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On the Fundamental Polynomials for Hermite–Fejér Interpolation of Lagrange Type on the Chebyshev Nodes

SIMON J. SMITH *

Division of Mathematics, La Trobe University, P.O. Box 199, Bendigo, Victoria 3552, Australia

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For a fixed integer $m \geq 0$ and $1 \leq k \leq n$, let $A_{k,2m,n}(T,x)$ denote the kth fundamental polynomial for $(0, 1, \ldots, 2m)$ Hermite–Fejér interpolation on the Chebyshev nodes $\{x_{j,n} = \cos[(2j-1)\pi/(2n)]: 1 \leq j \leq n\}$. (So $A_{k,2m,n}(T,x)$ is the unique polynomial of degree at most $(2m+1)n-1$ which satisfies $A_{k,2m,n}(T, x_{j,n}) = \delta_{k,j}$, and whose first $2m$ derivatives vanish at each $x_{j,n}$.) In this paper it is established that

$$|A_{k,2m,n}(T,x)| \leq A_{1,2m,n}(T,1), \quad 1 \leq k \leq n, \ -1 \leq x \leq 1.$$

It is also shown that $A_{1,2m,n}(T,1)$ is an increasing function of n, and the best possible bound C_m so that $|A_{k,2m,n}(T,x)| < C_m$ for all k, n and $x \in [-1,1]$ is obtained. The results generalise those for Lagrange interpolation, obtained by P. Erdős and G. Grünwald in 1938.

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* E-mail: S.Smith@bendigo.latrobe.edu.au.
Web: http://www.bendigo.latrobe.edu.au/mte/maths/staff/smith/.

1 INTRODUCTION

Suppose f is a continuous real-valued function defined on the interval $[-1, 1]$, and let

$$X = \{x_{k,n}\colon\ k = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots\}$$

be an infinite triangular matrix such that for all n,

$$1 \geq x_{1,n} > x_{2,n} > \cdots > x_{n,n} \geq -1.$$

Then for each integer $m \geq 0$ there exists a unique polynomial $H_{m,n}(X, f, x)$ of degree at most $(m+1)n-1$ which satisfies

$$H^{(r)}_{m,n}(X, f, x_{k,n}) = \delta_{0,r} f(x_{k,n}), \quad 1 \leq k \leq n,\ 0 \leq r \leq m.$$

$H_{m,n}(X, f, x)$ is known as as the $(0, 1, \ldots, m)$ Hermite–Fejér (HF) interpolation polynomial of $f(x)$, and it can be expressed as

$$H_{m,n}(X, f, x) = \sum_{k=1}^{n} f(x_{k,n}) A_{k,m,n}(X, x),$$

where $A_{k,m,n}(X, x)$ is the unique polynomial of degree at most $(m+1)n - 1$ such that

$$A^{(r)}_{k,m,n}(X, x_{j,n}) = \delta_{0,r}\, \delta_{k,j}, \quad 1 \leq k, j \leq n,\ 0 \leq r \leq m.$$

The $A_{k,m,n}(X, x)$ are referred to as the fundamental polynomials for $(0, 1, \ldots, m)$ HF interpolation on X, and the quantities

$$\lambda_{m,n}(X, x) = \sum_{k=1}^{n} \left| A_{k,m,n}(X, x) \right|$$

and

$$\Lambda_{m,n}(X) = \max_{-1 \leq x \leq 1} \lambda_{m,n}(X, x),$$

which are known as the Lebesgue function and Lebesgue constant, respectively, for $(0, 1, \ldots, m)$ HF interpolation on X, play a crucial role in determining the convergence behaviour of $H_{m,n}(X, f, x)$ to $f(x)$ as $n \to \infty$.

If $m = 0$, we obtain the familiar Lagrange interpolation process. Here it is known (cf. [10, Section 1.3]) that there is a positive constant c so that

$$\Lambda_{0,n}(X) > \frac{2}{\pi}\log n + c \tag{1}$$

for any X. This leads to the classic result [5] that for any X there exists $f \in C[-1,1]$ so that $H_{0,n}(X,f,x)$ does not converge uniformly to $f(x)$ on $[-1,1]$ as $n \to \infty$. On the other hand, if T denotes the matrix of Chebyshev nodes

$$T = \left\{\cos\left(\frac{2k-1}{2n}\pi\right):\ k = 1,2,\ldots,n;\ n = 1,2,3,\ldots\right\},$$

then

$$\Lambda_{0,n}(T) \le \frac{2}{\pi}\log n + 1, \quad n = 1,2,3,\ldots$$

(See [10, Theorem 1.2].) Further, $H_{0,n}(X,f,x)$ converges uniformly to $f(x)$ on $[-1,1]$ if f satisfies the relatively mild Dini–Lipschitz condition $\omega_f(1/n)\log n \to 0$ as $n \to \infty$. (See, for example, [9, Section 4.1].) Here ω_f denotes the modulus of continuity of f, defined by

$$\omega_f(\delta) = \max\{|f(s) - f(t)|:\ \{s,t\} \subset [-1,1],\ |s-t| \le \delta\}.$$

Thus, in terms of the magnitude of Lebesgue constants, and convergence properties of the interpolation polynomials, the Chebyshev nodes are close to optimal for Lagrange interpolation.

The fundamental polynomials $A_{k,0,n}(T,x)$ and Lebesgue function $\lambda_{0,n}(T,x)$ have been studied extensively. For example, Erdős and Grünwald [4] obtained the following result.

THEOREM 1 *For* $n = 1,2,3,\ldots$,

$$\max_{1\le k\le n}\ \max_{-1\le x\le 1}|A_{k,0,n}(T,x)| = \frac{1}{n}\cot\frac{\pi}{4n}.$$

The maximum is attained if and only if $k = 1$ *and* $x = 1$ *or* $k = n$ *and* $x = -1$. *Furthermore, since the right-hand side is monotonic increasing to*

$4/\pi$ as $n \to \infty$, it follows that for all k and n,

$$|A_{k,0,n}(T,x)| < \frac{4}{\pi}, \quad -1 \le x \le 1,$$

and the constant on the right-hand side is best possible.

With regard to the Lebesgue constant, Ehlich and Zeller [3] demonstrated that for $n = 1, 2, 3, \dots,$

$$\Lambda_{0,n}(T) = \lambda_{0,n}(T, \pm 1). \tag{2}$$

A proof of this result is also developed in Rivlin [10, Section 1.3], and closely related results are presented in Brutman [1] and Günttner [6].

For higher-order HF interpolation, there are many similarities between the Lagrange and $(0, 1, \dots, m)$ HF processes for even values of m. For instance, Szabados [12] extended (1) by showing that there are positive constants c_m so that for any X,

$$\Lambda_{2m,n}(X) \ge c_m \log n, \quad n = 1, 2, 3, \dots$$

Thus, for any node system X, there exists $f \in C[-1, 1]$ so that $H_{2m,n}(X, f, x)$ does not converge uniformly to $f(x)$ on $[-1, 1]$ as $n \to \infty$. With regard to Lebesgue constants for the Chebyshev nodes, Byrne *et al.* [2] generalized (2) by showing that

$$\Lambda_{2m,n}(T) = \lambda_{2m,n}(T, \pm 1) = \frac{2}{\pi}\frac{(2m)!}{2^{2m}(m!)^2}\log n + \mathcal{O}(1) \quad \text{as } n \to \infty.$$

The aim of this paper is to study the fundamental polynomials for $(0, 1, \dots, 2m)$ HF interpolation on the Chebyshev nodes. We obtain the following generalization of Theorem 1.

THEOREM 2 *If $m \ge 0$ is fixed, and $n = 1, 2, 3, \dots,$ then*

$$\max_{1 \le k \le n}\ \max_{-1 \le x \le 1} |A_{k,2m,n}(T,x)|$$

is attained if and only if $k = 1$ and $x = 1$ or $k = n$ and $x = -1$. Furthermore, $|A_{1,2m,n}(T,1)|$ is an increasing function of n, and for all k and n,

$$|A_{k,2m,n}(T,x)| < 2\sum_{r=0}^{m} a_{r,m}\left(\frac{2}{\pi}\right)^{2m+1-2r}, \quad -1 \le x \le 1, \tag{3}$$

where the $a_{r,m}$ are the coefficients in the Laurent expansion

$$\frac{1}{\sin^{2m+1} z} = \frac{1}{z^{2m+1}} \sum_{r=0}^{\infty} a_{r,m} z^{2r}, \quad 0 < |z| < \pi. \tag{4}$$

The constant on the right-hand side of (3) *is best possible.*

Thus, for example, for all k, n and $x \in [-1, 1]$,

$$|A_{k,2m,n}(T, x)| < \begin{cases} 4/\pi & \text{if } m = 0, \\ 2/\pi + 16/\pi^3 & \text{if } m = 1, \\ 3/(2\pi) + 40/(3\pi^3) + 64/\pi^5 & \text{if } m = 2, \end{cases}$$

and the constants on the right-hand side are best possible.

Note that the corresponding problem to that considered in Theorem 2 for odd-order HF interpolation on the Chebyshev nodes is answered by a result of Smith [11] which states that the $A_{k,2m+1,n}(T, x)$ are non-negative for $-1 \le x \le 1$. Thus, since $\sum_{k=1}^{n} A_{k,2m+1,n}(T, x) = 1$ for all x, it follows that

$$\max_{1 \le k \le n} \max_{-1 \le x \le 1} |A_{k,2m+1,n}(T, x)| = 1,$$

and for each k the maximum is attained if and only if $x = \cos[(2k-1)\pi/(2n)]$.

The proof of Theorem 2 will be presented in Section 2. For the proof of Theorem 1, Erdős and Grünwald used Riesz's lemma [8], which provides a lower bound for the separation of the maximum point for the absolute value of a trigonometric polynomial and the zeros of the polynomial, and explicit formulas for the fundamental polynomials $A_{k,1,n}(T, x)$ for (0, 1) HF interpolation on T. Because the formulas for the $A_{k,m,n}(T, x)$ become increasingly more complicated with increasing m, our method for proving Theorem 2 relies as much as possible on zero-counting techniques and adaptations of Riesz's method to trigonometric polynomials with multiple roots. Specific formulas for the fundamental polynomials are used only for the final part of the proof.

2 PROOF OF THEOREM 2

For fixed m and $1 \le k \le n$, define the cosine polynomial $t_{k,2m,n}(\theta)$ by

$$t_{k,2m,n}(\theta) = A_{k,2m,n}(T, \cos\theta),$$

and for given j put

$$\theta_j = \theta_{j,n} = (2j-1)\frac{\pi}{2n}.$$

Then $t_{k,2m,n}(\theta)$ is the unique trigonometric polynomial of degree at most $(2m+1)n-1$ which satisfies

$$t^{(r)}_{k,2m,n}(\theta_j) = \delta_{0,r}\,\delta_{k,j}, \quad 1 \le k, j \le n, \ 0 \le r \le 2m. \tag{5}$$

In the following sequence of lemmas the problem of finding $\max_{0\le\theta\le\pi}|t_{k,2m,n}(\theta)|$ is studied. Since this problem is equivalent to that of finding $\max_{-1\le x\le 1}|A_{k,2m,n}(T,x)|$, the lemmas provide a proof of Theorem 2.

LEMMA 1 *For* $1 \le k \le n$, $t'_{k,2m,n}(\theta)$ *has zeros of order* $2m$ *at* $\theta_1, \theta_2, \ldots, \theta_n$, *and of order* 1 *at* 0 *and* π. *If* $k=1$, $t'_{k,2m,n}(\theta)$ *also has a single zero in each interval* (θ_j, θ_{j+1}) *for* $2 \le j \le n-1$; *if* $k=n$, *then* $t'_{k,2m,n}(\theta)$ *has a single zero in each interval* (θ_j, θ_{j+1}) *for* $1 \le j \le n-2$; *if* $2 \le k \le n-1$, *then* $t'_{k,2m,n}(\theta)$ *has a single zero in each interval* (θ_j, θ_{j+1}) *for* $1 \le j \le n-1$, $j \ne k-1, k$, *and has a zero in* $(\theta_{k-1}, \theta_{k+1})$ *that is additional to the* $2m$ *zeros at* θ_k. *In all cases,* $t'_{k,2m,n}(\theta)$ *has no other zeros in* $[0,\pi]$. *Further,* $t_{k,2m,n}(\theta)$ *has no zeros in* $[0,\pi]$ *apart from those given by* (5), *and changes sign at each of its zeros.* (*Hence* $t_{k,2m,n}(\theta) > 0$ *on* $(\theta_{k-1},\theta_{k+1})$.)

Proof Suppose $2 \le k \le n-1$. By (5), $t'_{k,2m,n}(\theta)$ has zeros of order $2m$ at $\theta_1, \theta_2, \ldots, \theta_n$ in $(0,\pi)$, and (by Rolle's theorem) has a zero in each interval (θ_j, θ_{j+1}) for $1 \le j \le n-1$, $j \ne k-1, k$. Further, since $t_{k,2m,n}(\theta_{k-1}) = t_{k,2m,n}(\theta_{k+1})$, $t'_{k,2m,n}(\theta)$ has an odd number of zeros in $(\theta_{k-1},\theta_{k+1})$, and so there is at least one zero of $t'_{k,2m,n}(\theta)$ in $(\theta_{k-1}, \theta_{k+1})$ in addition to those already identified. We have thus located $2mn+n-2$ zeros of $t'_{k,2m,n}(\theta)$ in $(0,\pi)$. Since $t'_{k,2m,n}(\theta)$ is odd, it also has zeros at 0 and π. Hence, because $t'_{k,2m,n}(\theta)$ has degree at most $(2m+1)n-1$, we have identified all zeros of $t'_{k,2m,n}(\theta)$ in $[0,\pi]$. The remaining parts of the lemma now follow immediately. The cases $k=1$ and $k=n$ are handled in a similar (indeed, slightly simpler) fashion.

LEMMA 2 *For* $1 \le k \le n$, *the maximum of* $|t_{k,2m,n}(\theta)|$ *on the interval* $0 \le \theta \le \pi$ *is achieved at a unique point* $\phi_k = \phi_{k,n}$, *where* $\phi_1 = 0$, $\phi_n = \pi$, *and* $\phi_k \in (\theta_{k-1}, \theta_{k+1})$ *for* $2 \le k \le n-1$. *Further,* $t_{k,2m,n}(\phi_k) > 0$.

Proof Fix k so that $2 \le k \le n-1$, and let $\alpha \in [0, \pi]$ be such that

$$\max_{0 \le \theta \le \pi} \left| t_{k,2m,n}(\theta) \right| = M = \left| t_{k,2m,n}(\alpha) \right|.$$

Suppose $\alpha \in (\theta_r, \theta_{r+1})$ for some r so that $0 \le r \le n$ and $r \ne k-1, k$. (Note that if $r = 0$ or n, then $\alpha = 0$ or π, respectively.) If $\operatorname{sgn}(t_{k,2m,n}(\theta)) = \varepsilon$ for $\theta_r < \theta < \theta_{r+1}$, consider

$$f(\theta) = t_{k,2m,n}(\theta) - (-1)^r \varepsilon M \cos^{2m+1} n\theta.$$

Now, $f'(\theta)$ has zeros of order $2m$ at $\theta_1, \theta_2, \ldots, \theta_n$, and by Rolle's theorem it has a zero in each interval (θ_j, θ_{j+1}) for $1 \le j \le n-1$ and $j \ne k-1, k$. Also,

$$f(\theta_{k-1/2}) = t_{k,2m,n}(\theta_{k-1/2}) - (-1)^{r+k-1} \varepsilon M,$$
$$f(\theta_{k+1/2}) = t_{k,2m,n}(\theta_{k+1/2}) + (-1)^{r+k-1} \varepsilon M,$$

so $f(\theta_{k-1/2}) f(\theta_{k+1/2}) \le 0$. Thus f has a zero in $[\theta_{k-1/2}, \theta_{k+1/2}]$, and hence (by Rolle's theorem), f' has at least 2 zeros in $(\theta_{k-1}, \theta_{k+1})$ in addition to those already identified at θ_k. So we have located $2mn + n - 1$ zeros of $f'(\theta)$ in $[\theta_1, \theta_n]$.

Note that if $r = 0$, then $f(0) = 0$, and so $f'(\theta)$ has an additional zero in $(0, \theta_1)$. Similarly, if $r = n$, then $f'(\theta)$ has an additional zero in (θ_n, π). Finally, if $1 \le r \le n-1$ and $r \ne k-1, k$, then

$$f(\alpha) = \varepsilon M - \varepsilon M |\cos^{2m+1} n\alpha|,$$
$$f(\theta_{r+1/2}) = \varepsilon |t_{k,2m,n}(\theta_{r+1/2})| - \varepsilon M.$$

If $f(\theta_{r+1/2}) = 0$, then $f'(\theta)$ has 2 zeros in (θ_r, θ_{r+1}), while if $f(\theta_{r+1/2}) \ne 0$, then f has a zero between α and $\theta_{r+1/2}$, and so again $f'(\theta)$ has 2 zeros in (θ_r, θ_{r+1}). In either case we have found an additional zero of $f'(\theta)$ in (θ_r, θ_{r+1}) to that already located. Overall, then, for all choices of r with $0 \le r \le n$ and $r \ne k-1, k$, we have located $2mn + n$ zeros of $f'(\theta)$ in $(0, \pi)$, and since $f'(\theta)$ is odd (so has zeros at 0, π), we have identified $4mn + 2n + 2$ zeros of $f'(\theta)$ in $(-\pi, \pi]$. This provides a contradiction, since $f'(\theta)$ has degree $2mn + n$, and so the assumption that $\alpha \in (\theta_r, \theta_{r+1})$ for some r such that $0 \le r \le n$ and $r \ne k-1, k$ is incorrect. Therefore $\alpha \in (\theta_{k-1}, \theta_{k+1})$. Further, since $t_{k,2m,n}(\theta)$ has only one turning point in $(\theta_{k-1}, \theta_{k+1})$, α is unique.

The cases $k=1$ and $k=n$ are resolved in a similar manner, and so the proof of Lemma 2 is complete.

Note that, by symmetry, $t_{n,2m,n}(\theta)=t_{1,2m,n}(\pi-\theta)$ for all θ. Thus the following lemma completes the proof of the first part of Theorem 2.

LEMMA 3 *For* $2\leq k\leq n-1$, $t_{1,2m,n}(0)>t_{k,2m,n}(\phi_k)$.

Proof For fixed k so that $2\leq k\leq n-1$, consider

$$g(\theta) = t_{1,2m,n}(\theta) - t_{k,2m,n}(\theta+\theta_{k-1/2}),$$

which is a trigonometric polynomial of degree at most $(2m+1)n-1$. Also, let I denote the interval $[\theta_{-2k+1},\theta_{2n-2k+1})$ of width 2π. Note that $g(\theta)$ satisfies $g(\theta_0)=1$, $g(\theta_{-2k+2})=-1$, and $g(\theta_j)=0$ for $-2k+1\leq j\leq 2n-2k+1$ and $j\neq 0,-2k+2$.

In I, $g'(\theta)$ has zeros of order $2m$ at θ_j for $-2k+1\leq j\leq 2n-2k$, and by Rolle's theorem $g'(\theta)$ has a zero in each interval (θ_j,θ_{j+1}) for $-2k+3\leq j\leq 2n-2k$, $j\neq -1,0$. So we have identified $(4m+2)n-4$ zeros of $g'(\theta)$ in I. Because $g(\theta_{-2k+1})=g(\theta_{-2k+3})$, g' has an odd number of zeros in $(\theta_{-2k+1},\theta_{-2k+3})$, and so g' has a zero in $(\theta_{-2k+1},\theta_{-2k+3})$ in addition to the $2m$ zeros at θ_{-2k+2} that have already been mentioned. Similarly, g' has an additional zero in (θ_{-1},θ_1), and so all $(4m+2)n-2$ zeros of $g'(\theta)$ in I have been located.

By the above discussion, the only zeros of $g'(\theta)$ on (θ_{-1},θ_1) are a zero of order $2m+1$ at θ_0 or else a zero of order $2m$ at θ_0 and a zero of order 1 at another point in the interval. In either case, g has only one turning point on (θ_{-1},θ_1), and since $g(\theta_{-1})=g(\theta_1)=0$ and $g(\theta_0)>0$, it follows that $g(\theta)>0$ if $\theta_{-1}<\theta<\theta_1$. In particular, this gives $t_{1,2m,n}(\theta)>t_{k,2m,n}(\theta+\theta_{k-1/2})$ for $\theta_0\leq\theta<\theta_1$. Since the maximum value of $t_{1,2m,n}(\theta)$ in $[\theta_0,\theta_1]$ is $t_{1,2m,n}(0)$, we obtain

$$t_{1,2m,n}(0) > t_{k,2m,n}(\theta), \quad \theta_{k-1}\leq\theta\leq\theta_k.$$

(Strict inequality holds at θ_k because $t_{1,2m,n}(0)>1=t_{k,2m,n}(\theta_k)$.)

A similar zero-counting argument applied to

$$h(\theta) = t_{1,2m,n}(\theta) - t_{k,2m,n}(\theta+\theta_{k+1/2})$$

on the interval $[\theta_{-2k},\theta_{2n-2k})$ shows that

$$t_{1,2m,n}(0) > t_{k,2m,n}(\theta), \quad \theta_k\leq\theta\leq\theta_{k+1}.$$

Thus $t_{1,2m,n}(0) > t_{k,2m,n}(\theta)$ for $\theta_{k-1} \leq \theta \leq \theta_{k+1}$, and since $\phi_k \in (\theta_{k-1}, \theta_{k+1})$, then $t_{1,2m,n}(0) > t_{k,2m,n}(\phi_k)$.

The following lemma concludes the proof of Theorem 2 by establishing the monotone increasing property and limiting value of $t_{1,2m,n}(0)$ as a function of n.

LEMMA 4 *For fixed integer $m \geq 0$, $t_{1,2m,n}(0)$ is an increasing function of n, and*

$$\lim_{n\to\infty} t_{1,2m,n}(0) = 2\sum_{r=0}^{m} a_{r,m}\left(\frac{2}{\pi}\right)^{2m+1-2r} \tag{6}$$

where the $a_{r,m}$ are given by (4).

Proof By [7, Theorem 1.1], the function

$$T_{2m,n}(\theta) = \frac{\sin^{2m+1} n\theta}{2n^{2m+1}} \sum_{r=0}^{m} \frac{a_{r,m}\, n^{2r}}{(2m-2r)!} \frac{\mathrm{d}^{2m-2r}}{\mathrm{d}\theta^{2m-2r}} \cot\frac{\theta}{2} \tag{7}$$

is the unique trigonometric polynomial of the form

$$T_{2m,n}(\theta) = \sum_{k=0}^{(2m+1)n} b_k \cos k\theta + \sum_{k=1}^{(2m+1)n-1} c_k \sin k\theta \tag{8}$$

such that

$$T^{(r)}_{2m,n}\left(\frac{j\pi}{n}\right) = \delta_{0,r}\,\delta_{0,j}, \quad 0 \leq j \leq 2n-1,\ 0 \leq r \leq 2m.$$

Now, from (7) it follows that $T_{2m,n}(\theta)$ is even, and so

$$S_{2m,n}(\theta) = T_{2m,n}\left(\theta + \frac{\pi}{2n}\right) + T_{2m,n}\left(\theta - \frac{\pi}{2n}\right) \tag{9}$$

is a cosine polynomial of degree no greater than $(2m+1)n$ which satisfies

$$S^{(r)}_{2m,n}(\theta_j) = \delta_{0,r}\,\delta_{1,j}, \quad 1 \leq j \leq n,\ 0 \leq r \leq 2m.$$

Furthermore, by (8) and (9), the $\cos(2m+1)n\theta$ term in $S_{2m,n}(\theta)$ vanishes, and so $S_{2m,n}(\theta)$ is of degree $(2m+1)n-1$. By uniqueness considerations it follows that $S_{2m,n}(\theta) = t_{1,2m,n}(\theta)$ for all θ, and hence from (7),

$$t_{1,2m,n}(0) = 2\,T_{2m,n}\left(\frac{\pi}{2n}\right) = \frac{1}{n^{2m+1}}\sum_{r=0}^{m}\frac{a_{r,m}\,n^{2r}}{(2m-2r)!}\left[\frac{\mathrm{d}^{2m-2r}}{\mathrm{d}\theta^{2m-2r}}\cot\frac{\theta}{2}\right]_{\theta=\pi/(2n)}.$$

Now, from the well-known Laurent series for $\cot\theta$ about 0, we obtain

$$\cot\frac{\theta}{2} = \frac{2}{\theta} - 2\sum_{j=1}^{\infty}|B_{2j}|\frac{\theta^{2j-1}}{(2j)!},\qquad 0<|\theta|<2\pi,$$

where the B_j are Bernoulli numbers. Consequently, we can write

$$\left[\frac{\mathrm{d}^{2m-2r}}{\mathrm{d}\theta^{2m-2r}}\cot\frac{\theta}{2}\right]_{\theta=\pi/(2n)} = 2(2m-2r)!\left(\frac{2n}{\pi}\right)^{2m+1-2r} - \sum_{j=1}^{\infty}\frac{c_{j,r}}{n^{2j-1}},$$

where the $c_{j,r}$ are all positive, and so

$$t_{1,2m,n}(0) = 2\sum_{r=0}^{m} a_{r,m}\left(\frac{2}{\pi}\right)^{2m+1-2r} - \sum_{r=0}^{m}\sum_{j=1}^{\infty}\frac{a_{r,m}\,c_{j,r}}{(2m-2r)!}\,\frac{1}{n^{2j+2m-2r}}.\qquad(10)$$

By the definition (4) of the $a_{r,m}$, and the expansion

$$\frac{1}{\sin\theta} = \frac{1}{\theta} + \sum_{j=1}^{\infty}\frac{|B_{2j}|}{(2j)!}\left(2^{2j}-2\right)\theta^{2j-1},\qquad 0<|\theta|<\pi,$$

it follows that the $a_{r,m}$ are all positive. Thus, by (10), $t_{1,2m,n}(0)$ is an increasing function of n, with limit as given by the right-hand side of (6).

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