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The Generalized Hardy Operator with Kernel and Variable Integral Limits in Banach Function Spaces

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Let we have an integral operator

$$Kf(x) := v(x) \int_{a(x)}^{b(x)} k(x,y)u(y)f(y)\,\mathrm{d}y \quad \text{for } x > 0$$

where a and b are nondecreasing functions, u and v are non-negative and finite functions, and $k(x,y) \geq 0$ is nondecreasing in x, nonincreasing in y and $k(x,z) \leq D[k(x,b(y)) + k(y,z)]$ for $y \leq x$ and $a(x) \leq z \leq b(y)$. We show that the integral operator $K\colon X \to Y$ where X and Y are Banach functions spaces with l-condition is bounded if and only if $A < \infty$. Where $A := A_0 + A_1$ and

$$A_0 := \sup_{x \leq y, a(y) \leq b(x)} \|\chi_{(x,y)}(.)v(.)k(.,b(x))\|_Y \|\chi_{(a(y),b(x))}u\|_{X'}$$
$$A_1 := \sup_{x \leq y, a(y) \leq b(x)} \|\chi_{(x,y)}v\|_Y \|\chi_{(a(y),b(x))}(.)K(x,.)u(.)\|_{X'}.$$

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1 INTRODUCTION

Let X and Y be two Banach function spaces on (c, d) and $\mathbb{R}$ respectively. We define the general Hardy operator

$$Kf(x) := v(x) \int_{a(x)}^{b(x)} k(x, y)u(y)f(y)\, \mathrm{d}y \quad \text{for } x \in \mathbb{R} \tag{1.1}$$

where a, b are nondecreasing functions on $\mathbb{R}$, $-\infty \leq c \leq a(x) \leq b(x) \leq d \leq \infty$, and v and u are non-negative measurable and finite functions a.e. on $\mathbb{R}$ and (c, d). The kernel $k(x, y) \geq 0$ is defined a.e. on $\{(x, y);\, x \in \mathbb{R},\, a(x) \leq y \leq b(x)\}$ and satisfies the following conditions:

(i) it is nondecreasing in x and nonincreasing in y;

(ii) $k(x, z) \leq D[k(x, b(y)) + k(y, z)]$ for every $y \leq x$ and $a(x) \leq z \leq b(y)$, where the constant $D > 1$ is independent of x, y, z. (1.2)

In this paper we describe the necessary and sufficient condition for the boundedness of the operator (1.1) in Banach function spaces.

This paper extends results of Lomakina and Stepanov [3] and Opic and Kufner [4]. In these papers the operator (1.1) was characterized for $a(x) = 0$ and $b(x) = x$.

Sections 2 and 3 contain the definitions, formulations of the main results and some comments. In Section 4 we treat the simpler case when the kernel $k(x, y)$ is equal to 1 and the spaces X, Y satisfy the l-condition. We use this result in Section 5 to deal with the general kernel satisfying (1.2).

2 DEFINITIONS

In this section we recall the definition and some basic properties of the Banach function spaces. In what follows $\mathcal{M}(\Omega)$ will be the set of all measurable functions on Ω, where Ω is any measurable subset of $\mathbb{R}$.

DEFINITION 2.1 A normed linear space $(X, \|.\|_X)$ on Ω is called a Banach function space (BFS) on Ω if the following conditions are satisfied:

(2.1) the norm $\|f\|_X$ is defined for all $f \in \mathcal{M}(\Omega)$ and $f \in X$ if and only if $\|f\|_X < \infty$;

(2.2) $\|f\|_X = \|\,|f|\,\|_X$ for every $f \in \mathcal{M}(\Omega)$;

(2.3) if $0 \leq f_n \nearrow f$ a.e. in Ω then $\|f_n\|_X \nearrow \|f\|_X$;
(2.4) if $|E| < \infty$, $E \subset \Omega$, then $\chi_E \in x$;
(2.5) for every set E, $|E| < \infty$, $E \subset \Omega$, there exists a positive constant C_E such that $\int_E |f(x)| \mathrm{d}x \leq C_E \|f\|_X$.

By l we denote a Banach sequence space (BSS), which means that the axioms (2.1)–(2.5) are fulfilled with respect to the counting measure and $\{e_k\}$ denotes the standard basis in l.

Recall that the condition (2.3) immediately yields the following property:

(2.6) if $0 \leq f \leq g$ then $\|f\|_X \leq \|g\|_X$.

DEFINITION 2.2 The set

$$X' = \left\{ f; \left| \int_\Omega fgv \right| < \infty \text{ for every } g \in X \right\},$$

equipped with the norm

$$\|f\|_{X'} = \sup\left\{ \left| \int_\Omega fgv \right| ; \|g\|_X \leq 1 \right\},$$

is called the associate space of X. It is known from Bennett and Sharpley [1] that $X'' = X$ and that X' is again a BFS.

Let T be a linear operator from a BFS X into a BFS Y. Then T' is an associate operator to the operator T if $\int_\Omega (Tf)g = \int_\Omega f(T'g)$ for all $f \in X$ and $g \in Y$.

LEMMA 2.3 (Bennett and Sharpley [1]) *Let T be a linear operator from a BFS X into a BFS Y. Then $\|Tf\|_Y \leq C\|f\|_X$ for all $f \in X$ with a finite positive constant C, if and only if $\|T'g\|_{X'} \leq C\|g\|_{Y'}$ for all $g \in Y'$.*

Moreover $\|T\|_{X \to Y} = \|T'\|_{Y' \to X'}$.

DEFINITION 2.4 (Lomakina and Stepanov [3]) Given a BFS X and a BSS l, X is l-concave, if for any sequence of disjoint intervals $\{J_k\}$ such that $\cup J_k = \Omega$, and for all $f \in x$

$$\left\| \sum_k e_k \|\chi_{J_k} f\|_X \right\|_l \leq d_1 \|f\|_X,$$

where is d_1 a finite positive constant independent on $f \in X$ and $\{J_k\}$. Analogously, a BFS Y is said to be l-convex, if for any sequence of disjoint intervals $\{I_k\}$, $\cup I_k = \Omega$ and for all $g \in Y$

$$\|g\|_Y \leq d_2 \left\| \sum_k e_k \|\chi_{I_k} g\|_Y \right\|_l$$

with a finite positive constant d_2 independent on $g \in Y$ and $\{I_k\}$.

We say, that BFS X, Y satisfy the l-condition, if there exist a BSS l such that X is l-concave and Y is l-convex simultaneously.

LEMMA 2.5 (Lomakina and Stepanov [3]) *Let Y be a l-convex BFS. Then Y' is an l'-concave BFS and*

$$\left\| \sum_k e_k \|\chi_{I_k} f\|_{Y'} \right\|_{l'} \leq d_2 \|f\|_{Y'}$$

for all $f \in Y'$ and $\{I_k\}$, $\cup I_k = \Omega$.

3 MAIN RESULTS

Assume X and Y are two BFS on (c, d) and $\mathbb{R}$, respectively. Then we denote

$$A_0 := \sup_{x \leq y, a(y) \leq b(x)} \|\chi_{(x,y)}(.)v(.)k(., b(x))\|_Y \|\chi_{(a(y),b(x))} u\|_{X'}$$
$$A_1 := \sup_{x \leq y, a(y) \leq b(x)} \|\chi_{(x,y)} v\|_Y \|\chi_{(a(y),b(x))}(.)k(x, .)u(.)\|_{X'}$$

and $A := A_0 + A_1$.

Note that $A_0 = A_1$ if $k(x, y) = 1$.

THEOREM 3.1 *Let X and Y be two BFS on (c, d) and $\mathbb{R}$, respectively, satisfying the l-condition. Let K be the integral operator of the form* (1.1) *with kernel $k(x, y) \geq 0$ satisfying* (1.2). *Then $K: X \to Y$ is bounded, if and only if, A is finite. Moreover*

$$\|K\|_{X \to Y} \asymp A.$$

To prove Theorem 3.1 we need a corresponding result for the general Hardy operator with kernel $k(x,y)=1$.

Let

$$Hf(x) := v(x)\int_{a(x)}^{b(x)} u(y)f(y)\,\mathrm{d}y \tag{3.1}$$

where $-\infty \le c \le a(x) \le b(x) \le d \le \infty$ are nondecreasing functions on $\mathbb{R}$, v and u are real measurable and finite functions a.e. on $\mathbb{R}$ and (α,β), respectively.

THEOREM 3.2 *Let X and Y be two BFS on (c,d) and $\mathbb{R}$, respectively, satisfying the l-condition, and let H be the operator defined by* (3.1). *Then $H: X \to Y$ is bounded, if and only if,*

$$A_H := \sup_{x\le y,a(y)\le b(x)} \|\chi_{(x,y)}v\|_Y \|\chi_{(a(y),b(x))}u\|_{X'} < \infty.$$

Moreover $A_H \asymp \|H\|_{X\to Y}$.

4 BOUNDEDNESS OF THE OPERATOR H

In this section we prove Theorem 3.2. At first we prove a lemma.

DEFINITION 4.1 Let v be a non-negative measurable function on an interval (α,β) where $-\infty \le \alpha < \beta \le \infty$. Let $c \in \mathbb{R}$, let $-\infty \le a(x) \le c \le b(x) \le \infty$ be nondecreasing functions, and let u be a non-negative measurable function on (e,d) where $e := \liminf_{x\to\alpha} a(x)$ and $d := \limsup_{x\to\beta} b(x)$. Then we define

$$H_b f(x) := v(x)\int_c^{b(x)} u(t)f(t)\,\mathrm{d}t$$

for every measurable function f on (c,d), and

$$H_a f(x) := v(x)\int_{a(x)}^{c} u(t)f(t)\,\mathrm{d}t$$

for every measurable function f on (e,c).

LEMMA 4.2 *Let X and Y be two BFS on (e, d) and (α, β), respectively, satisfying the l-condition. Then $H_b: X \to Y$ is bounded, if and only if,*

$$A_b := \sup_{\alpha \leq x \leq \beta} \|v\chi_{(x,\beta)}\|_Y \|u\chi_{(c,b(x))}\|_{X'} < \infty.$$

Moreover

$$\|H_b\|_{X\to Y} \asymp A_b.$$

Also $H_a: X \to Y$ is bounded, if and only if,

$$A_a := \sup_{\alpha \leq x \leq \beta} \|v\chi_{(x,\beta)}\|_Y \|u\chi_{(a(x),c)}\|_{X'} < \infty.$$

Moreover

$$\|H_a\|_{X\to Y} \asymp A_a.$$

Proof We will give the proof only for H_b. The proof for H_a is similar.

Necessity Given $x \in (\alpha, \beta)$ and $f \in X$ such that $fu \geq 0$, we have

$$\begin{aligned} \left\| v(.) \int_c^{b(.)} u(t)f(t)\,\mathrm{d}t \right\|_Y &\geq \left\| v(.)\chi_{(x,\beta)}(.) \int_c^{b(.)} u(t)f(t)\,\mathrm{d}t \right\|_Y \\ &\geq \left\| v(.)\chi_{(x,\beta)}(.) \int_c^{b(x)} u(t)f(t)\,\mathrm{d}t \right\|_Y \\ &= \|v\chi_{(x,\beta)}\|_Y \int_c^{b(x)} |u(t)f(t)|\,\mathrm{d}t. \end{aligned}$$

Taking the supremum over all such f and $x \in (\alpha, \beta)$ we obtain

$$\|H_b\|_{X\to Y} \geq \|v\chi_{(x,\beta)}\|_Y \|u\chi_{(c,b(x))}\|_{X'}$$

and so,

$$\|H_b\|_{X\to Y} \geq A_b.$$

Sufficiency If $A_b = \infty$ then $\|H\|_{X\to Y} \leq A_b$. If $A_b = 0$ then $\|H_b\|_{X\to Y} = 0$.

Let $0 < A_b < \infty$. Choose $f \in X$ such that $\|f\|_X = 1$. Define $C_i = \{t;\ t \in (\alpha, \beta),\ \int_c^{b(t)} |fu| \geq 2^i\}$, $D_i = C_i \backslash C_{i+1}$ and $E_i = \{x;\ x \in (c, d),\ b(x) \in C_i\}$, $B_i = E_i \backslash E_{i+1}$. Then $|(c, d) \backslash \cup_{i \in \mathbf{Z}} B_i| = 0$ and $|(\alpha, \beta) \backslash \cup_{i \in \mathbf{Z}} D_i| = 0$ and we have

$$\begin{aligned}
\int_\alpha^\beta gH_bf &\leq \sum_{i \in \mathbf{Z}} \int_{D_i} 2^{i+1} gv \leq \sum_{i \in \mathbf{Z},\, |D_i|>0} 2^{i+1} \|v\chi_{D_i}\|_Y \|g\chi_{D_i}\|_{Y'} \\
&\leq \sum_{i \in \mathbf{Z},\, |D_i|>0} 2^{i+1} \|v\chi_{C_i}\|_Y \|g\chi_{D_i}\|_{Y'} \\
&\leq \sum_{i \in \mathbf{Z},\, |D_i|>0} 2^{i+1} \frac{A_b}{\|u\chi_{(c, \sup E_{i-1})}\|_{X'}} \|g\chi_{D_i}\|_{Y'} \\
&\quad \left(\text{using } 2^{i-1} \leq \int_c^d |f\chi_{B_{i-1}}||u| \right. \\
&\quad \left. \leq \|f\chi_{B_{i-1}}\|_X \|u\chi_{(c, \sup B_{i-1})}\|_{X'}\right) \\
&\leq \sum_{i \in \mathbf{Z},\, |D_i|>0} A_b 2^{i+1} \frac{1}{2^{i-1}} \|f\chi_{B_{i-1}}\|_X \|g\chi_{D_i}\|_{Y'} \\
&\quad (\text{using Holders inequality and } l\text{-condition}) \\
&\leq 4A_b \left\| \sum_{i \in \mathbf{Z},\, |D_i|>0} e_i \|f\chi_{(B_{i-1})}\|_X \right\|_l \left\| \sum_{i \in \mathbf{Z},\, |D_i|>0} e_i \|g\chi_{D_i}\|_{Y'} \right\|_{l'} \\
&\leq 4d_1 d_2 A_b \|f\|_X \|g\|_{Y'}.
\end{aligned}$$

Then we have

$$\|H_bf\|_Y = \sup_{\|g\|_{Y'} \leq 1} \int_a^b gH_bf \leq 4d_1 d_2 A_b \|f\|_X.$$

Now we prove Theorem 3.2.

Proof of Theorem 3.2 Necessity Let $f \in X$ be such that $fu \geq 0$ and $\|f\|_X = 1$, and let x, y be such that $\alpha \leq x \leq y \leq \beta$ and $b(x) \geq a(y)$. Then

$$\begin{aligned}
\|Hf\|_Y &\geq \|v(.)\chi_{(x,y)}(.) \int_{a(.)}^{b(.)} u(t)f(t)\, \mathrm{d}t\|_Y \\
&= \|v\chi_{(x,y)}\|_Y \int_{a(y)}^{b(x)} u(t)f(t)\, \mathrm{d}t.
\end{aligned}$$

Taking the supremum over all such x, y and f we obtain

$$\|H\|_{X\to Y} \geq \|v\chi_{(x,y)}\|_Y \|u\chi_{(a(y),b(x))}\|_{X'}.$$

Sufficiency Define $M := \{(x,t);\ x \in \mathbb{R},\ a(x) < t < b(x)\}$. M is a measurable set. If $|M| = 0$ then it is easy to see that $\|H\|_{X\to Y} = 0$.

Suppose that $|M| > 0$. We set $M_y := \{x;\ (x,t) \in M,\ t = y\}$, $y \in \mathbb{R}$, and $P := \{y;\ (x,t) \in M,\ |M_y| > 0\}$. Then $P = \cup_{i=1}^m P_i$, where P_i are intervals, $|P_i| > 0$, and $m \leq \infty$.

Let $y_0 \in \operatorname{int} P_i$; then we have a set M_{y_0} and $c_0 = a(\inf M_{y_0})$, $d_0 = b(\sup M_{y_0})$.

Suppose we have defined y_i, c_i, d_i and M_{y_i}.

If $i \geq 0$ and $d_i \in \operatorname{int} P_i$ then we define $y_{i+1} = d_i$, $c_{i+1} = a(\inf M_{y_i+1})$, $d_{i+1} = b(\sup M_{y_i+1})$. If $i \leq 0$ and $c_i \in \operatorname{int} P_i$ then we define $y_{i-1} = c_i$, $c_{i-1} = a(\inf M_{y_{i-1}})$, $d_{i-1} = b(\sup M_{y_{i-1}})$.

By using this method we can construct, for every P_i sequences $\{y_j^i\}_{j=m_i}^{n_i}$, $\{c_j^i\}_{j=m_i}^{n_i}$, $\{d_j^i\}_{j=m_i}^{n_i}$, $\{M_{y_j^i}\}_{j=m_i}^{n_i}$, where $-\infty \leq m_i \leq n_i \leq \infty$.

We can rewrite all these sequences in the following way: $\{y_i\}_{i=1}^k$, $\{c_i\}_{i=1}^k$, $\{d_i\}_{i=1}^k$ and $\{M_{y_i}\}_{i=1}^k$ where $k = \sum_{i=1}^m (n_i - m_i + 1)$.

Then we have

$$Hf(x) = \sum_{i=1}^k \chi_{M_{y_i}}(x)\left(v(x)\int_{y_i}^{b(x)} u(t)f(t)\,\mathrm{d}t + v(x)\int_{a(x)}^{y_i} u(t)f(t)\,\mathrm{d}t \right) \text{ a.e.}$$

and

$$\begin{aligned}
\int_{\mathbb{R}} gHf &= \sum_{i=1}^k \int_{M_{y_i}} gHf \\
&= \sum_{i=1}^k \int_{M_{y_i}} \left(\left[v(x)\int_{y_i}^{b(x)} f(t)u(t)\,\mathrm{d}t + v(x)\int_{a(x)}^{y_i} f(t)u(t)\,\mathrm{d}t \right] g(x) \right) \mathrm{d}x \\
&= \sum_{i=1}^k \left[\int_{M_{y_i}} \left(v(x)\int_{y_i}^{b(x)} f(t)\chi_{(y_i,d_i)} u(t)\,\mathrm{d}t \right) g(x)\,\mathrm{d}x \right. \\
&\qquad \left. + \int_{M_{y_i}} \left(v(x)\int_{a(x)}^{y_i} f(t)\chi_{(c_i,y_i)} u(t)\,\mathrm{d}t \right) g(x)\,\mathrm{d}x \right]
\end{aligned}$$

(Using Lemma 4.2 and $A_a + A_b \leq A_H$)

$$\int_{\mathbb{R}} gHf \leq 4d_1d_2 \sum_{i=1}^{k} A_H \|f\chi_{(c_i,y_i)}\|_X \|g\chi_{M_i}\|_{Y'}$$
$$+ 4d_1d_2 \sum_{i=1}^{k} A_H \|f\chi_{(y_i,d_i)}\|_X \|g\chi_{M_i}\|_{Y'}$$
$$\leq 8d_1d_2A_H \sum_{i=1}^{k} \|f\chi_{(c_i,d_i)}\|_X \|g\chi_{M_i}\|_{Y'}$$
(use Hölder's inequality and l-condition)
$$\leq 8A_Hd_1^2d_2^2\|f\|_X\|g\|_{Y'}.$$

5 BOUNDEDNESS OF THE OPERATOR K

In this section we prove Theorem 3.1.

LEMMA 5.1 *Let $b(x)$ be a nondecreasing right continuous function on (α,β) and let $b(\alpha)=c, b(\beta)=d$. Let $k_0(x,y)\geq 0$ be a kernel satisfying*(1.2), *and $k_0(x,y)>0$ on set of positive measure. Suppose that $k_0(x,y)$ is right continuous with respect to x for all $x\in[\alpha,\beta]$ and for a.e. $y\in(c,b(x))$.*

Let u,f be measurable functions on (c,d), $fu\geq 0$, and

$$G_0(x)=\int_c^{b(x)} k_0(x,y)u(y)f(y)\,\mathrm{d}y.$$

For a fixed number $\delta>D$ (where D is a constant from(1.2)), *we define $\Delta_k:=\{x\in(\alpha,\beta); G_0(x)\geq(\delta+1)^k\}$, $k\in\mathbf{Z}$, and $N=\sup\{k;\Delta_k\neq\emptyset\}$. Then there exist sequences $\{x_k\}$, $\{\gamma_k\}$ such that $\alpha<\cdots<x_{k-1}<x_k<\cdots<\beta$ and the inequality*

$$(\delta+1)^{\gamma_k-1}\leq \int_{b(x_{k-1})}^{b(x_k)} k_0(x_k,y)u(y)f(y)\,\mathrm{d}y$$
$$+D\int_{b(x_{k-2})}^{b(x_{k-1})} k_0(x_{k-1},y)u(y)f(y)\,\mathrm{d}y$$
$$+Dk_0(x_k,b(x_{k-1}))\int_{b(x_{k-1})}^{b(x_k)} u(y)f(y)\,\mathrm{d}y$$
$$+Dk_0(x_k,b(x_{k-2}))\int_c^{b(x_{k-2})} u(y)f(y)\,\mathrm{d}y.$$

holds for all $k\leq N$, and $G_0(x)\leq(1+\delta)^{\gamma_{k-1}+1}$ when $x\in[x_{k-1},x_k)$.

Proof By the Lebesgue Dominated Convergence Theorem $G_0(x)$ is a nondecreasing right continuous function for all $\alpha \le x \le \beta$ and $\lim_{x\to\alpha} G_0(x) = 0$.

Set $a_k = \inf \Delta_k$, for $k \le N$.

Fix $i \in \mathbf{Z}$ such that $|\Delta_i| > 0$. We set $x_0 = a_i$, $\gamma_0 = \max\{i;\, a_i = x_0\}$, $x_k = a_{\gamma_k}$ where $\gamma_k = \max\{i; a_i = a_{\gamma_{k-1+1}}\}$ for $k > 0$ and $\gamma_k = \max\{i; a_i = a_{\gamma_{k+1}}\}$ for $k < 0$.

It is obvious that $\{\gamma_k\}$ is an increasing sequence of integers, therefore $\gamma_{k_1} \le \gamma_k - 1$, $G(x_k) = G(a_{\gamma_k}) \ge (1+\delta)^{\gamma_k}$.

If $x \in [x_k, x_{k+1})$, then we have $a_{\gamma_k+1} = x_{k+1}$, and therefore $x < a_{\gamma_k+1}$ $G(x) < (1+\delta)^{\gamma_k+1}$. Next on using (1.2) we find that

$$\begin{aligned}
(1+\delta)^{\gamma_k} &= \int_c^{b(x_k)} k_0(x_k,y)u(y)f(y)\,\mathrm{d}y \\
&= \int_{b(x_{k-1})}^{b(x_k)} k_0(x_k,y)u(y)f(y)\,\mathrm{d}y + \int_{b(x_{k-2})}^{b(x_{k-1})} k_0(x_k,y)u(y)f(y)\,\mathrm{d}y \\
&\quad + \int_c^{b(x_{k-2})} k_0(x_k,y)u(y)f(y)\,\mathrm{d}y \\
&\le \int_{b(x_{k-1})}^{b(x_k)} k_0(x_k,y)u(y)f(y)\,\mathrm{d}y \\
&\quad + D\int_{b(x_{k-2})}^{b(x_{k-1})} k_0(x_{k-1},y)u(y)f(y)\,\mathrm{d}y \\
&\quad + Dk_0(x_k,b(x_{k-1}))\int_{b(x_{k-2})}^{b(x_{k-1})} u(y)f(y)\,\mathrm{d}y \\
&\quad + Dk_0(x_k,b(x_{k-2}))\int_c^{b(x_{k-2})} u(y)f(y)\,\mathrm{d}y + DG_0(x_{k-2}).
\end{aligned}$$

As $DG_0(x_{k-2}) \le D(1+\delta)^{\gamma_{k-2}+1} \le D(1+\delta)^{\gamma_k-1} \le \delta(1+\delta)^{\gamma_k-1}$ and $(1+\delta)^{\gamma_k} - \delta(1+\delta)^{\gamma_k-1} = (1+\delta)^{\gamma_k-1}$ the lemma follows.

THEOREM 5.2 *Let X and Y be two BFS on (c,d) and (α,β), respectively, (where $b(\beta)=d$ and $b(\alpha)=c$) satisfying the l-condition and*

$$K_b f(x) := v(x)\int_c^{b(x)} k(x,y)f(y)u(y)\,\mathrm{d}y,$$

where $k(x,y)$ satisfies (1.2). *Then*

$$\|K_b\|_{X\to Y} \asymp A_b^1 + A_b^0$$

where

$$A_b^1 := \sup_{\alpha<z<\beta} \|\chi_{(z,\beta)}v\|_Y \|\chi_{(c,b(z))}(.)k(z,.)u(.)\|_{X'},$$
$$A_b^0 := \sup_{\alpha<z<\beta} \|\chi_{(z,\beta)}(.)v(.)k(.,b(z))\|_Y \|\chi_{(c,b(z))}u\|_{X'}.$$

Proof *Necessity* Let $x>\alpha$. Then $b(x)\ge c$. Since $k(x,y)$ is nondecreasing in x and nonincreasing in y, for every $\alpha<x<z<\beta$ we have

$$Kf(x) \ge v(z)\int_c^{b(x)} k(x,t)f(t)u(t)\,\mathrm{d}t$$

and

$$Kf(z) \ge v(z)k(z,b(x))\int_c^{b(x)} u(t)f(t)\,\mathrm{d}t.$$

Hence,

$$\begin{aligned}\|Kf\|_Y &\ge \|\chi_{(x,\beta)}(.)v(.)\int_c^{b(x)} k(x,t)f(t)u(t)\,\mathrm{d}t\|_Y\\ &\ge \|\chi_{(x,\beta)}v\|_Y\|\chi_{(c,b(x))}(.)k(x,.)u(.)\|_{X'}\|f\chi_{(c,b(x))}\|_X\end{aligned}$$

for all $f\in X$ and $\alpha<x<\beta$, and

$$\begin{aligned}\|Kf\|_Y &\ge \|\chi_{(x,\beta)}(.)v(.)k(.,b(x))\int_c^{b(x)} u(t)f(t)\,\mathrm{d}t\|_Y\\ &\ge \|\chi_{(x,\beta)}(.)v(.)k(.,b(x))\|_Y\|\chi_{(c,b(x))}u\|_{X'}\|\chi_{(c,b(x))}f\|_X\end{aligned}$$

for all $f\in X$ and $\alpha<x<\beta$.

Sufficiency Let D be the constant from condition (1.2) and let $\delta>D$ be fixed. Without loss of generality we may assume that $k(x,y)$ and $b(x)$ satisfy the assumptions of Lemma 5.1. Otherwise we replace $k(x,y)$ by $k(x_+,y)$ and $b(x)$ by $b(x_+)$.

By the principle of duality it is sufficient to show that

$$J = \left| \int_\alpha^\beta v(t)G(t)g(t)\,\mathrm{d}t \right| \le A\|f\|_X\|g\|_{Y'}, \quad \text{for all } f \in X \text{ and } g \in Y',$$

where $G(t) = \int_c^{b(t)} |k(t,y)f(y)u(y)|\,\mathrm{d}y$. By Lemma 5.1 we obtain

$$\begin{aligned} J &\le \sum_{k\le N} \int_{x_k}^{x_{k+1}} |v(t)G(t)g(t)|\,\mathrm{d}t \\ &\le \sum_{k\le N} (1+\delta)^{\gamma_k+1} \int_{x_k}^{x_{k+1}} |v(t)g(t)|\,\mathrm{d}t \\ &\le (1+\delta)^2[J_{11}+J_{12}+J_{21}+J_{22}], \end{aligned}$$

where

$$\begin{aligned} J_{11} &:= \sum_{k\le N} \int_{b(x_{k-1})}^{b(x_k)} |k(x_k,t)u(t)f(t)|\,\mathrm{d}t \int_{x_k}^{x_{k+1}} |g(t)v(t)|\,\mathrm{d}t, \\ J_{12} &:= D\sum_{k\le N} \int_{b(x_{k-2})}^{b(x_{k-1})} |k(x_{k-1},t)u(t)f(t)|\,\mathrm{d}t \int_{x_k}^{x_{k+1}} |g(t)v(t)|\,\mathrm{d}t, \\ J_{21} &:= D\sum_{k\le N} k(x_k, b(x_{k-1})) \int_{b(x_{k-2})}^{b(x_{k-1})} |u(t)f(t)|\mathrm{d}t \int_{x_k}^{x_{k+1}} |g(t)v(t)|\,\mathrm{d}t, \\ J_{22} &:= D\sum_{k\le N} k(x_k, b(x_{k-2})) \int_c^{b(x_{k-2})} |u(t)f(t)|\,\mathrm{d}t \int_{x_k}^{x_{k+1}} |g(t)v(t)|\,\mathrm{d}t. \end{aligned}$$

Applying the Holder inequality and the l-condition we find

$$\begin{aligned} J_{11} &\le \sum_{k\le N} \big(\|\chi_{(b(x_{k-1}),b(x_k))}k(x_k,.)u\|_{X'}\|\chi_{(b(x_{k-1}),b(x_k))}f\|_X \\ &\quad \times \|\chi_{(x_k,x_{k+1})}g\|_{Y'}\|\chi_{(x_k,x_{k+1})}v\|_Y\big) \\ &\le \sum_{k\le N} \|\chi_{(c,b(x_k))}k(x_k,.)u\|_{X'}\|\chi_{(x_k,\beta)}v\|_Y\|\chi_{(b(x_{k-1}),b(x_k))}f\|_X\|\chi_{(x_k,x_{k+1})}g\|_{Y'} \\ &\le A_b^1 \sum_{k\le N} \|\chi_{(b(x_{k-1}),b(x_k))}f\|_X\|\chi_{(x_k,x_{k+1})}g\|_{Y'} \\ &\le A_b^1 \left\| \sum_{k\le N} e_k\|\chi_{(b(x_{k-1}),b(x_k))}f\|_X \right\|_l \left\| \sum_{k\le N} e_k\|\chi_{(x_k,x_{k+1})}g\|_{Y'} \right\|_{l'} \\ &\le d_1d_2A_b^1\|f\|_X\|g\|_{Y'}. \end{aligned}$$

Analogously, we obtain

$$J_{12} \leq d_1 d_2 A_b^1 \|f\|_X \|g\|_{Y'}.$$

The estimate for J_{21} is similar to that for J_{11} on applying the knowledge that $k(x, y)$ is nondecreasing in x and proceeding like for J_{11}: we find that

$$J_{21} \leq d_1 d_2 A_b^0 \|f\|_X \|g\|_{Y'}.$$

For J_{22} we write

$$\begin{aligned} J_{22} &= \int_\alpha^\beta \int_c^{b_1(x)} |u(t)f(t)|\,\mathrm{d}t \sum_{k\leq N} k(x_k, b(x_{k-2}))\chi_{(x_k,x_{k+1})}(x)|g(x)v(x)|\,\mathrm{d}x \\ &\leq \|K_1 f\|_Y \|g\|_{Y'}, \end{aligned}$$

where $K_1 f(x) := \phi(x)v(x)\int_c^{b_1(x)} |u(t)f(t)|\mathrm{d}t$ and $b_1(x) := \sum_{k\leq N} b(x_{k-2})\chi_{(x_k,x_{k+1})}(x)$ and $\phi(x) := \sum_{k\leq N} k(x_k, b(x_{k-2}))\chi_{(x_k,x_{k+1})}(x)$. By Theorem 3.2 we have that

$$\|K_1\|_{X\to Y} \leq C \sup_{\alpha<z<\beta} \|\chi_{(z,\beta)} v\phi\|_Y \|\chi_{(c,b_1(z))} u\|_{X'}$$

if $x_{k_0} < z < x_{k_0+1}$ when $b_1(z) = b(x_{k_0-2})$ and

$$\phi(t)\chi_{(z,\beta)}(t) \leq \sum_{k=k_0}^\infty k(x_k, b(x_{k-2}))\chi_{(x_k,x_{k+1})}(t) \leq k(t, b(x_{k_0-2})).$$

Therefore we have

$$\begin{aligned} \|\chi_{(z,\beta)}\phi v\|_Y \|\chi_{(c,b_1(z))} u\|_{X'} &\leq \|\chi_{(x_{k_0-2},\beta)} k(.,b(x_{k_0-2}))v\|_Y \|\chi_{(c,b(x_{k_0-2}))} u\|_{X'} \\ &\leq A_b^0. \end{aligned}$$

Thus $J_{22} \leq C A_b^0 \|f\|_X \|g\|_{Y'}$. Then we have that $\|K_b\|_{X\to Y} \leq C(A_b^1 + A_b^0)$.

Proof of Theorem 3.1 *Necessity* Let $fu \geq 0$ a.e., $x < y$ and $b(x) \geq a(y)$. Since $k(x, y)$ is nondecreasing in x and nonincreasing in y, for every $x < z < y$ we have

$$Kf(z) \geq v(z) \int_{a(y)}^{b(x)} k(x, t)f(t)u(t)\,\mathrm{d}t$$

and

$$Kf(z) \geq v(z)k(z, b(x)) \int_{a(y)}^{b(x)} f(t)u(t)\,\mathrm{d}t.$$

Therefore we get

$$\int_{a(y)}^{b(x)} k(x,t)u(t)f(t)\,\mathrm{d}t\|\chi_{(x,y)}\|_Y \leq \|Kf\|_Y \leq \|K\|_{X\to Y}\|f\|_X$$

and

$$\int_{a(y)}^{b(x)} u(t)f(t)\,\mathrm{d}t\|v\chi_{(x,y)}k(.,b(x))\|_Y \leq \|Kf\|_Y \leq \|K\|_{X\to Y}\|f\|_X$$

for all $f \in X$ such that $fu \geq 0$.

Then by duality we have

$$A_0 + A_1 \leq 2\|K\|_{X\to Y}.$$

Sufficiency We use the same technique as in the proof of Theorem 3.2. For $a(x)$, $b(x)$ we define $\{c_i\}_{i=1}^k$, $\{d_i\}_{i=1}^k$, $\{y_i\}_{i=1}^k$ and $\{M_{y_i}\}_{i=1}^k$ as in that proof. Then we have

$$\begin{aligned} Kf(x) &= \sum_{i=1}^{k} \chi_{M_{y_i}}(x)(v(x)\int_{y_i}^{b(x)} k(x,t)u(t)f(t)\mathrm{d}t \\ &\quad + v(x)\int_{a(x)}^{y_i} k(x,t)u(t)f(t)\mathrm{d}t) \\ &= \sum_{i=1}^{k} \chi_{M_{y_i}}(x)K_i^1 f(x) + \sum_{i=1}^{k} \chi_{M_{y_i}}(x)K_i^2 f(x), \end{aligned}$$

where $K_i^1 = v(x)\int_{y_i}^{b(x)} k(x,t)u(t)f(t)\mathrm{d}t\chi_{M_{y_i}}(x)$ and $K_i^2 = v(x)\int_{a(x)}^{y_i} k(x,t)$ $u(t)f(t)\mathrm{d}t\chi_{M_{y_i}}(x)$.

By the l-condition we obtain

$$\begin{aligned} \|Kf\|_Y &\leq \left\|\sum_{i=1}^{k} e_i\|K_i^1(f)\chi_{M_{y_i}}\|_Y\right\|_l + \left\|\sum_{i=1}^{k} e_i\|K_i^2(f)\chi_{M_{y_i}}\|_Y\right\|_l \\ &= I_1 + I_2. \end{aligned}$$

By Theorem 5.2 we have

$$\|K_i^1(f)\chi_{M_{y_i}}\|_Y \le CA\|\chi_{(c_i,d_i)}f\|_X$$

and therefore we obtain

$$I_1 \le CA\|\sum_{i=1}^{k} e_i\|\chi_{(c_i,d_i)}f\|_X\|_l \le CA\|f\|_X.$$

To estimate I_2 we use the condition (1.2) for $x_i=\inf(M_{y_i})$ and $a(x)\le t\le y_i=b(x_i)$, $x_i<x$. Then $k(x,t)\le D[k(x,y_i)+k(x_i,t)]$ and we have

$$\begin{aligned}\chi_{M_{y_i}}(x)K_i^2f(x) &= \chi_{M_{y_i}}(x)v(x)\int_{a(x)}^{y_i} k(x,t)u(t)f(t)\,\mathrm{d}t\\ &\le D\chi_{M_{y_i}}(x)v(x)k(x,y_i)\int_{a(x)}^{y_i} u(t)f(t)\,\mathrm{d}t\\ &\quad + D\chi_{M_{y_i}}(x)v(x)\int_{a(x)}^{y_i} k(x_i,t)u(t)f(t)\,\mathrm{d}t.\end{aligned}$$

Theorem 3.2 yields

$$\left\|\chi_{M_{y_i}}v(x)k(x,y_i)\int_{a(x)}^{y_i} u(t)f(t)\,\mathrm{d}t\right\|_Y \le A\|f\chi_{(c_i,d_i)}\|_X$$

and

$$\left\|\chi_{M_{y_i}}v(x)\int_{a(x)}^{y_i} k(x_i,t)u(t)f(t)\mathrm{d}t\right\|_Y \le A\|f\chi_{(c_i,d_i)}\|_X.$$

Therefore

$$\|\chi_{M_{y_i}}K_i^2f\|_Y \le 2CA\|f\chi_{(c_i,d_i)}\|_X$$

and by the l-condition we obtain that

$$I_2 \le 2cA\left\|\sum_{i=1}^{k} e_i\|f\chi_{(c_i,d_i)}\|_X\right\|_l \le 2CA\|f\|.$$

Combining the estimates of I_1 and I_2 we arrive at

$$\|Kf\|_Y \leq AC\|f\|_X.$$

Theorem 3.1 is proved.

Remark When this paper was finished we learned (by oral communications) that this problem for Hardy operators in Lebesgue spaces was considered by Heinig and Sinnamon [2].

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