [6], are typical examples of inequalities connecting norms of derivatives of different orders of polynomials.

These facts motivated some author [1,8] to look for polynomial analogues of Landau's and Kolmogoroff's inequalities. In particular, Varma [8] established a sharp Landau type inequality and in a recent paper Bojanov and Varma [1] proved a Kolmogoroff type polynomial inequality for the weighted norm $\|f\|^2 = \int_{-\infty}^{\infty} f^2(x)\exp(-x^2)\,dx$. The extremal polynomials for which the inequalities in [8] and [1] reduce to equalities are the classical Hermite polynomials $H_n(x)$, orthogonal on $(-\infty,\infty)$ with respect to the weight function $\exp(-x^2)$.

In this paper we suggest a somehow more systematic approach than the one developed in [1], which allows us to establish the following Kolomogoroff type weighted polynomial inequalities.

THEOREM 1 *Let $j < m \le n$ be the positive integers and A and B positive constants.*

(i) *If*

$$\frac{A}{B} \le 2^{-m} \frac{(n-m)!}{n!} \frac{j}{m-j}, \tag{2}$$

then

$$\|f^{(j)}\|^2 \le \frac{A\|f^{(m)}\|^2 + B\|f\|^2}{A2^{m-j}(n-j)![(n-m)!]^{-1} + B2^{-j}(n-j)!(n!)^{-1}} \tag{3}$$

for every $f \in \pi_n$. Moreover, equality is attained if and only if $f(x)$ is a constant multiple of $H_n(x)$.

(ii) *If*

$$\frac{2^{-m}}{(m+1)!} \frac{j}{(m-j)} \le \frac{A}{B} \le 2^{-m} \frac{j}{m!(m-j)},$$

then

$$\|f^{(j)}\|^2 \le \frac{A\|f^{(m)}\|^2 + B\|f\|^2}{A2^{m-j}(m-j)! + B2^{-j}(m-j)!(m!)^{-1}} \tag{4}$$

for every $f \in \pi_n$. Moreover, equality is attained if and only if $f(x)$ is a constant multiple of $H_m(x)$.

(iii) *If*

$$\frac{A}{B} > 2^{-m} \frac{j}{m!(m-j)},$$

then

$$\|f^{(j)}\|^2 \le \frac{A\|f^{(m)}\|^2 + B\|f\|^2}{B2^{-j}(m-j-1)![(m-1)!]^{-1}} \tag{5}$$

for every $f \in \pi_n$. Moreover, equality is attained if and only if $f(x)$ is a constant multiple of $H_{m-1}(x)$.

(iv) *If $A/B = 2^{-m}j/((m-j)m!)$, then the inequalities* (4) *and* (5) *coincide and they hold for every $f \in \pi_n$. In this case equality is attained if and only if $f(x)$ is any linear combination of $H_{m-1}(x)$ and $H_m(x)$.*

As an immediate consequence of Theorem 1(i) we obtain

COROLLARY 1 *Let*

$$\alpha \leq \frac{(n-m)!j}{2^{m-j}(n-j)!m}.$$

Then the inequality

$$\|f^{(j)}\|^2 \leq \alpha\|f^{(m)}\|^2 + \left\{2^j\binom{n}{j}j! - \alpha 2^m \binom{n}{m} m!\right\}\|f\|^2$$

holds for every $f \in \pi_n$. *Moreover, equality is attained if and only if* $f(x)$ *is a constant multiple of* $H_n(x)$.

This is exactly the result of Bojanov and Varma [1] mentioned above. In this case $j=1$ and $m=2$ we provide a complete characterisation of the positive constants A and B, for which the corresponding Landau type polynomial inequalities hold.

THEOREM 2 *Let* A *and* B *be positive constants.*

(i) *If* $0 < A/B < (4n(n-1))^{-1}$, *then*

$$\|f'\|^2 \leq \frac{A}{2A(n-1)+B(2n)^{-1}}\|f''\|^2 + \frac{B}{2A(n-1)+B(2n)^{-1}}\|f\|^2, \tag{6}$$

for every $f \in \pi_n$. *Moreover, equality is attained if and only if* $f(x) = cH_n(x)$, *where* c *is a constant.*

(ii) *If* $(4k(k+1))^{-1} < A/B < (4k(k-1))^{-1}$, *where* $k \in \mathbb{N}$, $2 < k < n-1$, *then*

$$\|f'\|^2 \leq \frac{A}{2A(k-1)+B(2k)^{-1}}\|f''\|^2 + \frac{B}{2A(k-1)+B(2k)^{-1}}\|f\|^2, \tag{7}$$

for every $f \in \pi_n$. *Moreover, equality is attained if and only if* $f(x) = cH_k(x)$, *where* c *is a constant.*

(iii) *If* $1/8 < A/B < \infty$, *then*

$$\|f'\|^2 \leq \frac{2A}{B}\|f''\|^2 + 2\|f\|^2, \tag{8}$$

for every $f \in \pi_n$. *Moreover, equality is attained if and only if* $f(x) = cH_1(x)$.

(iv) *If* $A/B = (4k(k+1))^{-1}$ *for some integer* k, *then the inequalities*

$$\|f'\|^2 \leq \frac{A}{2A(k-1)+B(2k)^{-1}}\|f''\|^2 + \frac{B}{2A(k-1)+B(2k)^{-1}}\|f\|^2, \tag{9}$$

and

$$\|f'\|^2 \leq \frac{A}{2Ak+B[2(k+1)]^{-1}}\|f''\|^2 + \frac{B}{2Ak+B[2(k+1)]^{-1}}\|f\|^2, \tag{10}$$

coincide and they hold for every $f \in \pi_n$. *In this case equality in* (9) *and* (10) *is attained if and only if* $f(x)$ *is any linear combination of* $H_k(x)$ *and* $H_{k+1}(x)$.

Setting $B = 4n^2A$, in Theorem 3(i) we obtain the inequality

$$\|f'\|^2 \leq \frac{1}{2(2n-1)}\|f''\|^2 + \frac{2n^2}{2n-1}\|f\|^2, \quad f \in \pi_n,$$

where equality is attained only for the polynomials $f(x)$ that are constant multiples of $H_n(x)$. This is nothing but Varma's result [8].

2. A PRELIMINARY RESULT

Our idea is to study, for any given integers j, m, n, $0 < j < m \leq n$, and positive constants A and B, the extremal problem

$$\min\left\{\frac{A\|f^{(m)}\|^2 + B\|f\|^2}{\|f^{(j)}\|^2} : f \in \pi_n, f(x) \neq 0\right\},$$

where the objective function $F(A,B,f)=(A\|f^{(m)}\|^2+B\|f\|^2)/\|f^{(j)}\|^2$ depends on the parameters A and B. Denote by $f_*\in\pi_n$ the extremal polynomial, that is, the polynomial for which the minimum of $F(A,B,f)$ is attained. Thus we define

$$F(A,B)=F(A,B,f_*)=\min\{F(A,B,f):\ f\in\pi_n,\ f(x)\neq 0\}.$$

Let the sequences $\{\mu_i\}_{i=j}^n$, $\{\theta_i\}_{i=m}^n$ and $\{\gamma_k\}_{k=j}^n$ be defined by

$$\begin{aligned}&\mu_i=2^{-j}\frac{(i-j)!}{i!},\qquad \theta_i=2^{m-j}\frac{(i-j)!}{(i-m)!},\\ &\gamma_k=B\mu_k,\quad k=j,\dots,m-1,\\ &\gamma_k=A\theta_k+B\mu_k,\quad k=m,\dots,n.\end{aligned}$$

LEMMA 1 *For any given integers $j<m\le n$ and positive constants A and B*

$$F(A,B)=\min_{j\le i\le n}\gamma_i:=\gamma_k. \tag{11}$$

Proof We need two basic properties of the Hermite polynomials H_n (cf. (5.5.1) and (5.5.10) in [7]):

$$\int_{-\infty}^{\infty}H_k(x)H_i(x)\exp(-x^2)\,\mathrm{d}x=\sqrt{\pi}2^k k!\delta_{ik}, \tag{12}$$

where δ_{ik} is the Kronecker delta, and

$$H_i'(x)=2iH_{i-1}(x). \tag{13}$$

In what follows a different normalisation of the Hermite polynomials will be used. Set $\tilde{H}_i(x)=c_iH_i(x)$, where

$$c_i=1\quad \text{for } i=0,\dots,j-1,$$

and

$$c_i=\left(\sqrt{\pi}2^{i+j}\frac{(i!)^2}{(i-j)!}\right)^{-1/2}\qquad \text{for } i=j,\dots,n.$$

Since $\{H_n\}$ are orthogonal, the polynomials $\tilde{H}_0(x), \dots, \tilde{H}_n(x)$ form a basis in π_n. Then every $f \in \pi_n$ can be uniquely represented as a linear combination $f(x) = \sum_{k=0}^{n} a_k \tilde{H}_k(x)$. Hence the orthogonality relation (12) and the definition of the polynomials $\tilde{H}_i(x)$ yield

$$\|f\|^2 = \int_{-\infty}^{\infty} \left(\sum_{i=0}^{n} a_i \tilde{H}_i(x) \right)^2 \exp(-x^2)\,\mathrm{d}x = \sum_{i=0}^{n} \mu_i a_i^2,$$

where $\mu_i = \sqrt{\pi} 2^i i!$ for $i = 0, \dots, j-1$ and μ_i for $i = j, \dots, n$ are defined above.

Similarly, the relations (12), (13) and the definition of $\tilde{H}_i(x)$ imply

$$\|f^{(j)}\|^2 = \sum_{i=j}^{n} a_i^2 c_i^2 \|H_i^{(j)}\|^2 = \sum_{i=j}^{n} a_i^2.$$

In the same manner we obtain

$$\|f^{(m)}\|^2 = \sum_{i=m}^{n} \theta_i a_i^2,$$

where θ_i, $i = m, \dots, n$, are defined above.

Thus the problem formulated in the beginning of this section reduces to the following one:

$$\min\left\{ \left(\sum_{i=0}^{n} B\mu_i a_i^2 + \sum_{i=m}^{n} (A\theta_i + B\mu_i) a_i^2 \right) \Big/ \sum_{i=j}^{m} a_i^2 :\ a_0, \dots, a_n \in \mathbb{R} \right\}.$$

Obviously the above minimum is attained for $a_0 = \cdots = a_{j-1} = 0$, that is,

$$F(A, B)$$
$$= \min\left\{ \sum_{i=j}^{n} B\mu_i a_i^2 + \sum_{i=m}^{n} (A\theta_i + B\mu_i) a_i^2 :\ a_j, \dots, a_n \in \mathbb{R},\ \sum_{i=j}^{n} a_i^2 = 1 \right\}.$$

Let $\bar{a} = (a_j, \ldots, a_n)$. Therefore our problem reduces to determine the minimum of $\bar{a}^t C \bar{a}$, subject to $\bar{a}^t \bar{a} = 1$, where C is the diagonal matrix

$$\mathrm{diag}(B\mu_j, \ldots, B\mu_{m-1}, A\theta_m + B\mu_m, \ldots, A\theta_n + B\mu_n).$$

By the Rayleigh–Ritz Theorem (cf. Theorem 4.2.2 on p. 176 in Horn and Johnson [2]), $F(A,B)$ is equal to the smallest eigenvalue of C, that is,

$$F(A,B) = \min\{\gamma_j, \ldots, \gamma_n\}.$$

Moreover, if $F(A,B) = \gamma_k$, the extremal polynomial $f_*(x)$, for which $F(A,B) = F(A,B,f_*)$, is a constant multiple of $H_k(x)$.

3. PROOFS OF THE THEOREMS

Proof of Theorem 1 The sequence μ is decreasing. Indeed,

$$\mu_i - \mu_{i+1} = 2^{-j}\left(\frac{(i-j)!}{i!} - \frac{(i-j+1)!}{(i+1)!}\right) = 2^{-j}\frac{(i-j)!j}{(i+1)!} \geq 0.$$

Then the smallest among the numbers $B\mu_j, \ldots, B\mu_{m-1}$ is $B\mu_{m-1}$. Thus, according to Lemma 1, we need to find the smallest γ_ν among $\gamma_m, \ldots, \gamma_n$ and to compare γ_ν with $B\mu_{m-1}$.

Consider the monotonicity of the sequence $\{\gamma_k\}_{k=m}^n$. Since

$$\gamma_{k+1} - \gamma_k = A(\theta_{k+1} - \theta_k) + B(\mu_k - \mu_{k+1}),$$

then (a) $\{\gamma_k\}_{k=m}^n$ is increasing if $A/B \geq (\mu_k - \mu_{k+1})/(\theta_{k+1} - \theta_k) =: S_k$ for $k = m, \ldots, n-1$ and (b) $\{\gamma_k\}_{k=m}^n$ is decreasing if $A/B \leq S_k$ for $k = m, \ldots, n-1$.

Straightforward calculations show that

$$S_k = \frac{\mu_k - \mu_{k+1}}{\theta_{k+1} - \theta_k} = 2^{-m}\frac{(k-m+1)!}{(k+1)!}\frac{j}{m-j},$$

and then $S_{k+1}/S_k = (k-m+2)/(k+2) < 1$. This means that $\{S_k\}$ is a decreasing sequence. Hence, if $A/B \geq S_m$, then, γ_k is increasing and then $\gamma_\nu = \gamma_m$. Thus, in this case we have

$$F(A,B) = \min_{j \leq k \leq n} \gamma_k = \min\{\gamma_{m-1}, \gamma_m\}.$$

In order to compare γ_{m-1} and γ_m, observe that $\gamma_{m-1} < \gamma_m$ if $(\mu_{m-1} - \mu_m)/\theta_m < A/B$ and $\gamma_{m-1} \geq \gamma_m$ otherwise. In view of the identity $(\mu_{m-1} - \mu_m)/\theta_m = 2^{-m}j/((m-j)m!)$ we can conclude:

(1) If $A/B \geq S_m$ and $A/B > 2^{-m}j/((m-j)m!)$, then $\gamma_{m-1} < \gamma_m$ and $F(A, B) = \gamma_{m-1}$.
(2) If $S_m \leq A/B < 2^{-m}j/((m-j)m!)$, then $\gamma_{m-1} > \gamma_m$ and $F(A, B) = \gamma_m$.

It is worth mentioning that the interval $[S_m, 2^{-m}j/((m-j)m!)]$ is not empty because the inequality $S_m < 2^{-m}j/((m-j)m!)$ is equivalent to the obvious one $m > 0$.

The latter cases (1) and (2) correspond to the statements (iii) and (ii) of Theorem 1.

The above observation (b) and the monotonicity of S_k imply that the sequence $\{\gamma_k\}_{k=m}^n$ is decreasing provided $A/B \leq S_{n-1}$. Hence, in this case we have $F(A, B) = \min\{\gamma_{m-1}, \gamma_n\}$. In order to compare γ_{m-1} and γ_n, note that $\gamma_{m-1} < \gamma_n$ if $(\mu_{m-1} - \mu_n)/\theta_n < A/B$ and $\gamma_{m-1} \leq \gamma_n$ otherwise.

In view of the identity

$$\frac{\mu_{m-1} - \mu_n}{\theta_n} = 2^{-m}\frac{(n-m)!}{(n-j)!}\frac{[(m-j-1)!n! - (n-j)!(m-1)!]}{(m-1)!n!},$$

we need a relation between the latter expression and A/B. On the other hand, the inequality

$$\binom{m}{j} < \binom{n}{j} \quad \text{for } j < m < n$$

yields

$$S_{n-1} < 2^{-m}\frac{(n-m)!}{(n-j)!}\frac{[(m-j-1)!n! - (n-j)!(m-1)!]}{(m-1)!n!},$$

which means that $(\mu_{m-1} - \mu_n)/\theta_n < S_{n-1}$. If

$$\frac{A}{B} \leq S_{n-1}$$

and

$$\frac{A}{B} < 2^{-m}\frac{(n-m)!}{(n-j)!}\frac{[(m-j-1)!n! - (n-j)!(m-1)!]}{(m-1)!n!},$$

then $\gamma_{m-1} > \gamma_n$ and $F(A,B) = \gamma_n$. This corresponds to the statement (i) of the theorem.

Proof of Theorem 2 Since $j=1$ and $m=2$, Lemma 1 shows that we need to determine

$$\min\{B\mu_1, A\theta_2 + B\mu_2, \dots, A\theta_n + B\mu_n\}.$$

In order to this, we shall find the smallest among the numbers $A\theta_2 + B\mu_2, \dots, A\theta_n + B\mu_n$, say γ_ν, and in each case we shall compare γ_ν to $B\mu_1$.

In what follows, up to the final observation in this proof, we shall assume that $2^m A + B = 1$. Then we have

$$\gamma_k = A\theta_k + B\mu_k = (B/k + (1-B)(k-1))/2 \quad \text{for } k = 2, \dots, n.$$

Define the function

$$g(x) = (B/x + (1-B)(x-1))/2 \quad \text{for } 2 \le x \le n.$$

Since $g(k) = \gamma_k$ for $k = 2, \dots, n$, then our problem reduces to investigate the behaviour of $g(x)$ when A and B belong to the segment $2^m A + B = 1$, $A, B > 0$. Note that $2g'(x) = -B/x^2 + (1-B) = 0$, if and only if $x = \pm(B/(1-B))^{1/2}$ and $g''(x) = B/x^3 > 0$ for $x > 0$. Hence $g(x)$ is convex on the positive half-line and it attains its absolute minimum there at $x = (B/(1-B))^{1/2}$. Thus, we can conclude that:

- If $\sqrt{B/(1-B)} < 2$, then $\gamma_{\min} = \gamma_2$.
- If $\sqrt{B/(1-B)} > n$, then $\gamma_{\min} = \gamma_n$.
- If $k \le \sqrt{B/(1-B)} < k+1$, where $2 \le k < n-1$, then $\gamma_\nu = \min\{\gamma_k, \gamma_{k+1}\}$.

In order to determine the smaller among γ_k and γ_{k+1}, observe that

$$\gamma_{k+1} < \gamma_k \quad \text{if } \frac{k(k+1)}{k^2+k+1} < B$$

and $\gamma_{k+1} \ge \gamma_k$ otherwise. It is clear that $\gamma_k = \gamma_{k+1}$ if and only if $B = k(k+1)/(k^2+k+1)$.

Set $y := (B/(1-B))^{1/2}$ for any $B, 0 \leq B < 1$. If $B = k(k+1)/(k^2+k+1)$ the point of minimum of $g(x)$ is

$$y_k := \sqrt{k(k+1)}.$$

Obviously $k < y_k < k+1$. Since the function $g(x)$ is convex, then $\gamma_\nu = \gamma_{k+1}$ if and only if $y_k < y < y_{k+1}$ and this conclusion holds for $k = 1, \dots, n-2$. The latter inequality is itself equivalent to

$$\frac{k(k+1)}{k^2+k+1} < B < \frac{(k+1)(k+2)}{k^2+3k+3}.$$

Let us compare, in each of these cases, γ_{k+1} to $\gamma_1 = B/2$.

For $B \in (0, 6/7)$, we need to compare γ_1 and γ_2. Since $\gamma_2 - \gamma_1 = (1 - 3B/2)/2$, then $\gamma_1 < \gamma_2$ for $0 < B < 2/3$, $\gamma_1 = \gamma_2$ for $B = 2/3$ and $\gamma_2 < \gamma_1$ for $2/3 < B < 6/7$.

Let now k be any integer, such that $2 \leq k \leq n$ and let

$$B \in \left(\frac{k(k+1)}{k^2+k+1}, \frac{(k+1)(k+2)}{k^2+3k+3} \right) = \Delta_k.$$

Since

$$\gamma_{k+1} - \gamma_1 = \frac{k}{2(k+1)}((k+1) - 2B(k+2)) \leq 0$$

if and only if $(k+1)/(k+2) < B$ and this latter inequality always holds for $B \in \Delta_k$, then we have $\gamma_{k+1} < \gamma_1$ for every $B \in \Delta_k$.

Finally, we have $\gamma_n < \gamma_1$ for every

$$B \in \left(\frac{n(n-1)}{n^2-n+1}, 1 \right) = \Delta_{n-1}$$

because $\gamma_n - \gamma_1 = (n-1)(n - B(n+1))/n < 0$ is equivalent to $n/(n+1) < B$ and obviously $n/(n+1) < n(n-1)/(n^2-n+1)$.

Recall that all considerations have been done under the restriction $2^m A + B = 1$. The restrictions $B \in \Delta_k$ can be easily transformed into equivalent restrictions for A/B. We omit this detail. The result is:

- If $1/8 < A/B < \infty$, then $\gamma_{\min} = \gamma_1$.
- If $(4(k+1)(k+2))^{-1} < A/B < (4k(k+1))^{-1}$, then $\gamma_{\min} = \gamma_{k+1}$, $k = 1, \dots, n-2$.
- If $0 < A/B < (4n(n-1))^{-1}$, then $\gamma_{\min} = \gamma_n$.

Our final observation is that we can remove the restriction $2^m A + B = 1$. Indeed, we have proved inequalities of the form

$$\|f'\| \leq \frac{A\|f''\| + B\|f\|}{A\theta_k + B\mu_k}.$$

The quotient on the right-hand side is homogeneous with respect to A, B, so this quotient has the same value for A, B and for dA, dB, whatever the positive constant d is.

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