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*Research Article*

# Optimality Conditions and Duality in Nonsmooth Multiobjective Programs

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We study nonsmooth multiobjective programming problems involving locally Lipschitz functions and support functions. Two types of Karush-Kuhn-Tucker optimality conditions with support functions are introduced. Sufficient optimality conditions are presented by using generalized convexity and certain regularity conditions. We formulate Wolfe-type dual and Mond-Weir-type dual problems for our nonsmooth multiobjective problems and establish duality theorems for (weak) Pareto-optimal solutions under generalized convexity assumptions and regularity conditions.

## 1. Introduction

Multiobjective programming problems arise when more than one objective function is to be optimized over a given feasible region. Pareto optimum is the optimality concept that appears to be the natural extension of the optimization of a single objective to the consideration of multiple objectives.

In 1961, Wolfe [1] obtained a duality theorem for differentiable convex programming. Afterwards, a number of different duals distinct from the Wolfe dual are proposed for the nonlinear programs by Mond and Weir [2]. Duality relations for multiobjective programming problems with generalized convexity conditions were given by several authors [3–10]. Majumdar [11] gave sufficient optimality conditions for differentiable multiobjective programming which modified those given in Singh [12] under the assumption of convexity, pseudoconvexity, and quasiconvexity of the functions involved at the Pareto-optimal solution. Subsequently, Kim et al. [13] gave a counterexample showing that some theorems of Majumdar [11] are incorrect and establish sufficient optimality theorems for (weak) Pareto-optimal solutions by using modified conditions. Later on, Kim and Schaible [6] introduced nonsmooth multiobjective programming problems involving locally Lipschitz functions for inequality and equality constraints. They extended sufficient optimality conditions in Kim et al. [13] to the nonsmooth case and established duality theorems for nonsmooth multiobjective programming problems involving locally Lipschitz functions.

In this paper, we apply the results in Kim and Schaible [6] for this problem to nonsmooth multiobjective programming problem involving support functions. We introduce nonsmooth multiobjective programming problems involving locally Lipschitz functions and support functions for inequality and equality constraints. Two kinds of sufficient optimality conditions under various convexity assumptions and certain regularity conditions are presented. We propose a Wolfe-type dual and a Mond-Weir-type dual for the primal problem and establish duality results between the primal problem and its dual problems under generalized convexity and regularity conditions.

## 2. Notation and Definitions

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and $\mathbb{R}^n_+$ its nonnegative orthant.

We consider the following nonsmooth multiobjective programming problem involving locally Lipschitz functions:

$$\begin{aligned} &\text{minimize} \quad f(x) + s(x \mid C) = \left(f_1(x) + s(xC_1), \dots, f_m(x) + s(x \mid C_m)\right) \\ &\text{subject to} \quad g_j(x) \leqq 0, \quad j \in P, \quad h_k(x) = 0, \quad k \in Q, \quad x \in \mathbb{R}^n, \end{aligned} \tag{MP}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in M = \{1, 2, \dots, m\}$, $g_j : \mathbb{R}^n \to \mathbb{R}$, $j \in P = \{1, 2, \dots, p\}$, and $h_k : \mathbb{R}^n \to \mathbb{R}$, $k \in Q = \{1, 2, \dots, q\}$, are locally Lipschitz functions. Here, $C_i$, $i \in M$, is compact convex sets in $\mathbb{R}^n$. We accept the formal writing $C = (C_1, C_2, \dots, C_m)^t$ with the convention that $s(x \mid C) = (s(x \mid C_1), \dots, s(x \mid C_m))$, where $s(x \mid C_i)$ is the support function of $C_i$ (see Definition 2.2).

Throughout the article the following notation for order relations in $\mathbb{R}^n$ will be used:

$$\begin{gathered} x \leqq u \Longleftrightarrow u - x \in \mathbb{R}^n_+, \\ x \leq u \Longleftrightarrow u - x \in \mathbb{R}^n_+ \setminus \{0\}, \\ x < u \Longleftrightarrow u - x \in \operatorname{int} \mathbb{R}^n_+, \\ x \nleq u \text{ is the negation of } x \leq u, \\ x \nless u \text{ is the negation of } x < u. \end{gathered} \tag{2.1}$$

*Definition 2.1.* (i) A real-valued function $F : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz if for each $z \in \mathbb{R}^n$, there exist a positive constant $K$ and a neighborhood $N$ of $z$ such that, for every $x, y \in N$,

$$\left|F(x) - F(y)\right| \leqq K\|x - y\|, \tag{2.2}$$

where $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$.

(ii) The Clarke generalized directional derivative [14] of a locally Lipschitz function $F$ at $x$ in the direction $d \in \mathbb{R}^n$, denoted by $F^0(x;d)$, is defined as follows:

$$F^0(x;d) = \limsup_{y \to x,\, t \to 0+} t^{-1}(F(y+td) - F(y)), \tag{2.3}$$

where $y$ is a vector in $\mathbb{R}^n$.

(iii) The Clarke generalized subgradient [14] of $F$ at $x$ is denoted by

$$\partial F(x) = \left\{ \xi \in \mathbb{R}^n : F^0(x;d) \geqq \xi^t d,\ \forall d \in \mathbb{R}^n \right\}. \tag{2.4}$$

(iv) $F$ is said to be regular at $x$ if for all $d \in \mathbb{R}^n$ the one-sided directional derivative $F'(x;d)$ exists and $F'(x;d) = F^0(x;d)$.

*Definition 2.2* (see [10]). Let $C$ be a compact convex set in $\mathbb{R}^n$. The support function $s(x \mid C)$ of $C$ is defined by

$$s(x \mid C) := \max\{x^t y : y \in C\}. \tag{2.5}$$

The support function $s(x \mid C)$, being convex and everywhere finite, has a subdifferential, that is, for every $x \in \mathbb{R}^n$ there exists $z$ such that

$$s(y \mid C) \geq s(x \mid C) + z^t(y - x), \quad \forall y \in C. \tag{2.6}$$

Equivalently,

$$z^t x = s(x \mid C). \tag{2.7}$$

The subdifferential of $s(x \mid C)$ is given by

$$\partial s(x \mid C) := \{z \in C : z^t x = s(x \mid C)\}. \tag{2.8}$$

For any set $S \subset \mathbb{R}^n$, the normal cone to $S$ at a point $x \in S$ is defined by

$$N_S(x) := \{y \in \mathbb{R}^n : y^t(z - x) \leq 0,\ \forall z \in S\}. \tag{2.9}$$

It is readily verified that for a compact convex set $C$, $y$ is in $N_C(x)$ if and only if $s(y \mid C) = x^t y$, or equivalently, $x$ is in the subdifferential of $s$ at $y$.

In the notation of the problem (MP), we recall the definitions of convexity, affine, pseudoconvexity, and quasiconvexity for locally Lipschitz functions.

*Definition 2.3.* (i) $f = (f_1, f_2, \ldots, f_m)$ is convex (strictly convex) at $x^0 \in X$ if for each $x \in X$ and any $\xi_i \in \partial f_i(x^0)$, $f_i(x) - f_i(x^0) \geqq (>) \xi_i^t(x - x^0)$, for all $i \in M$.

(ii) $g_A$ is convex at $x^0 \in X$ if for each $x \in X$ and any $\zeta_j \in \partial g_j(x^0)$, $g_j(x) - g_j(x^0) \geqq \zeta_j^t(x - x^0)$, where $j \in A$, and $g_A$ denotes the active constraints at $x^0$.

(iii) $h = (h_1, h_2, \ldots, h_k)$ is convex at $x^0 \in X$ if for each $x \in X$ and any $\alpha_k \in \partial h_k(x^0)$, $h_k(x) - h_k(x^0) \geqq \alpha_k^t(x - x^0)$, for all $k \in Q$.

(iv) $h$ is affine at $x^0 \in X$ if for each $x \in X$ and any $\alpha_k \in \partial h_k(x^0)$, $h_k(x) - h_k(x^0) = \alpha_k^t(x - x^0)$, for all $k \in Q$.

(v) $f$ is pseudoconvex at $x^0 \in X$ if for each $x \in X$ and any $\xi_i \in \partial f_i(x^0)$, $\xi_i^t(x - x^0) \geqq 0$ implies $f_i(x) \geqq f_i(x^0)$, for all $i \in M$.

(vi) $f$ is strictly pseudoconvex at $x^0 \in X$ if for each $x \in X$ with $x \neq x^0$ and any $\xi_i \in \partial f_i(x^0)$, $\xi_i^t(x - x^0) \geqq 0$ implies $f_i(x) > f_i(x^0)$, for all $i \in M$.

(vii) $f$ is quasiconvex at $x^0 \in X$ if for each $x \in X$ with $x \neq x^0$ and any $\xi_i \in \partial f_i(x^0)$, $f_i(x) \leqq f_i(x^0)$ implies $\xi_i^t(x - x^0) \leqq 0$, for all $i \in M$.

Finally, we recall the definition of Pareto-optimal (efficient, nondominated) and weak Pareto-optimal solutions of (MP).

*Definition 2.4.* (i) A point $x^0 \in X$ is said to be a Pareto-optimal solution of (MP) if there exists no other $x \in X$ with $f(x) \leq f(x^0)$.

(ii) A point $x^0 \in X$ is said to be a weak Pareto-optimal solution of (MP) if there exists no other $x \in X$ with $f(x) < f(x^0)$.

## 3. Sufficient Optimality Conditions

In this section, we introduce the following two types of (KKT) conditions which differ only in the nonnegativity of the multipliers for the equality constraint and neither of which includes a complementary slackness condition, common in necessary optimality conditions [14].

$$\begin{gathered}
\exists u^0 \geq 0,\ v^0 \geqq 0,\ w^0 \geqq 0 \ \left(u^0 \in \mathbb{R}^m,\ v^0 \in \mathbb{R}^p,\ w^0 \in \mathbb{R}^q\right) \text{ such that} \\
0 \in \sum_{i \in M} u_i^0\left(\partial f_i\left(x^0\right) + z_i\right) + \sum_{j \in A} v_j^0 \partial g_j\left(x^0\right) + \sum_{k \in Q} w_k^0 \partial h_k\left(x^0\right), \\
g\left(x^0\right) \leqq 0, \qquad h\left(x^0\right) = 0, \qquad z_i^t x_i^0 = s\left(x^0 \mid C_i\right), \quad i \in M, \\
\text{where } A = \left\{j \in P : g_j\left(x^0\right) = 0\right\}.
\end{gathered} \tag{KKT}$$

$$\begin{gathered}
\exists u^0 \geq 0,\ v^0 \geqq 0,\ w^0 \ \left(u^0 \in \mathbb{R}^m,\ v^0 \in \mathbb{R}^p,\ w^0 \in \mathbb{R}^q\right) \text{ such that} \\
0 \in \sum_{i \in M} u_i^0\left(\partial f_i\left(x^0\right) + z_i\right) + \sum_{j \in A} v_j^0 \partial g_j\left(x^0\right) + \sum_{k \in Q} w_k^0 \partial h_k\left(x^0\right), \\
g\left(x^0\right) \leqq 0, \qquad h\left(x^0\right) = 0, \qquad z_i^t x_i^0 = s\left(x^0 \mid C_i\right), \quad i \in M, \\
\text{where } A = \left\{j \in P : g_j\left(x^0\right) = 0\right\}.
\end{gathered} \tag{KKT$'$}$$

In Theorems 3.1 and 3.2 and Corollaries 3.3 and 3.4 below we present new versions including support functions of the results by Kim and Schaible in [6] for smooth problems (MP) involving (KKT).

**Theorem 3.1.** *Let* $(x^0, u^0, v^0, w^0)$ *satisfy* (KKT). *If* $f(\cdot) + z^t(\cdot)$ *is pseudoconvex at* $x^0$, $g_A$ *and* $h$ *are quasiconvex at* $x^0$, *and* $f$ *is regular at* $x^0$, *then* $x^0$ *is a weak Pareto-optimal solution of* (MP).

*Proof.* Let $(x^0, u^0, v^0, w^0)$ satisfy (KKT). Then $g(x^0) \leqq 0$, $h(x^0) = 0$,

$$0 \in \sum_{i \in M} u_i^0 \left( \partial f_i \left( x^0 \right) + z_i \right) + \sum_{j \in A} v_j^0 \partial g_j \left( x^0 \right) + \sum_{k \in Q} w_k^0 \partial h_k \left( x^0 \right). \tag{3.1}$$

From (3.1), there exist $\xi_i \in \partial f_i(x^0)$, $\zeta_j \in \partial g_j(x^0)$, and $\alpha_k \in \partial h_k(x^0)$ such that

$$\sum_{i \in M} u_i^0 (\xi_i + z_i) + \sum_{j \in A} v_j^0 \zeta_j + \sum_{k \in Q} w_k^0 \alpha_k = 0. \tag{3.2}$$

Suppose that $x^0 \in X$ is not a weak Pareto-optimal solution of (MP). Then there exists $\overline{x} \in X$ such that $f(\overline{x}) + s(\overline{x} \mid C) < f(x^0) + s(x^0 \mid C)$ that implies $f(\overline{x}) + z^t \overline{x} < f(x^0) + z^t x^0$ because of $z^t x \leqq s(x \mid C)$ and the assumption $z^t x^0 = s(x^0 \mid C)$ which means that this function $s(x \mid C)$ is subdifferentiable and regular at $x^0$. By pseudoconvexity of $f(\cdot) + z^t(\cdot)$ at $x^0$, we have

$$(\xi_i + z_i)^t \left( \overline{x} - x^0 \right) < 0 \quad \text{for any } \xi_i \in \partial f_i \left( x^0 \right), \ i \in M. \tag{3.3}$$

Since $g_j(\overline{x}) \leqq 0 = g_j(x^0)$, $j \in A$, we obtain the following inequality with the help of quasiconvexity of $g_A$ at $x^0$:

$$\zeta_j^t \left( \overline{x} - x^0 \right) \leqq 0 \quad \text{for any } \zeta_j \in \partial g_j \left( x^0 \right), \ j \in A. \tag{3.4}$$

Also, since $h_k(\overline{x}) = 0 = h_k(x^0)$, $k \in Q$, it follows from quasiconvexity of $h$ at $x^0$ that

$$\alpha_k^t \left( \overline{x} - x^0 \right) \leqq 0 \quad \text{for any } \alpha_k \in \partial h_k \left( x^0 \right), \ k \in Q. \tag{3.5}$$

From (3.3)–(3.5), we obtain

$$\left[ \sum_{i \in M} u_i^0 (\xi_i + z_i) + \sum_{j \in A} v_j^0 \zeta_j + \sum_{k \in Q} w_k^0 \alpha_k \right]^t \left( \overline{x} - x^0 \right) < 0, \tag{3.6}$$

which contradicts (3.2). Hence $x^0$ is a weak Pareto-optimal solution of (MP). □

**Theorem 3.2.** *Let* $(x^0, u^0, v^0, w^0)$ *satisfy* (KKT). *If* $f(\cdot) + z^t(\cdot)$ *is strictly pseudoconvex at* $x^0$, $g_A$ *and* $h$ *are quasiconvex at* $x^0$, *and* $f$ *is regular at* $x^0$, *then* $x^0$ *is a Pareto-optimal solution of* (MP).

**Corollary 3.3.** *Let $(x^0, u^0, v^0, w^0)$ satisfy* (KKT). *If $f(\cdot) + z^t(\cdot)$, $g_A$ and $h$ are convex at $x^0$, and $f$ is regular at $x^0$, then $x^0$ is a weak Pareto-optimal solution of* (MP).

**Corollary 3.4.** *Let $(x^0, u^0, v^0, w^0)$ satisfy* (KKT). *If $f(\cdot) + z^t(\cdot)$ is strictly convex at $x^0$, $g_A$ and $h$ are convex at $x^0$, and $f$ is regular at $x^0$, then $x^0$ is a Pareto-optimal solution of* (MP).

**Theorem 3.5.** *Let $(x^0, u^0, v^0, w^0)$ satisfy* (KKT′). *If $f(\cdot) + z^t(\cdot)$ is quasiconvex at $x^0$, $(v^0)^t g_A + (w^0)^t h$ is strictly pseudoconvex at $x^0$, and $f$, $g_A$, and $h$ are regular at $x^0$, then $x^0$ is a Pareto-optimal solution of* (MP).

*Proof.* Suppose that $x^0 \in X$ is not a Pareto-optimal solution of (MP). Then there exists $\overline{x} \in X$ such that $f(\overline{x}) + s(\overline{x} \mid C) \leq f(x^0) + s(x^0 \mid C)$, that implies $f(\overline{x}) + z^t\overline{x} \leq f(x^0) + z^t x^0$ because of $z^t x \leqq s(x \mid C)$ and the assumption $z^t x^0 = s(x^0 \mid C)$. By quasiconvexity of $f(\cdot) + z^t(\cdot)$ at $x^0$, we have

$$(\xi_i + z_i)^t \left(\overline{x} - x^0\right) \leqq 0 \quad \text{for any } \xi_i \in \partial f_i\left(x^0\right),\ i \in M. \tag{3.7}$$

Since $(x^0, u^0, v^0, w^0)$ satisfy (KKT′), we obtain $[\sum_{j\in A} v_j^0 \zeta_j + \sum_{k\in Q} w_k^0 \alpha_k]^t (\overline{x} - x^0) \geqq 0$ for some $\zeta_j \in \partial g_j(x^0)$, $j \in A$, and $\alpha_k \in \partial h_k(x^0)$, $k \in Q$. By regularity of $g_A$ and $h$ at $x^0$, there exists $\beta \in \partial (v^0)^t g_A + (w^0)^t h$ such that $\beta^t(\overline{x} - x^0) \geqq 0$. With the help of a strict pseudoconvexity of $(v^0)^t g_A + (w^0)^t h$, we have

$$\left(v^0\right)^t g_A(\overline{x}) + \left(w^0\right)^t h(\overline{x}) > \left(v^0\right)^t g_A\left(x^0\right) + \left(w^0\right)^t h\left(x^0\right). \tag{3.8}$$

Since $\overline{x} \in X$, we obtain

$$\left(v^0\right)^t g_A(\overline{x}) + \left(w^0\right)^t h(\overline{x}) \leqq 0. \tag{3.9}$$

Since $g_A(x^0) = h(x^0) = 0$, we obtain

$$\left(v^0\right)^t g_A\left(x^0\right) + \left(w^0\right)^t h\left(x^0\right) = 0. \tag{3.10}$$

Substituting (3.9) and (3.10) for (3.8), we arrive at a contradiction. Hence $x^0$ is a Pareto-optimal solution of (MP). □

Now we present a result for nonsmooth problems (MP) which in the smooth case is similar to Singh's earlier result in [12] under generalized convexity.

**Theorem 3.6.** *Let $(x^0, u^0, v^0, w^0)$ satisfy* (KKT′). *If $(u^0)^t(f(\cdot) + z^t(\cdot)) + (v^0)^t g_A + (w^0)^t h$ is pseudoconvex at $x^0$, and $f$, $g_A$, and $h$ are regular at $x^0$, then $x^0$ is a weak Pareto-optimal solution of* (MP).

*Proof.* Suppose that $x^0 \in X$ is not a weak Pareto-optimal solution of (MP). Then there exists $\overline{x} \in X$ such that $f(\overline{x}) + s(\overline{x} \mid C) < f(x^0) + s(x^0 \mid C)$, that implies $f(\overline{x}) + z^t\overline{x} < f(x^0) + z^t x^0$ because of $z^t x \leqq s(x \mid C)$ and the assumption $z^t x^0 = s(x^0 \mid C)$. Since $\overline{x} \in X$, we have

$g_A(\overline{x}) \leqq 0 = g_A(x^0)$ and $h(\overline{x}) = 0 = h(x^0)$. Therefore, $f(\overline{x}) + s(\overline{x} \mid C) - f(x^0) + s(x^0 \mid C) < 0$, $g_A(\overline{x}) - g_A(x^0) \leqq 0$, and $h(\overline{x}) - h(x^0) = 0$. Hence $(u^0)^t(f(\overline{x}) + z^t(\overline{x})) + (v^0)^t g_A(\overline{x}) + (w^0)^t h(\overline{x}) < (u^0)^t(f(x^0) + z^t(x^0)) + (v^0)^t g_A(x^0) + (w^0)^t h(x^0)$. Since $f$, $g_A$, and $h$ are regular at $x^0$, we obtain

$$\begin{aligned} &\partial\left(\sum_{i\in M} u_i^0\left(f_i\left(x^0\right) + s\left(x^0 \mid C_i\right)\right) + \sum_{j\in A} v_j^0 g_j\left(x^0\right) + \sum_{k\in Q} w_k^0 h_k\left(x^0\right)\right) \\ &\quad = \sum_{i\in M} u_i^0\left(\partial f_i\left(x^0\right) + \partial s\left(x^0 \mid C_i\right)\right) + \sum_{j\in A} v_j^0 \partial g_j\left(x^0\right) + \sum_{k\in Q} w_k^0 \partial h_k\left(x^0\right). \end{aligned} \tag{3.11}$$

By pseudoconvexity of $(u^0)^t(f(\cdot) + z^t(\cdot)) + (v^0)^t g_A + (w^0)^t h$, we have $\beta^t(\overline{x} - x^0) < 0$ for any $\beta \in \partial(\sum_{i\in M} u_i^0(f_i(x^0) + z_i^t x^0) + \sum_{j\in A} v_j^0 g_j(x^0) + \sum_{k\in Q} w_k^0 h_k(x^0))$. We easily see that this contradicts $0 \in \sum_{i\in M} u_i^0(\partial f_i(x^0) + z_i) + \sum_{j\in A} v_j^0 \partial g_j(x^0) + \sum_{k\in Q} w_k^0 \partial h_k(x^0)$. Hence $x^0$ is a weak Pareto-optimal solution of (MP). □

**Theorem 3.7.** *Let $(x^0, u^0, v^0, w^0)$ satisfy* (KKT′). *If $(u^0)^t(f(\cdot) + z^t(\cdot)) + (v^0)^t g_A + (w^0)^t h$ is strictly pseudoconvex at $x^0$, and $f$, $g_A$, and $h$ are regular at $x^0$, then $x^0$ is a Pareto-optimal solution of* (MP).

The proof is similar to the one used for the previous theorem.

## 4. Duality

Following Mond and Weir [2], in this section we formulate a Wolfe-type dual problem (WD) and a Mond-Weir-type dual problem (MD) of the nonsmooth problem (MP) and establish duality theorems. We begin with a Wolfe-type dual problem:

$$\begin{aligned} &\text{maximize} \quad f(y) + z^t y + v^t g(y)e + w^t h(y)e \\ &\text{subject to} \quad 0 \in \sum_{i\in M} u_i(\partial f_i(y) + z_i) + \sum_{j\in P} v_j \partial g_j(y) + \sum_{k\in Q} w_k \partial h_k(y), \\ &\qquad\qquad y \in \mathbb{R}^n,\ u \geq 0 \quad \text{with } u^t e = 1,\ v \geqq 0,\ w \geqq 0. \end{aligned} \tag{WD}$$

Here $e = (1, \ldots, 1)^t \in \mathbb{R}^m$.

We now derive duality relations.

**Theorem 4.1.** *Let $x$ be feasible for* (MP) *and $(y, u, v, w)$ feasible for* (WD). *If $f(\cdot) + z^t(\cdot)$, $g$ and $w^t h$ are convex, and $f$ is a regular function, then $f(x) + s(x \mid C) \nleq f(y) + z^t y + v^t g(y)e + w^t h(y)e$.*

*Proof.* Let $x$ be feasible for (MP) and $(y, u, v, w)$ feasible for (WD). Then $g(x) \leqq 0$, $h(x) = 0$,

$$0 \in \sum_{i\in M} u_i(\partial f_i(y) + z_i) + \sum_{j\in P} v_j \partial g_j(y) + \sum_{k\in Q} w_k \partial h_k(y). \tag{4.1}$$

According to (4.1), there exist $\xi_i \in \partial f_i(y)$, $\zeta_j \in \partial g_j(y)$, and $\alpha_k \in \partial h_k(y)$ such that

$$\sum_{i\in M} u_i(\xi_i + z_i) + \sum_{j\in P} v_j\zeta_j + \sum_{k\in Q} w_k\alpha_k = 0. \tag{4.2}$$

Assume that

$$f(x) + s(x \mid C) < f(y) + z^t y + v^t g(y)e + w^t h(y)e. \tag{4.3}$$

Multiplying (4.3) by $u$ and using the equality $z^t x = s(x \mid C)$, we have

$$u^t(f(x) + z^t x) - u^t(f(y) + z^t y) - v^t g(y) - w^t h(y) < 0 \tag{4.4}$$

since $u \geq 0$ and $u^t e = 1$. Now by convexity of $f(\cdot) + z^t(\cdot)$, $g$ and $w^t h$, we obtain

$$\begin{aligned} u^t(f(x) + z^t x) - u^t(f(y) + z^t y) &\geqq \sum_{i\in M} u_i(\xi_i + z_i)^t(x - y), \quad \forall \xi_i \in \partial f_i(y), \\ v^t g(x) - v^t g(y) &\geqq \sum_{j\in P} v_j\zeta_j^t(x - y), \quad \forall \zeta_j \in \partial g_j(y), \\ w^t h(x) - w^t h(y) &\geqq \sum_{k\in Q} w_k\alpha_k^t(x - y), \quad \forall \alpha_k \in \partial h_k(y). \end{aligned} \tag{4.5}$$

Since $v^t g(x) \leqq 0$ and $w^t h(x) = 0$, we obtain the following inequality from (4.2), (4.5):

$$u^t(f(x) + z^t x) - u^t(f(y) + z^t y) - v^t g(y) - w^t h(y) \geqq 0, \tag{4.6}$$

which contradicts (4.4).

Hence the weak duality theorem holds. □

Now we derive a strong duality theorem.

**Theorem 4.2.** *Let $\overline{x}$ be a weak Pareto-optimal solution of* (MP) *at which a constraint qualification holds [14]. Then there exist $\overline{u} \in \mathbb{R}^m$, $\overline{v} \in \mathbb{R}^p$, and $\overline{w} \in \mathbb{R}^q$ such that $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ is feasible for* (WD) *and $z^t\overline{x} = s(\overline{x} \mid C)$. In addition, if $f(\cdot) + z^t(\cdot)$, $g$ and $w^t h$ are convex, and $f$ is a regular function, then $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ is a weak Pareto-optimal solution of* (WD) *and the optimal values of* (MP) *and* (WD) *are equal.*

*Proof.* From the (KKT) necessary optimality theorem [14], there exist $u \in \mathbb{R}^m$, $v \in \mathbb{R}^p$, and $w \in \mathbb{R}^q$ such that

$$\begin{gathered} 0 \in \sum_{i\in M} u_i(\partial f_i(\overline{x}) + z_i) + \sum_{j\in P} v_j\partial g_j(\overline{x}) + \sum_{k\in Q} w_k\partial h_k(\overline{x}), \\ v^t g(\overline{x}) = 0, \quad u \geq 0, \quad v \geqq 0. \end{gathered} \tag{4.7}$$

Since $u \geq 0$, we can scale the $u_i's$, $v_j's$ and $w_k's$ as follows:

$$\overline{u}_i = \frac{u_i}{\sum_{i\in M} u_i}, \qquad \overline{v}_j = \frac{v_j}{\sum_{i\in M} u_i}, \qquad \overline{w}_k = \frac{w_k}{\sum_{i\in M} u_i}. \tag{4.8}$$

Then $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ is feasible for (WD). Since $\overline{x}$ is feasible for (MP), it follows from Theorem 4.1 that

$$\begin{aligned} f(\overline{x}) + s(\overline{x} \mid C) &= f(\overline{x}) + s(\overline{x} \mid C) + v^t g(\overline{x})e + w^t h(\overline{x})e \\ &\not< f(y) + z^t y + v^t g(y)e + w^t h(y)e \end{aligned} \tag{4.9}$$

for any feasible solution $(x, u, v, w)$ of (WD). Hence $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ is a weak Pareto-optimal solution of (WD) and the optimal values of (MP) and (WD) are equal. □

*Remark 4.3.* If we replace the convexity hypothesis of $f(\cdot) + z^t(\cdot)$ by strict convexity in Theorems 4.1 and 4.2, then these theorems hold for the case of a Pareto-optimal solution.

*Remark 4.4.* If we replace the convexity hypothesis of $w^t h$ by affineness of $h$ in Theorems 4.1 and 4.2, then these theorems are also valid.

**Theorem 4.5.** *Let $x$ be feasible for* (MP) *and $(y, u, v, w)$ feasible for* (WD). *If $u^t(f(\cdot) + z^t(\cdot)) + v^t g + w^t h$ is pseudoconvex and $f$, $g$, and $h$ are regular functions, then $f(x) + s(x \mid C) \nleq f(y) + z^t y + v^t g(y)e + w^t h(y)e$.*

*Proof.* Suppose to the contrary that $f(x) + s(x \mid C) \leq f(y) + z^t y + v^t g(y)e + w^t h(y)e$. By feasibility of $x$, we obtain

$$u^t(f(x) + z^t(x)) + v^t g(x) + w^t h(x) < u^t(f(y) + z^t(y)) + v^t g(y) + w^t h(y). \tag{4.10}$$

Since $f$, $g$, and $h$ are regular functions, we have $\beta^t(x - y) < 0$ by the pseudoconvexity of $u^t(f(\cdot) + z^t(\cdot)) + v^t g + w^t h$ for any $\beta \in \partial(\sum_{i\in M} u_i(f_i(y) + z_i^t y) + \sum_{j\in P} v_j g_j(y) + \sum_{k\in Q} w_k h_k(y))$. This contradicts the feasibility of $(y, u, v, w)$. Hence the weak duality theorem holds. □

**Theorem 4.6.** *Let $\overline{x}$ be a weak Pareto-optimal solution of* (MP) *at which a constraint qualification holds [14]. Then there exist $\overline{u} \in \mathbb{R}^m$, $\overline{v} \in \mathbb{R}^p$, and $\overline{w} \in \mathbb{R}^q$ such that $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ is feasible for* (WD) *and $z^t\overline{x} = s(\overline{x} \mid C)$. If in addition $u^t(f(\cdot) + z^t(\cdot)) + v^t g + w^t h$ is pseudoconvex and $f$, $g$, and $h$ are regular functions, then $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ is a weak Pareto-optimal solution of* (WD) *and the optimal values of* (MP) *and* (WD) *are equal.*

The proof is similar to the one used for Theorem 4.2.

*Remark 4.7.* If we replace the pseudoconvexity hypothesis of $u^t(f(\cdot) + z^t(\cdot)) + v^t g + w^t h$ by strictly pseudoconvexity in Theorems 4.5 and 4.6, then these results hold for the case of a Pareto-optimal solution.

We now prove duality relations between (MP) and the following Mond-Weir-type dual problem:

$$\begin{aligned}
&\text{maximize} \quad f(y) + z^t y \\
&\text{subject to} \quad 0 \in \sum_{i\in M} u_i(\partial f_i(y) + z_i) + \sum_{j\in P} v_j \partial g_j(y) + \sum_{k\in Q} w_k \partial h_k(y), \\
&\qquad\qquad v^t g(y) + w^t h(y) \geqq 0, \\
&\qquad\qquad y \in \mathbb{R}^n,\ u \geq 0 \quad \text{with } u^t e = 1,\ v \geqq 0,\ w \geqq 0.
\end{aligned} \tag{MD}$$

**Theorem 4.8.** *Let $x$ be feasible for* (MP) *and $(y,u,v,w)$ feasible for* (MD). *If $f(\cdot)+z^t(\cdot)$, $g$ and $w^t h$ are convex, and $f$ is a regular function, then $f(x) + s(x \mid C) \nless f(y) + z^t y$.*

*Proof.* Let $x$ be feasible for (MP) and $(y,u,v,w)$ feasible for (MD). Then $g(x) \leqq 0$, $h(x) = 0$,

$$0 \in \sum_{i\in M} u_i(\partial f_i(y) + z_i) + \sum_{j\in P} v_j \partial g_j(y) + \sum_{k\in Q} w_k \partial h_k(y), \tag{4.11}$$

$$v^t g(y) + w^t h(y) \geqq 0. \tag{4.12}$$

According to (4.11), there exist $\xi_i \in \partial f_i(y)$, $\zeta_j \in \partial g_j(y)$, and $\alpha_k \in \partial h_k(y)$ such that

$$\sum_{i\in M} u_i(\xi_i + z_i) + \sum_{j\in P} v_j \zeta_j + \sum_{k\in Q} w_k \alpha_k = 0. \tag{4.13}$$

Assume that

$$f(x) + s(x \mid C) < f(y) + z^t y. \tag{4.14}$$

Multiplying (4.14) by $u$ and using the inequality $z^t x \leqq s(x \mid C)$, we have

$$u^t(f(x) + z^t x) < u^t(f(y) + z^t y). \tag{4.15}$$

By convexity of $f(\cdot) + z^t(\cdot)$, $g$ and $w^t h$, we obtain

$$u^t(f(x) + z^t x) + v^t g(x) + w^t h(x) \geqq u^t(f(y) + z^t y) + v^t g(y) + w^t h(y). \tag{4.16}$$

From (4.12) and (4.16), we obtain

$$u^t(f(x) + z^t x) - u^t(f(y) + z^t y) \geqq 0, \tag{4.17}$$

since $v^t g(x) \leqq 0$ and $w^t h(x) = 0$. However, (4.17) contradicts (4.15). Hence the proof is complete. □

**Theorem 4.9.** *Let $\overline{x}$ be a weak Pareto-optimal solution of* (MP) *at which a constraint qualification holds. Then there exist $\overline{u} \in \mathbb{R}^m$, $\overline{v} \in \mathbb{R}^p$, and $\overline{w} \in \mathbb{R}^q$ such that $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ is feasible for* (MD) *and $z^t\overline{x} = s(\overline{x} \mid C)$. If in addition $f(\cdot) + z^t(\cdot)$, $g$ and $w^t h$ are convex, and $f$ is a regular function, then $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ is a weak Pareto-optimal solution of* (MD) *and the optimal values of* (MP) *and* (MD) *are equal.*

*Proof.* Let $\overline{x}$ be a weak Pareto-optimal solution of (MP) such that a constraint qualification is satisfied at $\overline{x}$. According to the (KKT) necessary optimality theorem, there exist $u \in \mathbb{R}^m$, $v \in \mathbb{R}^p$, and $w \in \mathbb{R}^q$ such that

$$
\begin{gathered}
0 \in \sum_{i \in M} u_i \left( \partial f_i(\overline{x}) + z_i \right) + \sum_{j \in P} v_j \partial g_j(\overline{x}) + \sum_{k \in Q} w_k \partial h_k(\overline{x}), \\
v^t g(\overline{x}) = 0, \quad u \geq 0, \quad v \geqq 0.
\end{gathered}
\tag{4.18}
$$

Since $u \geq 0$, we can scale the $u_i' s$, $v_j' s$, and $w_k' s$ as in the proof of Theorem 4.2 such that $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ is feasible for (MD), it follows from Theorem 4.8 that $f(\overline{x}) + s(\overline{x} \mid C) \nless f(y) + z^t y$ for any feasible solution $(y, u, v, w)$ of (MD). Hence $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ is a weak Pareto-optimal solution of (MD) and the optimal values of (MP) and (MD) are equal. □

*Remark 4.10.* If we replace the convexity hypothesis of $f(\cdot) + z^t(\cdot)$ by strict convexity in Theorems 4.8 and 4.9, then these theorems hold in the sense of a Pareto-optimal solution.

*Remark 4.11.* If we replace the convexity hypothesis of $w^t h$ by affineness of $h$ in Theorems 4.8 and 4.9, then these theorems are also valid.

**Theorem 4.12.** *Let $x$ be feasible for* (MP) *and $(y, u, v, w)$ feasible for* (MD). *If $u^t(f(\cdot) + z^t(\cdot)) + v^t g + w^t h$ is pseudoconvex and $f$, $g$, and $h$ are regular functions, then $f(x) + s(x \mid C) \nleq f(y) + z^t y$.*

*Proof.* Suppose that $f(x) + s(x \mid C) \leq f(y) + z^t y$. By using the feasibility assumptions and $z^t x \leqq s(x \mid C)$, we obtain

$$
u^t \left( f(x) + z^t x \right) + v^t g(x) + w^t h(x) < u^t \left( f(y) + z^t y \right) + v^t g(y) + w^t h(y). \tag{4.19}
$$

By the same argument as in the proof of Theorem 4.5, we arrive at a contradiction. □

**Theorem 4.13.** *Let $\overline{x}$ be a weak Pareto-optimal solution of* (MP) *at which a constraint qualification holds. Then there exist $\overline{u} \in \mathbb{R}^m$, $\overline{v} \in \mathbb{R}^p$, and $\overline{w} \in \mathbb{R}^q$ such that $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ is feasible for* (MD) *and $z^t\overline{x} = s(\overline{x} \mid C)$. If in addition $u^t(f(\cdot) + z^t(\cdot)) + v^t g + w^t h$ is pseudoconvex and $f$, $g$, and $h$ are regular functions, then $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ is a weak Pareto-optimal solution of* (MD) *and the optimal values of* (MP) *and* (MD) *are equal.*

The proof is similar to the one used for the previous theorem.

*Remark 4.14.* If we replace the pseudoconvexity hypothesis of $u^t(f(\cdot) + z^t(\cdot)) + v^t g + w^t h$ by strict pseudoconvexity in Theorems 4.12 and 4.13, then these results hold for the case of a Pareto-optimal solution.

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