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Research Article

Optimality Conditions of Vector Set-Valued Optimization Problem Involving Relative Interior

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Firstly, a generalized weak convexlike set-valued map involving the relative interior is introduced in separated locally convex spaces. Secondly, a separation property is established. Finally, some optimality conditions, including the generalized Kuhn-Tucker condition and scalarization theorem, are obtained.

1. Introduction

In mathematical programming, set-valued optimization is a very important topic. Since the 1980s, many authors have paid attention to it. Some international journals such as Set-Valued and Variational Analysis (original name: Set-Valued Analysis) were also established. Theories and applications are widely developed. Rong and Wu [1], Li [2], and Yang [3] and Yang [4] introduced cone convexlikeness, subconvexlikeness, generalized subconvexlikeness, and nearly subconvexlikeness, respectively. In these generalized convex set-valued maps, it is clear that nearly subconvexlikeness is the weakest. We find that, in the above-mentioned papers, the convex cone has a nonempty topological interior. However, it is possible that the topological interior of the convex cone is empty. For instance, if $C = \{(r, 0) \mid r \geq 0\} \subseteq R^2$, then the topological interior of C is empty. In order to study some optimization problems which the convex cone has empty topological interior, we have to weaken the concept of the topological interior. Rockafellar [5] introduced the relative interior, which is the generalization of the topological interior. Based on the relative interior, Frenk and Kassay [6, 7] obtained Lagrangian duality theorems and Bot et al. [8] studied strong duality for generalized convex optimization problems. Borwein and Lewis [9] introduced the quasi-relative interior. Bot et al. [10] studied the regularity conditions via quasi-relative interior in convex programming. However, we find that only a few papers [11, 12] are about set-valued optimization involving the relative interior. In this paper, we will further study set-valued optimization problems involving relative interior.

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, a kind of generalized weak convexlike set-valued map involving relative interior is introduced, and a separation property is established. In Section 4, some optimality conditions, including the generalized Kuhn-Tucker condition and scalarization theorem, are obtained.

2. Preliminaries

Let X, Y, and Z be three separated locally convex spaces, and let 0 denote the zero element for every space. Let K be a nonempty subset of Y. The generated cone of K is defined as $\operatorname{cone} K = \{\lambda a \mid a \in K,\ \lambda \geq 0\}$. A cone $K \subseteq Y$ is said to be pointed if $K \cap (-K) = \{0\}$. A cone $K \subseteq Y$ is said to be nontrivial if $K \neq \{0\}$ and $K \neq Y$.

Let Y^* and Z^* stand for the topological dual space of Y and Z, respectively. From now on, let C and D be nontrivial pointed closed-convex cones in Y and Z, respectively. The topological dual cone C^+ and strict topological dual cone C^{+i} of C are defined as

$$\begin{aligned} C^+ &= \{y^* \in Y^* \mid \langle y, y^* \rangle \geqslant 0,\ \forall y \in C\}, \\ C^{+i} &= \{y^* \in Y^* \mid \langle y, y^* \rangle > 0,\ \forall y \in C \setminus \{0\}\}, \end{aligned} \tag{2.1}$$

where $\langle y, y^* \rangle$ denotes the value of the linear continuous functional y^* at the point y. The meanings of D^+ and D^{+i} are similar.

Let K be a nonempty subset of Y. We denote by $\operatorname{cl} K$, $\operatorname{int} K$, and $\operatorname{aff} K$ the closed hull, topological interior, and affine hull of K, respectively.

Definition 2.1 (see [11, 13]). Let K be a subset of Y. The relative interior of K is the set

$$\operatorname{ri} K = \{x \in K \mid \text{there exists } U, \text{ a neighborhood of } x, \text{ such that } U \cap \operatorname{aff} K \subseteq K\}. \tag{2.2}$$

Now, we give some basic properties about the relative interior.

Lemma 2.2. *Let K be a subset of Y. Let $k_0 \in K$, $\overline{k} \in \operatorname{ri} K$, $\alpha \in R$, and $\lambda \in (0,1]$. Then,*

(a) $\alpha \operatorname{ri} K = \operatorname{ri}(\alpha K)$;

(b) *if K is convex, then*

$$(1-\lambda)k_0 + \lambda \overline{k} \in \operatorname{ri} K. \tag{2.3}$$

Proof. (a) Since $\alpha \operatorname{aff} K = \operatorname{aff}(\alpha K)$, it is clear that $\alpha \operatorname{ri} K = \operatorname{ri}(\alpha K)$;

(b) since $\overline{k} \in \operatorname{ri} K$, there exists V, a neighborhood of 0, such that

$$\left(\overline{k} + V\right) \cap \operatorname{aff} K \subseteq K. \tag{2.4}$$

By (2.4), we have

$$\left(\lambda\overline{k} + \lambda V\right) \cap (\lambda \operatorname{aff} K) \subseteq \lambda K. \tag{2.5}$$

It follows from (2.5) that

$$\left((1-\lambda)k_0 + \lambda\overline{k} + \lambda V\right) \cap ((1-\lambda)k_0 + \lambda \operatorname{aff} K) \subseteq (1-\lambda)k_0 + \lambda K. \tag{2.6}$$

It is clear that

$$(1-\lambda)k_0 + \lambda \operatorname{aff} K = \operatorname{aff} K. \tag{2.7}$$

Since K is convex, we have

$$(1-\lambda)k_0 + \lambda K \subseteq K. \tag{2.8}$$

By (2.6), (2.7), and (2.8), we obtain

$$\left((1-\lambda)k_0 + \lambda\overline{k} + \lambda V\right) \cap \operatorname{aff} K \subseteq K, \tag{2.9}$$

which implies that

$$(1-\lambda)k_0 + \lambda\overline{k} \in \operatorname{ri} K. \tag{2.10}$$

□

Remark 2.3. By Lemma 2.2, if K is a convex cone, then $\operatorname{ri} K \cup \{0\}$ is a convex cone.

Lemma 2.4. *If K is a convex cone of Y, then*

$$K + \operatorname{ri} K \subseteq \operatorname{ri} K. \tag{2.11}$$

Proof. If $\operatorname{ri} K = \phi$, it is clear that the conclusion holds. If $\operatorname{ri} K \neq \phi$, we have

$$K + \operatorname{ri} K = 2\left(\frac{1}{2}K + \frac{1}{2}\operatorname{ri} K\right) \subseteq 2\operatorname{ri} K = \operatorname{ri} 2K = \operatorname{ri} K, \tag{2.12}$$

where Lemma 2.2(b) is used in the first inclusion relation and Lemma 2.2(a) is used in the second equality. □

Lemma 2.5 (see [14, 15]). *Let W be a linear topological space and w^* be a linear functional on W. w^* is continuous if and only if $H = \{w \mid \langle w, w^*\rangle = 0, w \in W\}$ is closed. If H is not closed, H is dense in W.*

We will close this section by giving a separation theorem based on the relative interior.

Lemma 2.6 (see [11]). *Let $K \subseteq Y$ be a closed-convex set with* $\operatorname{ri} K \neq \phi$. *If* $0 \notin \operatorname{ri} K$, *then there exists* $y^* \in Y^* \setminus \{0\}$ *such that* $\langle k, y^* \rangle \geq 0$ *for each* $k \in K$.

Remark 2.7. The following example will show that the closeness of K cannot be deleted in Lemma 2.6.

Example 2.8. Let Y be an infinite-dimensional normed space and k^* be a non-continuous linear functional on Y. K is defined as

$$K = \{k \mid \langle k, k^* \rangle = 1,\ k \in Y\}. \tag{2.13}$$

Since $\operatorname{aff} K = K$, it is clear that $0 \notin \operatorname{ri} K = K$. By Lemma 2.5, K is not closed and $\operatorname{cl} K = Y$. Therefore, for any $y^* \in Y^* \setminus \{0\}$, y^* cannot separate 0 and K.

Remark 2.9. Example 2.8 shows that, even if K is a convex subset of Y, the expression that $\operatorname{ri}(\operatorname{cl} K) = \operatorname{ri} K$ does not hold generally.

3. Separation Property

From now on, we suppose that $\operatorname{ri} C \neq \phi$ and $\operatorname{ri} D \neq \phi$. Let A be a nonempty subset of X and $F : A \to 2^Y$ be a set-valued map on A. Write $F(A) = \cup_{x \in A} F(x)$.

Definition 3.1 (see [1]). Let A be a nonempty subset of X. A set-valued map $F : A \to 2^Y$ is called C-convexlike on A if the set $F(A) + C$ is convex.

In [2, 3, 16, 17], when $\operatorname{int} C \neq \phi$, C-subconvexlike map and generalized C-subconvexlike map were introduced, respectively. The following two definitions are generalizations of C-subconvexlike map and generalized C-subconvexlike map, respectively.

Definition 3.2 (see [12]). Let A be a nonempty subset of X. A set-valued map $F : A \to 2^Y$ is called C-weak convexlike on A if the set $F(A) + \operatorname{ri} C$ is convex.

Definition 3.3 (see [12]). Let A be a nonempty subset of X. A set-valued map $F : A \to 2^Y$ is called generalized C-weak convexlike on A if the set $\operatorname{cone} F(A) + \operatorname{ri} C$ is convex.

Remark 3.4. By [12, Theorems 3.1 and 3.2], we have the following implications:
C-convexlikeness $\Rightarrow$ C-weak convexlikeness $\Rightarrow$ generalized C-weak convexlikeness.

However, the following two examples show that the converse of the above implications is not generally true.

Example 3.5. Let $X = Y = R^2$, $C = \{(y_1, 0) \mid y_1 \geq 0\}$, and $A = \{(1,0), (0,2)\}$. The set-valued map $F : A \to 2^Y$ is defined as follows:

$$\begin{aligned} F(1,0) &= \{(y_1, y_2) \mid 1 < y_1 \leq 2,\ 0 \leq y_2 \leq 1\} \cup \{(1,0),\ (1,1)\}, \\ F(0,2) &= \{(y_1, y_2) \mid 1 < y_1 \leq 2,\ 1 \leq y_2 \leq 2\} \cup \{(1,2),\ (1,1)\}. \end{aligned} \tag{3.1}$$

It is clear that $F(A) + \operatorname{ri} C$ is convex and $F(A) + C$ is not convex. Therefore, F is C-weak convexlike on A. However, F is not C-convexlike on A.

Example 3.6. Let $X = Y = R^2$, $C = \{(y_1, 0) \mid y_1 \geq 0\}$, and $A = \{(1,0), (0,2)\}$. The set-valued map $F : A \to 2^Y$ is defined as follows:

$$\begin{aligned} F(1,0) &= \{(y_1, y_2) \mid y_1 \geq 0,\ 1 \leq y_2 \leq -y_1 + 2\}, \\ F(0,2) &= \{(y_1, y_2) \mid y_1 \geq 1,\ 0 \leq y_2 \leq -y_1 + 2\}. \end{aligned} \tag{3.2}$$

It is clear that $\operatorname{cone} F(A) + \operatorname{ri} C$ is convex and $F(A) + \operatorname{ri} C$ is not convex. Therefore, F is generalized C-weak convexlike on A. However, F is not C-weak convexlike on A.

Now, we consider the following two systems.

System 1: There exists $x_0 \in A$ such that $F(x_0) \cap (-\operatorname{ri} C) \neq \phi$.

System 2: There exists $y^* \in C^+ \setminus \{0\}$ such that $\langle y, y^* \rangle \geq 0$, for all $y \in F(A)$.

Theorem 3.7. *Let A be a nonempty subset of X.*

(i) *Suppose that $F : A \to 2^Y$ is generalized C-weak convexlike on A and* $\operatorname{ri}(\operatorname{cl}(\operatorname{cone} F(A) + \operatorname{ri} C)) = \operatorname{ri}(\operatorname{cone} F(A) + \operatorname{ri} C) \neq \phi$. *If System 1 has no solution, then System 2 has solution.*

(ii) *If $y^* \in C^{+i}$ is a solution of System 2, then System 1 has no solution.*

Proof. (i) Firstly, we assert that $0 \notin \operatorname{cone} F(A) + \operatorname{ri} C$. Otherwise, there exist $x_0 \in A$, $\alpha \geq 0$ such that $0 \in \alpha F(x_0) + \operatorname{ri} C$.

Case 1. If $\alpha = 0$, then $0 \in \operatorname{ri} C$. Thus, there exists U, a neighborhood of 0, such that

$$U \cap \operatorname{aff} C \subseteq C. \tag{3.3}$$

Without loss of generality, we suppose that U is symmetric. It follows from (3.3) that

$$U \cap (-\operatorname{aff} C) \subseteq (-C). \tag{3.4}$$

It is clear that $\operatorname{aff} C$ is a linear subspace of Y. Therefore, $\operatorname{aff} C = -\operatorname{aff} C$. By (3.4), we have

$$U \cap \operatorname{aff} C \subseteq (-C). \tag{3.5}$$

By (3.3) and (3.5), we obtain

$$U \cap \operatorname{aff} C \subseteq C \cap (-C). \tag{3.6}$$

Since C is nontrivial, there exists $\overline{c} \in C \setminus \{0\}$. By the absorption of U, there exists λ, a sufficiently small positive number, such that

$$\lambda \overline{c} \in U \cap \operatorname{aff} C \subseteq C \cap (-C), \tag{3.7}$$

which contradicts that C is pointed.

Case 2. If $\alpha > 0$, there exists $y_0 \in F(x_0)$ such that $-y_0 \in (1/\alpha)\operatorname{ri} C \subseteq \operatorname{ri} C$, which contradicts $F(x) \cap (-\operatorname{ri} C) = \phi$, for all $x \in A$.

Therefore, our assertion is true. Thus, we obtain

$$0 \notin \operatorname{ri}(\operatorname{cl}(\operatorname{cone} F(A) + \operatorname{ri} C)). \tag{3.8}$$

Since F is generalized C-weak convexlike on A, $\operatorname{cl}(\operatorname{cone} F(A) + \operatorname{ri} C)$ is a closed-convex set. By Lemma 2.6, there exists $y^* \in Y^* \setminus \{0\}$ such that

$$\langle y, y^* \rangle \geq 0, \quad \forall y \in \operatorname{cl}(\operatorname{cone} F(A) + \operatorname{ri} C). \tag{3.9}$$

So,

$$\langle \alpha F(x) + c, y^* \rangle \geq 0, \quad \forall x \in A,\ c \in \operatorname{ri} C,\ \alpha \geq 0. \tag{3.10}$$

Letting $\alpha = 0$ in (3.10), we obtain

$$\langle c, y^* \rangle \geq 0, \quad \forall c \in \operatorname{ri} C. \tag{3.11}$$

We assert that $y^* \in C^+$. Otherwise, there exists $c' \in C$ such that $\langle c', y^* \rangle < 0$, hence, $\langle \theta c', y^* \rangle < 0$, for all $\theta > 0$. By Lemma 2.4, we have

$$\theta c' + c \in \operatorname{ri} C, \quad \forall c \in \operatorname{ri} C. \tag{3.12}$$

It follows from (3.11) that

$$\langle \theta c' + c, y^* \rangle \geq 0, \quad \forall \theta > 0,\ c \in \operatorname{ri} C. \tag{3.13}$$

Thus, we obtain

$$\theta \langle c', y^* \rangle + \langle c, y^* \rangle \geq 0, \quad \forall \theta > 0,\ c \in \operatorname{ri} C. \tag{3.14}$$

On the other hand, (3.14) does not hold when $\theta > -\langle c, y^* \rangle / \langle c', y^* \rangle \geq 0$. Therefore, $\langle c, y^* \rangle \geq 0$, for all $c \in C$, that is, $y^* \in C^+$.

Letting $\alpha = 1$ in (3.10), we have

$$\langle F(x) + c, y^* \rangle \geq 0, \quad \forall x \in A,\ c \in \operatorname{ri} C. \tag{3.15}$$

Taking $c_0 \in \operatorname{ri} C$, $\lambda_n > 0$, $\lim_{n \to \infty} \lambda_n = 0$, we have

$$\langle F(x) + \lambda_n c_0, y^* \rangle \geq 0, \quad \forall x \in A,\ n \in N. \tag{3.16}$$

Limitting (3.16), we obtain $\langle F(x), y^* \rangle \geq 0$, for all $x \in A$.

(ii) Since $y^* \in C^{+i}$ is a solution of System 2, we have

$$\langle y, y^* \rangle \geq 0, \quad \forall y \in F(A). \tag{3.17}$$

Now, we suppose that System 1 has solution. Then, there exists $x_0 \in A$ such that $F(x_0) \cap (-\operatorname{ri} C) \neq \phi$. Thus, there exists $y_0 \in F(x_0)$ such that $-y_0 \in \operatorname{ri} C$. It is clear that $-y_0 \neq 0$. So, we have

$$\langle y_0, y^* \rangle < 0, \tag{3.18}$$

which contradicts (3.17). □

Remark 3.8. If $Y = R^n$, by [5, Theorems 6.2 and 6.3], the condition that $\operatorname{ri}(\operatorname{cl}(\operatorname{cone} F(A) + \operatorname{ri} C)) = \operatorname{ri}(\operatorname{cone} F(A) + \operatorname{ri} C) \neq \phi$ holds automatically. However, by Remark 2.9, it is possible that, the condition that $\operatorname{ri}(\operatorname{cl}(\operatorname{cone} F(A) + \operatorname{ri} C)) = \operatorname{ri}(\operatorname{cone} F(A) + \operatorname{ri} C) \neq \phi$ does not hold. Therefore, our assumption is reasonable.

4. Optimality Conditions

Let $F : A \to 2^Y$ and $G : A \to 2^Z$ be two set-valued maps from A to Y and Z, respectively. Now, we consider the following vector optimization problem of set-valued maps:

$$\begin{aligned} \min \quad & F(x) \\ \text{s.t.} \quad & -G(x) \cap D \neq \phi. \end{aligned} \tag{VP}$$

The feasible set of (VP) is defined by

$$S = \{x \in A \mid -G(x) \cap D \neq \phi\}. \tag{4.1}$$

Now, we define

$$\begin{gathered} W \operatorname{Min}(F(S), C) = \{y_0 \in F(S) \mid y_0 - y \notin \operatorname{ri} C,\ \forall y \in F(S)\}, \\ P \operatorname{Min}(F(S), C) = \{y_0 \in F(S) \mid (-C) \cap \operatorname{cl}(\operatorname{cone}(F(S) + C - y_0)) = \{0\}\}. \end{gathered} \tag{4.2}$$

Definition 4.1. A point x_0 is called a weakly efficient solution of (VP) if $x_0 \in S$ and $F(x_0) \cap W \operatorname{Min}(F(S), C) \neq \phi$. A point pair (x_0, y_0) is called a weak minimizer of (VP) if $y_0 \in F(x_0) \cap W \operatorname{Min}(F(S), C)$.

Definition 4.2. A point x_0 is called a Benson properly efficient solution of (VP) if $x_0 \in S$ and $F(x_0) \cap P \operatorname{Min}(F(S), C) \neq \phi$. A point pair (x_0, y_0) is called a Benson proper minimizer of (VP) if $y_0 \in F(x_0) \cap P \operatorname{Min}(F(S), C)$.

Let $I(x) = F(x) \times G(x)$, for all $x \in A$. It is clear that I is a set-valued map from A to $Y \times Z$, where $Y \times Z$ is a seperated local convex space with nontrivial pointed closed-convex cone $C \times D$. The topological dual space of $Y \times Z$ is $Y^* \times Z^*$, and the topological dual cone of $C \times D$ is $C^+ \times D^+$.

By Definition 3.3, we say that the set-valued map $I : A \to 2^{Y \times Z}$ is generalized $C \times D$-weak convexlike on A if cone $I(A) + \mathrm{ri}(C \times D)$ is a convex set of $Y \times Z$.

Theorem 4.3. *Let* $\mathrm{ri}(\mathrm{cl}(\mathrm{cone}\, I^*(A) + \mathrm{ri}(C \times D))) = \mathrm{ri}(\mathrm{cone}\, I^*(A) + \mathrm{ri}(C \times D)) \neq \phi$. *Suppose that the following conditions hold:*

(i) (x_0, y_0) *is a weak minimizer of* (VP);

(ii) $I^*(x)$ *is generalized* $C \times D$*-weak convexlike on* A, *where* $I^*(x) = (F(x) - y_0) \times G(x)$.

Then, there exists $(y^*, z^*) \in C^+ \times D^+$ *with* $(y^*, z^*) \neq (0, 0)$ *such that*

$$\inf_{x \in A}\left(\langle F(x), y^* \rangle + \langle G(x), z^* \rangle\right) = \langle y_0, y^* \rangle,$$
$$\inf \langle G(x_0), z^* \rangle = 0. \tag{4.3}$$

Proof. According to Definition 4.1, we have

$$\left(y_0 - F(S)\right) \cap \mathrm{ri}\, C = \phi. \tag{4.4}$$

It is clear that $I^*(x) = I(x) - (y_0, 0)$, for all $x \in A$. We assert that

$$-I^*(x) \cap \mathrm{ri}(C \times D) = \phi, \quad \forall x \in A. \tag{4.5}$$

Otherwise, there exists $\overline{x} \in A$ such that

$$-I^*(\overline{x}) \cap \mathrm{ri}(C \times D) \neq \phi. \tag{4.6}$$

It is easy to check that $\mathrm{ri}(C \times D) = \mathrm{ri}\, C \times \mathrm{ri}\, D$. Therefore,

$$-I^*(\overline{x}) \cap (\mathrm{ri}\, C \times \mathrm{ri}\, D) \neq \phi. \tag{4.7}$$

By (4.7), we obtain

$$\left(y_0 - F(\overline{x})\right) \cap \mathrm{ri}\, C \neq \phi, \tag{4.8}$$

$$-G(\overline{x}) \cap \mathrm{ri}\, D \neq \phi. \tag{4.9}$$

It follows from (4.9) that $\overline{x} \in S$. Thus, by (4.8), we have

$$\left(y_0 - F(S)\right) \cap \mathrm{ri}\, C \neq \phi, \tag{4.10}$$

which contradicts (4.4). Therefore, (4.5) holds.

By Theorem 3.7, there exists $(y^*, z^*) \in C^+ \times D^+$ with $(y^*, z^*) \neq (0,0)$ such that

$$\langle I^*(x), (y^*, z^*) \rangle \geq 0, \quad \forall x \in A. \tag{4.11}$$

That is,

$$\langle F(x), y^* \rangle + \langle G(x), z^* \rangle \geq \langle y_0, y^* \rangle, \quad \forall x \in A. \tag{4.12}$$

Since $x_0 \in S$, there exists $p \in G(x_0)$ such that $-p \in D$. Because $z^* \in D^+$, we obtain $\langle p, z^* \rangle \leq 0$. On the other hand, taking $x = x_0$ in (4.12), we get

$$\langle y_0, y^* \rangle + \langle p, z^* \rangle \geq \langle y_0, y^* \rangle. \tag{4.13}$$

It follows that $\langle p, z^* \rangle \geq 0$. So, $\langle p, z^* \rangle = 0$. Thus, we have

$$\langle y_0, y^* \rangle \in \langle F(x_0), y^* \rangle + \langle G(x_0), z^* \rangle. \tag{4.14}$$

Therefore, it follows from (4.12) and (4.14) that

$$\inf_{x \in A} \left(\langle F(x), y^* \rangle + \langle G(x), z^* \rangle \right) = \langle y_0, y^* \rangle. \tag{4.15}$$

Finally, taking again $x = x_0$ in (4.12), we obtain

$$\langle y_0, y^* \rangle + \langle G(x_0), z^* \rangle \geq \langle y_0, y^* \rangle. \tag{4.16}$$

So, $\langle G(x_0), z^* \rangle \geq 0$. We have shown that there exists $p \in G(x_0)$ such that $\langle p, z^* \rangle = 0$. Thus, we have

$$\inf \langle G(x_0), z^* \rangle = 0. \tag{4.17}$$

□

The following example will be used to illustrate Theorem 4.3.

Example 4.4. Let $X = Y = Z = R^2$, $C = D = \{(y_1, 0) \mid y_1 \geq 0\}$, and $A = \{(1,0), (1,2)\}$. The set-valued map $F : A \to 2^Y$ is defined as follows:

$$\begin{aligned} F(1,0) &= \{(y_1, y_2) \mid y_1 = 1,\ 0 \leq y_2 \leq 1\}, \\ F(1,2) &= \{(y_1, y_2) \mid y_1 > 1,\ 0 \leq y_2 \leq -y_1 + 2\}. \end{aligned} \tag{4.18}$$

The set-valued map $G : A \to 2^Y$ is defined as follows:

$$\begin{aligned} G(1,0) &= \{(y_1, y_2) \mid y_1 \leq 0,\ 0 \leq y_2 \leq y_1 + 1\}, \\ G(1,2) &= \{(y_1, y_2) \mid y_1 \geq -1,\ y_1 + 1 \leq y_2 \leq 1\}. \end{aligned} \tag{4.19}$$

Let $x_0 = (1,0)$ and $y_0 = (1,0) \in F(x_0)$. It is clear that all conditions of Theorem 4.3 are satisfied. Therefore, there exist $y^* : \langle (y_1, y_2), y^* \rangle = y_1 + y_2$ and $z^* : \langle (y_1, y_2), z^* \rangle = -y_1 + y_2$ such that

$$\inf_{x \in A} \left(\langle F(x), y^* \rangle + \langle G(x), z^* \rangle \right) = \langle y_0, y^* \rangle, \qquad \inf \langle G(x_0), z^* \rangle = 0. \tag{4.20}$$

Remark 4.5. Theorem 4.3 generalizes Theorem 3.1 of [2] and Theorem 4.2 of [3].

Theorem 4.6. *Suppose that the following conditions hold:*

(i) $x_0 \in S$;

(ii) *there exist* $y_0 \in F(x_0)$ *and* $(y^*, z^*) \in C^{+i} \times D^+$ *such that*

$$\inf_{x \in A} \left(\langle F(x), y^* \rangle + \langle G(x), z^* \rangle \right) \geq \langle y_0, y^* \rangle. \tag{4.21}$$

Then, x_0 *is a weakly efficient solution of* (VP).

Proof. By condition (ii), we have

$$\left\langle F(x) - y_0, y^* \right\rangle + \langle G(x), z^* \rangle \geq 0, \quad \forall x \in A. \tag{4.22}$$

Suppose to the contrary that x_0 is not a weakly efficient solution of (VP). Then, there exists $x' \in S$ such that $(y_0 - F(x')) \cap \operatorname{ri} C \neq \phi$. Therefore, there exists $t \in F(x')$ such that $y_0 - t \in \operatorname{ri} C \subsetneq C \setminus \{0\}$. Thus, we obtain

$$\left\langle t - y_0, y^* \right\rangle < 0. \tag{4.23}$$

Since $x' \in S$, there exists $q \in G(x')$ such that $-q \in D$. Hence,

$$\left\langle q, z^* \right\rangle \leq 0. \tag{4.24}$$

Adding (4.23) to (4.24), we have

$$\left\langle t - y_0, y^* \right\rangle + \left\langle q, z^* \right\rangle < 0, \tag{4.25}$$

which contradicts (4.22). Therefore, x_0 is a weakly efficient solution of (VP). □

The following example will be used to illustrate Theorem 4.6.

Example 4.7. Let $X = Y = Z = R^2$, $C = D = \{(y_1, 0) \mid y_1 \geq 0\}$, and $A = \{(1,0), (1,2)\}$. The set-valued map $F : A \to 2^Y$ is defined as follows:

$$F(1,0) = \left\{ (y_1, y_2) \mid y_1 \geq 1,\ y_1 \leq y_2 \leq 2 \right\}, \qquad F(1,2) = \left\{ (y_1, y_2) \mid y_1 \leq 2,\ 1 \leq y_2 \leq y_1 \right\}. \tag{4.26}$$

The set-valued map $G : A \to 2^Y$ is defined as follows:

$$G(1,0) = \{(y_1, y_2) \mid -1 \le y_1 \le 0,\ y_2 = 0\},$$
$$G(1,2) = \{(y_1, y_2) \mid -1 \le y_1 \le 0,\ 0 \le y_2 \le 1\}. \tag{4.27}$$

Let $x_0 = (1,0)$, $y_0 = (1,1) \in F(x_0)$, $\langle (y_1, y_2), y^* \rangle = y_1 + y_2$, and $\langle (y_1, y_2), z^* \rangle = -y_1$. It is clear that all conditions of Theorem 4.6 are satisfied. Therefore, $(1,0)$ is a weakly efficient solution of (VP).

Remark 4.8. Theorem 4.6 generalizes [2, Theorem 3.3].

Now, we consider the following scalar optimization problem $(\mathrm{VP})_\varphi$ of (VP):

$$\begin{aligned} \min \quad & \langle F(x), \varphi \rangle \\ \text{s.t.} \quad & x \in S, \end{aligned} \tag{VP$_\varphi$}$$

where $\varphi \in Y^* \setminus \{0\}$.

Definition 4.9. If $x_0 \in S$, $y_0 \in F(x_0)$ and

$$\langle y_0, \varphi \rangle \le \langle y, \varphi \rangle, \quad \forall y \in F(S), \tag{4.28}$$

then x_0 and (x_0, y_0) are called a minimal solution and a minimizer of $(\mathrm{VP})_\varphi$, respectively.

Lemma 4.10 (see [18]). *Let $U_1, U_2 \subset Y$ be two closed-convex cones such that $U_1 \cap U_2 = \{0\}$. If U_2 is pointed and locally compact, then $(-U_1^+) \cap U_2^{+i} \neq \phi$.*

Lemma 4.11. *If V is a subset of Y, then*

(i) $\mathrm{cl}(\mathrm{cone}(V + \mathrm{ri}\, C)) = \mathrm{cl}(\mathrm{cone}\, V + \mathrm{ri}\, C)$,

(ii) $\mathrm{cl}(\mathrm{cone}(V + \mathrm{ri}\, C)) = \mathrm{cl}(\mathrm{cone}(V + C))$.

Proof. (i) If $V = \phi$, it is obvious that

$$\mathrm{cl}(\mathrm{cone}(V + \mathrm{ri}\, C)) = \mathrm{cl}(\mathrm{cone}\, V + \mathrm{ri}\, C). \tag{4.29}$$

If $V \neq \phi$, there exists $c \in \mathrm{ri}\, C$. It is clear that

$$\lambda c \in \mathrm{cone}\, V + \mathrm{ri}\, C, \quad \forall \lambda \in (0, +\infty). \tag{4.30}$$

Letting $\lambda \to 0$ in (4.30), we have

$$0 \in \mathrm{cl}(\mathrm{cone}\, V + \mathrm{ri}\, C). \tag{4.31}$$

Now, we will show that

$$\operatorname{cone}(V + \operatorname{ri} C) \subseteq (\operatorname{cone} V + \operatorname{ri} C) \cup \{0\}. \tag{4.32}$$

Let $y \in \operatorname{cone}(V + \operatorname{ri} C)$.

Case 1. If $y = 0$, then $y \in (\operatorname{cone} V + \operatorname{ri} C) \cup \{0\}$.

Case 2. If $y \neq 0$, there exist $\alpha > 0$, $v \in V$, and $\overline{c} \in \operatorname{ri} C$ such that

$$y = \alpha(v + \overline{c}) = \alpha v + \alpha\overline{c} \in \operatorname{cone} V + \operatorname{ri} C \subseteq (\operatorname{cone} V + \operatorname{ri} C) \cup \{0\}. \tag{4.33}$$

Therefore, (4.32) holds. Since Y is separated, by (4.31) and (4.32), we obtain

$$\begin{aligned} \operatorname{cl}(\operatorname{cone}(V + \operatorname{ri} C)) &\subseteq \operatorname{cl}((\operatorname{cone} V + \operatorname{ri} C) \cup \{0\}) \\ &= \operatorname{cl}(\operatorname{cone} V + \operatorname{ri} C) \cup \operatorname{cl}\{0\} \\ &= \operatorname{cl}(\operatorname{cone} V + \operatorname{ri} C) \cup \{0\} \\ &= \operatorname{cl}(\operatorname{cone} V + \operatorname{ri} C). \end{aligned} \tag{4.34}$$

That is,

$$\operatorname{cl}(\operatorname{cone}(V + \operatorname{ri} C)) \subseteq \operatorname{cl}(\operatorname{cone} V + \operatorname{ri} C). \tag{4.35}$$

Using the technique of Lemma 2.1 in [19], we easily obtain

$$\operatorname{cone} V + \operatorname{ri} C \subseteq \operatorname{cl}(\operatorname{cone}(V + \operatorname{ri} C)). \tag{4.36}$$

So,

$$\operatorname{cl}(\operatorname{cone} V + \operatorname{ri} C) \subseteq \operatorname{cl}(\operatorname{cone}(V + \operatorname{ri} C)). \tag{4.37}$$

By (4.35) and (4.37), we have

$$\operatorname{cl}(\operatorname{cone}(V + \operatorname{ri} C)) = \operatorname{cl}(\operatorname{cone} V + \operatorname{ri} C). \tag{4.38}$$

(ii) It is obvious that

$$\operatorname{cl}(\operatorname{cone}(V + \operatorname{ri} C)) \subseteq \operatorname{cl}(\operatorname{cone}(V + C)). \tag{4.39}$$

We will show that

$$\operatorname{cone}(V + C) \subseteq \operatorname{cl}(\operatorname{cone}(V + \operatorname{ri} C)). \tag{4.40}$$

It is clear that (4.40) holds if $V = \phi$. Now, we suppose that $V \neq \phi$. Let $y \in \operatorname{cone}(V + C)$, then there exist $\lambda \geq 0$, $v \in V$, and $c \in C$ such that

$$y = \lambda(v + c). \tag{4.41}$$

Since $\operatorname{ri} C \neq \phi$, there exists $c_0 \in \operatorname{ri} C$. It follows from Lemma 2.4 that

$$\frac{\lambda}{\alpha} c_0 + y = \lambda\left(\frac{1}{\alpha} c_0 + c + v\right) \in \operatorname{cone}(V + \operatorname{ri} C), \quad \forall \alpha > 0. \tag{4.42}$$

Letting $\alpha \to +\infty$ in (4.42), we have

$$y \in \operatorname{cl}(\operatorname{cone}(V + \operatorname{ri} C)), \tag{4.43}$$

which implies that (4.40) holds. By (4.40), we obtain

$$\operatorname{cl}(\operatorname{cone}(V + C)) \subseteq \operatorname{cl}(\operatorname{cone}(V + \operatorname{ri} C)). \tag{4.44}$$

By (4.39) and (4.44), we have

$$\operatorname{cl}(\operatorname{cone}(V + \operatorname{ri} C)) = \operatorname{cl}(\operatorname{cone}(V + C)). \tag{4.45}$$

□

Theorem 4.12. *Suppose that the following conditions hold:*

(i) $C \subseteq Y$ *is locally compact;*

(ii) (x_0, y_0) *is a Benson proper minimizer of* (VP)*;*

(iii) $F - y_0$ *is generalized C-weak convexlike on S.*

Then, there exists $\varphi \in C^{+i}$ such that (x_0, y_0) is a minimizer of $(VP)_\varphi$.

Proof. By condition (ii), we have

$$(-C) \cap \operatorname{cl}\big(\operatorname{cone}\big(F(S) + C - y_0\big)\big) = \{0\}. \tag{4.46}$$

By Lemma 4.11 and condition (iii), we obtain that $\operatorname{cl}(\operatorname{cone}(F(S) + C - y_0))$ is a closed-convex cone. Thus, conditions of Lemma 4.10 are satisfied. Therefore, there exists $\varphi \in C^{+i}$ such that

$$\varphi \in \big(\operatorname{cl}\big(\operatorname{cone}\big(F(S) + C - y_0\big)\big)\big)^+. \tag{4.47}$$

Since $F(S) - y_0 \subseteq \operatorname{cl}(\operatorname{cone}(F(S) + C - y_0))$, we obtain

$$\langle y - y_0, \varphi \rangle \geq 0, \quad \forall y \in F(S). \tag{4.48}$$

That is,

$$\langle y, \varphi \rangle \geq \langle y_0, \varphi \rangle, \quad \forall y \in F(S). \tag{4.49}$$

So, (x_0, y_0) is a minimizer of $(\mathrm{VP})_\varphi$. □

The following example will be used to illustrate Theorem 4.12.

Example 4.13. Let $X = Y = Z = R^2$, $C = D = \{(y_1, 0) \mid y_1 \geq 0\}$, and $A = \{(1,0),(1,2)\}$. The set-valued map $F : A \to 2^Y$ is defined as follows:

$$\begin{gathered} F(1,0) = \{(y_1, y_2) \mid y_1 \geq 1, 2 \leq y_2 \leq -y_1 + 4\} \cup \{(1,1)\}, \\ F(1,2) = \{(y_1, y_2) \mid y_1 \geq 2, 1 \leq y_2 \leq -y_1 + 4\}. \end{gathered} \tag{4.50}$$

The set-valued map $G : A \to 2^Z$ is defined as follows:

$$\begin{gathered} G(1,0) = \{(y_1, y_2) \mid y_1 \leq 0,\ 0 \leq y_2 \leq y_1 + 1\}, \\ G(1,2) = \{(y_1, y_2) \mid y_1 \geq -1,\ y_1 + 1 \leq y_2 \leq 1\}. \end{gathered} \tag{4.51}$$

Let $x_0 = (1,0)$, $y_0 = (1,1) \in F(x_0)$. Thus, all conditions of Theorem 4.12 are satisfied. Therefore, there exists $\varphi : \langle (y_1, y_2), \varphi \rangle = y_1 + y_2$ such that (x_0, y_0) is a minimizer of $(\mathrm{VP})_\varphi$.

Remark 4.14. Theorem 4.12 generalizes Theorem 4.2 of [16] and the necessity of Theorem 4.1 of [17].

In this paper, our results improve some results in the literature, and our results are very useful to form Lagrange multipliers rule and establish duality theory.

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