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Inequalities for Eigenvalue Functionals

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We give sharp estimates for some eigenvalue functionals, and we indicate the optimal solutions.

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1 INTRODUCTION

Consider a vibrating string having density function $\rho(x)$ with fixed points. Its characteristic frequencies of vibration are determined by the eigenvalues $\lambda = \lambda(\rho)$ of the system

$$y'' + \lambda\rho(x)y = 0, \quad y(0) = y(l) = 0, \ 0 \leq x \leq l, \tag{1}$$

l being a positive real number. There will be an infinite sequence of positive eigenvalues $\lambda_1(\rho) \leq \lambda_2(\rho) \leq \cdots$ which increase without limit. M.G. Krein [4] has solved the following problem: Let $E(0, H, M)$ denote the class of all integrable functions ρ on $(0, l)$ such that $\int_0^l \rho(x)\,\mathrm{d}x = M$ and $0 \leq \rho(x) \leq H$ a.e. $x \in (0, l)$, where here and below H and M are given constants satisfying $Hl > M$. Then for all $\rho \in E(0, H, M)$ and each

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integer n he obtained

$$\frac{4n^2H}{M^2}X\left(\frac{M}{Hl}\right) \le \lambda_n(\rho) \le \frac{n^2\pi^2H}{M^2}, \tag{2}$$

where $X(t)$ is the smallest positive root of the equation $X^{1/2}tgX^{1/2} = t(1-t)^{-1}$. Estimates (2) are sharp for all n. For $n = 1$, the upper bound is attained only at the step function ρ_0 defined as

$$\rho_0(x) = \begin{cases} H & \text{if } x \in [0, M/(2H)], \\ 0 & \text{if } x \in (M/(2H), l - M/(2H)), \\ H & \text{if } x \in [l - M/(2H), l]. \end{cases} \tag{3}$$

Inequalities (2) are recently extented to some classes of differential equations [2]. Let for example $\lambda_n(q, \rho)$ denote the nth eigenvalue of the boundary-value problem

$$y'' - q(x)y + \lambda\rho(x)y = 0, \quad y(0) = y(l) = 0,$$

where $q(x) \in L^1(0, l)$ is nonnegative. Put $M' = \int_0^l q(x)\,\mathrm{d}x$. Then

$$\frac{4n^2H}{M^2}X\left(\frac{M}{Hl}\right) \le \lambda_n(q, \rho) \le \frac{n^2\pi^2H}{M^2}\left[\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{MM'}{n^2H\pi^2}}\right]^2.$$

In [1] and [3] the authors have studied the problem of determining the shape of the column hinged at its both extremities and having the smallest (largest) buckling load among all columns of length l and volume V. This problem is equivalent to that of finding a nonnegative function (*cross-sectional area*) $A(x)$ which minimizes (maximizes) the first eigenvalue of the following problem:

$$(A^\beta(x)y'')'' + \lambda y'' = 0, \tag{4}$$

$$y(0) = (A^\beta y'')(0) = 0, \quad y(l) = (A^\beta y'')(l) = 0, \tag{5}$$

under the condition that

$$\int_0^l A(x)\,\mathrm{d}x = V, \tag{6}$$

where $V > 0$ and β is a nonzero real number. It is proven that the infimum of $\lambda_1(A)$ over condition (6) is zero if β is outside the interval $[-1, 0)$. For each $\beta \in [-1, 0)$ the inequality

$$\lambda_1(A) \geq \Gamma_\beta l^{-2-\beta} V^\beta \tag{7}$$

holds for all A satisfying (6). Γ_β is a constant given by

$$\Gamma_\beta = (2+\beta)\left(\frac{1+\beta}{2+\beta}\right)^{1-\beta} \mathcal{B}^2\left(\frac{1}{2}, \frac{1}{2}+\frac{\beta}{2}\right),$$

if $\beta \in (-1, 0)$ and $\Gamma_{-1} = 4$. Here $\mathcal{B}$ is Euler's Beta function. In addition, there exists an optimal shape $\tilde{A}_\beta$ such that $\lambda_1(\tilde{A}_\beta) = \Gamma_\beta l^{-2-\beta} V^\beta$. For $\beta > 0$ and $\beta < -2$ they found that

$$\lambda_1(A) \leq C_\beta l^{-2-\beta} V^\beta \tag{8}$$

for all A satisfying (6), and there exists a best shape $\bar{A}_\beta$ such that $\lambda_1(\bar{A}_\beta) = \Gamma_\beta l^{-2-\beta} V^\beta$. In (8), C_β is also a constant depending only on β. Notice that if V is fixed and $l \to 0$ then $\lambda_1(\bar{A}_\beta) \to \infty$. For other works concerning extremal problems for eigenvalues see [2] and the references quoted there.

2 SHARP INEQUALITIES

Suppose a number of strings (and perhaps some rods as well) are all vibrating together and that they each have density function $\rho_i(x)$, length l_i and eigenvalues $\lambda_n(\rho_i)$. The fundamental frequencies of vibration Λ and the total mass of the system are determined by

$$\Lambda = \min_i\{\lambda_1(\rho_i)\}, \quad M = \sum_i \int_0^{l_i} \rho_i(x)\,dx.$$

One might maximize Λ subject to a given mass constraint. In a similar way, consider a single large load, being supported by several columns of length l_i and total volume V. The critical buckling load of each column is determined by an eigenvalue problem similar to (4) and (5).

To minimize the load-carrying capacity of the structure, one would want to minimize the sum† of all the first eigenvalues for each of the systems. To solve the first problem we shall use two lemmas. The first one is based on the convexity of the functional $x \mapsto x^p$.

LEMMA 1 *Let $x_1, \ldots, x_k$ be positive numbers such that $x_1 + \cdots + x_k = 1$. If $p \geq 1$ or $p < 0$, then the inequality*

$$x_1^p + \cdots + x_k^p \geq k^{1-p}$$

holds. If $0 < p \leq 1$, then

$$x_1^p + \cdots + x_k^p \leq k^{1-p}.$$

Moreover, in each case the extremum of the function $x_1^p + \cdots + x_k^p$ is attained only at the point $x_1 = \cdots = x_k = 1/k$ if $p \neq 1$.

A system of k strings $\{\rho_i\}_{i=1}^k$ is said to be admissible if each member $\rho_i \in L^1(0, l_i)$, $l_i < l$, $0 \leq \rho_i(x) \leq H$ a.e. $x \in (0, l_i)$ and $\sum_i l_i = l$ and $\sum_i \int_0^{l_i} \rho_i \, \mathrm{d}x = M$.

LEMMA 2 *For each real $\gamma \geq \frac{1}{2}$ and for all admissible system $\{\rho_i\}_{i=1}^k$, we have*

$$\sum_1^k \frac{1}{\lambda_1(\rho_i)^\gamma} \geq k^{1-2\gamma}\left(\frac{M^2}{\pi^2 H}\right)^\gamma.$$

Equality is attained at every system $\{\rho_i\}_{i=1}^k$ whose each member ρ_i is of the form

$$\rho_i(x) = \begin{cases} H & \text{if } x \in [0, M/(2Hk)], \\ 0 & \text{if } x \in (M/(2Hk), l_i - M/(2Hk)), \\ H & \text{if } x \in [l_i - M/(2Hk), l_i]. \end{cases}$$

Proof From (2), it follows that for all ρ_i we have

$$\lambda_1(\rho_i) \leq \pi^2 H\left(\int_0^{l_i} \rho_i \, \mathrm{d}x\right)^{-2},$$

† In view of the remark given in the end of the last section, we cannot consider the problem of maximizing this sum.

so for each $\alpha > 0$, we obtain

$$\left(\int_0^{l_i} \rho_i \, \mathrm{d}x\right)^{2\gamma} \le \pi^{2\gamma} H^\gamma \lambda_1(\rho_i)^{-\gamma}.$$

Hence

$$\sum_1^k \left(\int_0^{l_i} \rho_i \, \mathrm{d}x\right)^{2\gamma} \le \pi^{2\gamma} H^\gamma \sum_1^k \frac{1}{\lambda_1(\rho_i)^\gamma}. \tag{9}$$

If $\gamma \ge \frac{1}{2}$, then by the first part of Lemma 1, the left hand side of (9) is greater than $M^{2\gamma} k^{1-2\gamma}$ and therefore

$$M^{2\gamma} k^{1-2\gamma} \le \pi^{2\gamma} H^\gamma \sum_1^k \frac{1}{\lambda_1(\rho_i)^\gamma},$$

which proves the inequality in the lemma. If $\gamma > \frac{1}{2}$, then by Lemma 1, (9) becomes equality only if $\int_0^{l_i} \rho_i(x)\, \mathrm{d}x = M/k$, $i = 1, \dots, k$.

THEOREM 1 *Let* $\{\rho_i\}_{i=1}^k$ *be an arbitrary admissible system. Put* $\Lambda = \min_i \{\lambda_1(\rho_i)\}$. *Then we have*

$$\Lambda \le \frac{k^2 \pi^2 H}{M^2},$$

and equality is reached only by the optimal systems indicated in Lemma 2.

Proof We have

$$\frac{1}{\Lambda} \ge \frac{1}{k} \sum_1^k \frac{1}{\lambda_1(\rho_i)}$$

$$\ge \frac{1}{k}\left(\frac{M^2 k^{-1}}{\pi^2 H}\right)$$

by virtue of Lemma 2 applied for $\gamma = 1$. Consequently, $\Lambda \le \pi^2 H/(M/k)^2$.

Let now $\{A_i\}_{i=1}^k$ denote a system of k columns, each one is of length l_i and hinged at both extremities. The system is said to be admissible if $\sum_i l_i = l$ and $\sum_i \int_0^{l_i} A_i \, \mathrm{d}x = V$, where l and V are given constants.

To give answer to the second problem, we shall use the following lemma:

LEMMA 3 *Let r, s be nonzero real numbers. Denote by E the set of all vectors $X=(x_1, x_2, \cdots, x_n) \in (]0,1[)^n$ satisfying $\sum_{i=1}^n x_i < 1$ and define the function $F: E \times E \mapsto R$ by*

$$\begin{aligned} F(X, Y) =& x_1^r y_1^s + x_2^r y_2^s + \cdots + x_n^r y_n^s \\ &+ (1 - x_1 - x_2 - \cdots - x_n)^r (1 - y_1 - y_2 - \cdots - y_n)^s. \end{aligned}$$

If $rs < 0$ and $r + s \geq 1$, then the function F attains its minimal value $F_{\min} = (n+1)^{1-(r+s)}$ when $X = Y = (n+1)^{(-1)}(1, 1, \ldots, 1)$. If $rs > 0$ and $0 < r + s \leq 1$, then F attains its maximal value $F_{\max} = (n+1)^{1-(r+s)}$ when $X = Y = (n+1)^{(-1)}(1, 1, \ldots, 1)$. Furthermore, in each case the extremum point is unique if $r + s \neq 1$.

The idea of the proof is to show that F attains its minimal (maximal) value inside a square $[\delta, 1-\delta] \times [\delta, 1-\delta]$, where δ is a small positive number, and next to establish the standard necessary conditions for optimality. We mention that if r and s do not satisfy the first (second) conditions of the lemma, then F is not bounded below (above) i.e. it can take arbitrary small (large) positive values. Note finally that in general the extremum point in the lemma is not unique if $r + s = 1$. Indeed, for this case, one may easily verify that the function F achieves its maximum $F_{\max} = 1$ at every couple $(X, Y) \in E \times E$ satisfying $x_i = y_i$ for $i = 1, \ldots, n$.

THEOREM 2 *For each $\beta \in [-1, 0)$ and for all admissible system $\{A_i\}_{i=1}^k$, we have*

$$\sum_1^k \lambda_1(A_i) \geq \frac{k^3 \Gamma_\beta V^\beta}{l^{2+\beta}}.$$

Moreover, equality holds only when $l_i = l/k$ and $A_i(x) = \tilde{A}_\beta(kx)$ for all $x \in (0, l/k)$, $i = 1, \ldots, k$, where $\tilde{A}_\beta$ is the shape of the weakest column subjected to condition (6).

Proof Let $\{A_i\}_{i=1}^k$ be an admissible system of columns. Then (7) reads for each i

$$\lambda_1(A_i) \geq \Gamma_\beta l_i^{-2-\beta} \left(\int_0^{l_i} A_i \, \mathrm{d}x \right)^\beta. \tag{10}$$

Remark that as $\beta \in [-1, 0)$, we cannot use the first part of Lemma 3 to obtain a lower bound for $\sum_i \lambda_1(A_i)$. We shall then proceed as follows: Inequality (10) may be written as

$$\frac{\Gamma_\beta}{\lambda_1(A_i)} \leq l_i^{2+\beta} \left(\int_0^{l_i} A_i \, \mathrm{d}x \right)^{-\beta}$$

or

$$\frac{\Gamma_\beta^{1/2}}{\lambda_1(A_i)^{1/2}} \leq l_i^{1+\beta/2} \left(\int_0^{l_i} A_i \, \mathrm{d}x \right)^{-\beta/2}.$$

By the second part of Lemma 3, we get

$$\Gamma_\beta^{1/2} \sum_i \frac{1}{\lambda_1(A_i)^{1/2}} \leq l^{1+\beta/2} V^{-\beta/2}. \tag{11}$$

From Lemma 1 and (11), we may deduce that

$$\left(\sum_i \lambda_1(A_i) \right)^{1/2} \geq k^{1+1/2} \left(\sum_i \frac{1}{\lambda_1(A_i)^{1/2}} \right)^{-1} \geq \frac{k^{1+1/2} \Gamma_\beta^{1/2} V^{\beta/2}}{l^{1+\beta/2}},$$

and hence

$$\sum_i \lambda_1(A_i) \geq k^3 \Gamma_\beta V^\beta l^{-2-\beta}.$$

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