Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 897279, 11 pages
doi:10.1155/2010/897279

Research Article

On The Frobenius Condition Number of Positive Definite Matrices

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Received 19 February 2010; Revised 4 May 2010; Accepted 15 June 2010

Academic Editor: S. S. Dragomir

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We present some lower bounds for the Frobenius condition number of a positive definite matrix depending on trace, determinant, and Frobenius norm of a positive definite matrix and compare these results with other results. Also, we give a relation for the cosine of the angle between two given real matrices.

1. Introduction and Preliminaries

The quantity

$$\kappa(A) = \begin{cases} \|A\|\left\|A^{-1}\right\| & \text{if } A \textit{ is nonsingular,} \\ \infty & \text{if } A \text{ is singular} \end{cases} \tag{1.1}$$

is called the condition number for matrix inversion with respect to the matrix norm $\|\cdot\|$. Notice that $\kappa(A) = \|A^{-1}\|\|A\| \geq \|A^{-1}A\| = \|I\| \geq 1$ for any matrix norm (see, e.g., [1, page 336]). The condition number $\kappa(A) = \|A\|\|A^{-1}\|$ of a nonsingular matrix A plays an important role in the numerical solution of linear systems since it measures the sensitivity of the solution of linear systems $Ax = b$ to the perturbations on A and b. There are several methods that allow to find good approximations of the condition number of a general square matrix.

Let $\mathbb{C}^{n\times n}$ and $\mathbb{R}^{n\times n}$ be the space of $n \times n$ complex and real matrices, respectively. The identity matrix in $\mathbb{C}^{n\times n}$ is denoted by $I = I_n$. A matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if $A^* = A$, where A^* denotes the conjugate transpose of A. A Hermitian matrix A is said to be positive semidefinite or nonnegative definite, written as $A \geq 0$, if (see, e. g., [2], p.159)

$$x^* A x \geq 0, \quad \forall x \in \mathbb{C}^n, \tag{1.2}$$

A is further called positive definite, symbolized $A > 0$, if the strict inequality in (1.2) holds for all nonzero $x \in \mathbb{C}^n$. An equivalent condition for $A \in \mathbb{C}^{n\times n}$ to be positive definite is that A is Hermitian and all eigenvalues of A are positive real numbers.

The trace of a square matrix A (the sum of its main diagonal entries, or, equivalently, the sum of its eigenvalues) is denoted by $\mathrm{tr}A$. Let A be any $m \times n$ matrix. The Frobenius (Euclidean) norm of matrix A is

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|^2\right)^{1/2}. \tag{1.3}$$

It is also equal to the square root of the matrix trace of AA^*, that is,

$$\|A\|_F = \sqrt{\mathrm{tr}\, AA^*}. \tag{1.4}$$

The Frobenius condition number is defined by $\kappa_F(A) = \|A\|_F \|A^{-1}\|_F$. In $\mathbb{R}^{n\times n}$ the Frobenius inner product is defined by

$$\langle A, B\rangle_F = \mathrm{tr}\left(A^T B\right) \tag{1.5}$$

for which we have the associated norm that satisfies $\|A\|_F^2 = \langle A, A\rangle_F$. The Frobenius inner product allows us to define the cosine of the angle between two given real $n \times n$ matrices as

$$\cos(A, B) = \frac{\langle A, B\rangle_F}{\|A\|_F \|B\|_F}. \tag{1.6}$$

The cosine of the angle between two real $n \times n$ matrices depends on the Frobenius inner product and the Frobenius norms of given matrices. Then, the inequalities in inner product spaces are expandable to matrices by using the inner product between two matrices.

Buzano in [3] obtained the following extension of the celebrated Schwarz inequality in a real or complex inner product space $(H; \langle\cdot,\cdot\rangle)$:

$$|\langle a, x\rangle\langle x, b\rangle| \leq \frac{1}{2}[\|a\|\|b\| + |\langle a, b\rangle|]\|x\|^2, \tag{1.7}$$

for any $a, b, x \in H$. It is clear that for $a = b$, the above inequality becomes the standard Schwarz inequality

$$|\langle a, x\rangle|^2 \leq \|a\|^2\|x\|^2, \quad a, x \in H, \tag{1.8}$$

with equality if and only if there exists a scalar $\lambda \in K$ ($K = \mathbb{R}$ or $\mathbb{C}$) such that $x = \lambda a$. Also Dragomir in [4] has stated the following inequality:

$$\left| \frac{\langle a, x\rangle\langle x, b\rangle}{\|x\|^2} - \frac{\langle a, b\rangle}{2} \right| \leq \frac{\|a\|\|b\|}{2}, \tag{1.9}$$

where $a, b, x \in H$, $x \neq 0$. Furthermore, Dragomir [4] has given the following inequality, which is mentioned by Precupanu in [5], has been showed independently of Buzano, by Richard in [6]:

$$\frac{1}{2}[\langle a, b\rangle - \|a\|\|b\|]\|x\|^2 \leq \langle a, x\rangle\langle x, b\rangle \leq \frac{1}{2}[\langle a, b\rangle + \|a\|\|b\|]\|x\|^2. \tag{1.10}$$

As a consequence, in next section, we give some bounds for the Frobenius condition numbers and the cosine of the angle between two positive definite matrices by considering inequalities given for inner product space in this section.

2. Main Results

Theorem 2.1. *Let A be positive definite real matrix. Then*

$$2\frac{\operatorname{tr} A}{(\det A)^{1/n}} - n \leq \kappa_F(A), \tag{2.1}$$

where $\kappa_F(A)$ is the Frobenius condition number.

Proof. We can extend inequality (1.9) given in the previous section to matrices by using the Frobenius inner product as follows: Let $A, B, X \in \mathbb{R}^{n\times n}$. Then we write

$$\left| \frac{\langle A, X\rangle_F\langle X, B\rangle_F}{\|X\|_F^2} - \frac{\langle A, B\rangle_F}{2} \right| \leq \frac{\|A\|_F\|B\|_F}{2}, \tag{2.2}$$

where $\langle A, X\rangle_F = \operatorname{tr}(A^TX)$, and $\|\cdot\|_F$ denotes the Frobenius norm of matrix. Then we get

$$\left| \frac{\operatorname{tr}(A^TX)\operatorname{tr}(X^TB)}{\|X\|_F^2} - \frac{\operatorname{tr}(A^TB)}{2} \right| \leq \frac{\|A\|_F\|B\|_F}{2}. \tag{2.3}$$

In particular, in inequality (2.3), if we take $B = A^{-1}$, then we have

$$\left| \frac{\operatorname{tr}(A^TX)\operatorname{tr}(X^TA^{-1})}{\|X\|_F^2} - \frac{\operatorname{tr}(A^TA^{-1})}{2} \right| \leq \frac{\|A\|_F\|A^{-1}\|_F}{2}. \tag{2.4}$$

Also, if X and A are positive definite real matrices, then we get

$$\left| \frac{\operatorname{tr}(AX)\operatorname{tr}(XA^{-1})}{\|X\|_F^2} - \frac{n}{2} \right| \le \frac{\|A\|_F \left\|A^{-1}\right\|_F}{2} = \frac{\kappa_F(A)}{2}, \tag{2.5}$$

where $\kappa_F(A)$ is the Frobenius condition number of A.

Note that Dannan in [7] has showed the following inequality by using the well known arithmetic-geometric inequality, for n-square positive definite matrices A and B:

$$n(\det A \det B)^{m/n} \le \operatorname{tr}(A^m B^m), \tag{2.6}$$

where m is a positive integer. If we take $A = X$, $B = A^{-1}$, and $m = 1$ in (2.6), then we get

$$n\left(\det X \det A^{-1}\right)^{1/n} \le \operatorname{tr}\left(XA^{-1}\right). \tag{2.7}$$

That is,

$$n\left(\frac{\det X}{\det A}\right)^{1/n} \le \operatorname{tr}\left(XA^{-1}\right). \tag{2.8}$$

In particular, if we take $X = I$ in (2.5) and (2.8), then we arrive at

$$\begin{gathered} \left| \frac{\operatorname{tr} A \operatorname{tr} A^{-1}}{n} - \frac{n}{2} \right| \le \kappa_F(A), \\ n\left(\frac{1}{\det A}\right)^{1/n} \le \operatorname{tr} A^{-1}. \end{gathered} \tag{2.9}$$

Also, from the well-known Cauchy-Schwarz inequality, since $n^2 \le \operatorname{tr} A \operatorname{tr} A^{-1}$, one can obtain

$$0 < n \le 2\frac{\operatorname{tr}\ A\ \operatorname{tr}\ A^{-1}}{n} - n \le \kappa_F(A). \tag{2.10}$$

Furthermore, from arithmetic-geometric means inequality, we know that

$$n(\det A)^{1/n} \le \operatorname{tr} A. \tag{2.11}$$

Since $n \le \operatorname{tr} A/(\det\ A)^{1/n}$, we write $0 < n \le 2\,(\operatorname{tr} A/(\det\ A))^{1/n} - n$. Thus by combining (2.9) and (2.11) we arrive at

$$2\frac{\operatorname{tr}\ A}{(\det A)^{1/n}} - n \le \kappa_F(A). \tag{2.12}$$

□

Lemma 2.2. *Let A be a positive definite matrix. Then*

$$\frac{\operatorname{tr} A^{3/2} \operatorname{tr} A^{-(1/2)}}{\operatorname{tr} A} - \frac{n}{2} \geq 0. \tag{2.13}$$

Proof. Let λ_i be positive real numbers for $i = 1, 2, \ldots, n$. We will show that

$$\left(\sum_{i=1}^{k} \lambda_i^{3/2}\right)\left(\sum_{i=1}^{k} \lambda_i^{-(1/2)}\right) \geq \frac{k}{2}\left(\sum_{i=1}^{k} \lambda_i\right) \tag{2.14}$$

for all $k = 1, 2, \ldots, n$. The proof is by induction on k. If $k = 1$,

$$\lambda_1^{3/2} \cdot \lambda_1^{-(1/2)} = \lambda_1 \geq \frac{1}{2}\lambda_1. \tag{2.15}$$

Assume that inequality (2.14) holds for some k. that is,

$$\left(\sum_{i=1}^{k} \lambda_i^{3/2}\right)\left(\sum_{i=1}^{k} \lambda_i^{-(1/2)}\right) \geq \frac{k}{2}\left(\sum_{i=1}^{k} \lambda_i\right). \tag{2.16}$$

Then

$$\begin{aligned}
\left(\sum_{i=1}^{k+1} \lambda_i^{3/2}\right)\left(\sum_{i=1}^{k+1} \lambda_i^{-(1/2)}\right) &= \left(\sum_{i=1}^{k} \lambda_i^{3/2} + \lambda_{k+1}^{3/2}\right)\left(\sum_{i=1}^{k} \lambda_i^{-(1/2)} + \lambda_{k+1}^{-(1/2)}\right) \\
&= \left(\sum_{i=1}^{k} \lambda_i^{3/2}\right)\left(\sum_{i=1}^{k} \lambda_i^{-(1/2)}\right) + \sum_{i=1}^{k}\left(\lambda_i^{3/2}\lambda_{k+1}^{-(1/2)} + \lambda_i^{-(1/2)}\lambda_{k+1}^{(3/2)}\right) + \lambda_{k+1} \\
&\geq \frac{k}{2}\sum_{i=1}^{k} \lambda_i + \sum_{i=1}^{k}(\lambda_i + \lambda_{k+1}) + \lambda_{k+1} \\
&\geq \frac{k}{2}\sum_{i=1}^{k} \lambda_i + \frac{1}{2}\sum_{i=1}^{k}(\lambda_i + \lambda_{k+1}) + \frac{\lambda_{k+1}}{2} \\
&= \frac{k+1}{2}\left(\sum_{i=1}^{k+1} \lambda_i\right).
\end{aligned} \tag{2.17}$$

The first inequality follows from induction assumption and the inequality

$$\frac{a^2 + b^2}{a + b} \geq \frac{a + b}{2} \geq \sqrt{ab} \tag{2.18}$$

for positive real numbers a and b. □

Theorem 2.3. *Let A be positive definite real matrix. Then*

$$0 \leq 2n\frac{\operatorname{tr} A^{3/2}}{\operatorname{tr} A(\det A)^{1/2n}} - n \leq \kappa_F(A), \tag{2.19}$$

where $\kappa_F(A)$ is the Frobenius condition number.

Proof. Let $X > 0$ and $A > 0$. Then from inequality (1.9) we can write

$$\left| \frac{\langle A, X\rangle_F \langle X, A^{-1}\rangle_F}{\|X\|_F^2} - \frac{\langle A, A^{-1}\rangle_F}{2} \right| \leq \frac{\|A\|_F \left\|A^{-1}\right\|_F}{2} \tag{2.20}$$

where $\langle A, B\rangle_F = \operatorname{tr}(A^T B)$ and $\|\cdot\|$ denotes the Frobenius norm. Then we get

$$\left| \frac{\operatorname{tr}(AX)\operatorname{tr}\left(XA^{-1}\right)}{\|X\|_F^2} - \frac{n}{2} \right| \leq \frac{\kappa_F(A)}{2}. \tag{2.21}$$

Set $X = A^{1/2}$. Then

$$\left| \frac{\operatorname{tr} A^{3/2}\operatorname{tr} A^{-(1/2)}}{\operatorname{tr} A} - \frac{n}{2} \right| \leq \frac{\kappa_F(A)}{2}. \tag{2.22}$$

Since $(\operatorname{tr} A^{3/2}\operatorname{tr} A^{-(1/2)}/\operatorname{tr} A) - (n/2) \geq 0$ by Lemma 2.2 and $n(\det A^{-(1/2)})^{1/n} \leq \operatorname{tr} A^{-(1/2)}$,

$$\frac{\operatorname{tr} A^{3/2}}{\operatorname{tr} A} n\left(\det A^{-(1/2)}\right)^{1/n} - \frac{n}{2} \leq \frac{\operatorname{tr} A^{3/2}\operatorname{tr} A^{-(1/2)}}{\operatorname{tr} A} - \frac{n}{2} \leq \frac{\kappa_F(A)}{2}. \tag{2.23}$$

Hence

$$2n\frac{\operatorname{tr} A^{3/2}}{\operatorname{tr} A(\det A)^{1/2n}} - n \leq \kappa_F(A). \tag{2.24}$$

Let λ_i be positive real numbers for $i = 1, 2, \ldots, n$. Now we will show that the left side of inequality (2.19) is positive, that is,

$$2\sum_{i=1}^{n} \lambda_i^{3/2} \geq \left(\sum_{i=1}^{n} \lambda_i\right)\left(\prod_{i=1}^{n} \lambda_i^{1/2n}\right). \tag{2.25}$$

By the arithmetic-geometric mean inequality, we obtain the inequality

$$\frac{1}{n}\left(\sum_{i=1}^{n} \lambda_i\right)\left(\sum_{i=1}^{n} \lambda_i^{1/2}\right) \geq \left(\sum_{i=1}^{n} \lambda_i\right)\left(\prod_{i=1}^{n} \lambda_i^{1/2n}\right). \tag{2.26}$$

So, it is enough to show that

$$2\sum_{i=1}^{n}\lambda_i^{3/2} \geq \frac{1}{n}\left(\sum_{i=1}^{n}\lambda_i\right)\left(\sum_{i=1}^{n}\lambda_i^{1/2}\right). \tag{2.27}$$

Equivalently,

$$2n\sum_{i=1}^{n}\lambda_i^{3} \geq \left(\sum_{i=1}^{n}\lambda_i^2\right)\left(\sum_{i=1}^{n}\lambda_i\right). \tag{2.28}$$

We will prove by induction. If $k = 1$, then

$$2\lambda_1^3 \geq \lambda_1^2 \cdot \lambda_1 = \lambda_1^3. \tag{2.29}$$

Assume that the inequality (2.28) holds for some k. Then

$$\begin{aligned}
2(k+1)\left(\sum_{i=1}^{k+1}\lambda_i^3\right) &= 2k\sum_{i=1}^{k}\lambda_i^3 + 2\sum_{i=1}^{k}\lambda_i^3 + 2k\lambda_{k+1}^3 + 2\lambda_{k+1}^3 \\
&\geq \left(\sum_{i=1}^{k}\lambda_i^2\right)\left(\sum_{i=1}^{k}\lambda_i\right) + 2\left(\sum_{i=1}^{k}\lambda_i^3 + \lambda_{k+1}^3\right) + 2\lambda_{k+1}^3 \\
&\geq \left(\sum_{i=1}^{k}\lambda_i^2\right)\left(\sum_{i=1}^{k}\lambda_i\right) + 2\sum_{i=1}^{k}\left(\lambda_i^2\lambda_{k+1} + \lambda_i\lambda_{k+1}^2\right) + 2\lambda_{k+1}^3 \\
&\geq \left(\sum_{i=1}^{k}\lambda_i^2\right)\left(\sum_{i=1}^{k}\lambda_i\right) + \sum_{i=1}^{k}\lambda_i^2\lambda_{k+1} + \sum_{i=1}^{k}\lambda_i\lambda_{k+1}^2 + \lambda_{k+1}^3 \\
&= \left(\sum_{i=1}^{k+1}\lambda_i^2\right)\left(\sum_{i=1}^{k+1}\lambda_i\right).
\end{aligned} \tag{2.30}$$

The first inequality follows from induction assumption and the second inequality follows from the inequality

$$a^3 + b^3 \geq a^2b + ab^2 \tag{2.31}$$

for positive real numbers a and b. □

Theorem 2.4. *Let A and B be positive definite real matrices. Then*

$$\cos(A, I)\cos(B, I) \le \frac{1}{2}[\cos(A, B) + 1]. \tag{2.32}$$

In particular,

$$\cos\left(A, A^{-1}\right) \le \cos(A, I)\cos\left(A^{-1}, I\right) \le \frac{1}{2}\left[\cos\left(A, A^{-1}\right) + 1\right] \le 1. \tag{2.33}$$

Proof. We consider the right side of inequality (1.10):

$$\langle a, x\rangle\langle x, b\rangle \le \frac{1}{2}[\langle a, b\rangle + \|a\|\|b\|]\|x\|^2. \tag{2.34}$$

We can extend this inequality to matrices as follows:

$$\langle A, X\rangle_F\langle X, B\rangle_F \le \frac{1}{2}[\langle A, B\rangle_F + \|A\|_F\|B\|_F]\|X\|_F^2 \tag{2.35}$$

where $A, X, B \in \mathbb{R}^{n\times n}$. Since $\langle A, X\rangle_F = \operatorname{tr}(A^T X)$, it follows that

$$\operatorname{tr}\left(A^T X\right)\operatorname{tr}\left(X^T B\right) \le \frac{1}{2}\left[\operatorname{tr}\left(A^T B\right) + \|A\|_F\|B\|_F\right]\|X\|_F^2, \tag{2.36}$$

Let X be identity matrix and A and B positive definite real matrices. According to inequality (2.36), it follows that

$$\begin{gathered}\operatorname{tr} A \operatorname{tr} B \le \frac{1}{2}[\operatorname{tr} AB + \|A\|_F\|B\|_F]n,\\ \frac{\operatorname{tr} A \operatorname{tr} B}{\sqrt{n}\|A\|_F\sqrt{n}\|B\|_F} \le \frac{1}{2}\left[\frac{\operatorname{tr} AB}{\|A\|_F\|B\|_F} + 1\right].\end{gathered} \tag{2.37}$$

From the definition of the cosine of the angle between two given real $n \times n$ matrices, we get

$$\cos(A, I)\cos(B, I) \le \frac{1}{2}[\cos(A, B) + 1]. \tag{2.38}$$

In particular, for $B = A^{-1}$ we obtain that

$$\cos(A, I)\cos\left(A^{-1}, I\right) \le \frac{1}{2}\left[\cos\left(A, A^{-1}\right) + 1\right]. \tag{2.39}$$

Also, Chehab and Raydan in [8] have proved the following inequality for positive definite real matrix A by using the well-known Cauchy-Schwarz inequality:

$$\cos\left(A, A^{-1}\right) \leq \cos(A, I) \cos\left(A^{-1}, I\right). \tag{2.40}$$

By combining inequalities (2.39) and (2.40), we arrive at

$$\cos\left(A, A^{-1}\right) \leq \cos(A, I) \cos\left(A^{-1}, I\right) \leq \frac{1}{2}\left[\cos\left(A, A^{-1}\right) + 1\right] \tag{2.41}$$

and since $(1/2)[\cos(A, A^{-1}) + 1] = (n/2\|A\|_F\|A^{-1}\|_F) + (1/2)$ and $n \leq \kappa_F(A)$, we arrive at $(1/2)[\cos(A, A^{-1}) + 1] \leq 1$. Therefore, proof is completed. □

Theorem 2.5. *Let A be a positive definite real matrix. Then*

$$\frac{n\sqrt{n}\|A\|_F}{\operatorname{tr} A} \leq \kappa_F(A). \tag{2.42}$$

Proof. According to the well-known Cauchy-Schwarz inequality, we write

$$\left(\sum_{i=1}^{n}\lambda_i(A)\right)^2 \leq \left(\sum_{i=1}^{n}\lambda_i^2(A)\right)n, \tag{2.43}$$

where $\lambda_i(A)$ are eigenvalues of A. That is,

$$(\operatorname{tr} A)^2 \leq n \operatorname{tr} A^2. \tag{2.44}$$

Also, from definition of the Frobenius norm, we get

$$\operatorname{tr} A \leq \sqrt{n}\|A\|_F. \tag{2.45}$$

Then, we obtain that

$$\cos(A, I) = \frac{\operatorname{tr} A}{\sqrt{n}\|A\|_F} \leq 1. \tag{2.46}$$

Likewise,

$$\cos\left(A^{-1}, I\right) \leq 1. \tag{2.47}$$

When inequalities (2.40) and (2.47) are combined, they produce the following inequality:

$$\cos\left(A, A^{-1}\right) \leq \cos(A, I),$$
$$\frac{n}{\kappa_F(A)} \leq \frac{\operatorname{tr} A}{\sqrt{n}\|A\|_F}. \tag{2.48}$$

Therefore, finally we get

$$\frac{n\sqrt{n}\|A\|_F}{\operatorname{tr} A} \leq \kappa_F(A). \tag{2.49}$$

□

Note that Tarazaga in [9] has given that if A is symmetric matrix, a necessary condition to be positive semidefinite matrix is that $\operatorname{tr} A \geq \|A\|_F$.

Wolkowicz and Styan in [10] have established an inequality for the spectral condition numbers of symetric and positive definite matrices:

$$\kappa_2(A) \geq 1 + \frac{2s}{m - (s/p)}, \tag{2.50}$$

where $p = \sqrt{n-1}$, $m = \operatorname{tr} A/n$, and $s = (\|A\|_F^2/n - m^2)^{1/2}$.

Also, Chehab and Raydan in [8] have given the following practical lower bound for the Frobenius condition number $\kappa_F(A)$:

$$\kappa_F(A) \geq \max\left(n, \frac{\sqrt{n}}{\cos^2(A, I)}, 1 + \frac{2s}{m - s/p}\right). \tag{2.51}$$

Now let us compare the bound in (2.49) and the lower bound obtained by the authors in [8] for the Frobenius condition number of positive definite matrix A.

Since $0 \leq \|A\|_F/\operatorname{tr} A \leq 1$, $\|A\|_F^2/(\operatorname{tr} A)^2 \leq \|A\|_F/\operatorname{tr} A$. Thus, we get

$$\frac{n\sqrt{n}\|A\|_F^2}{(\operatorname{tr} A)^2} \leq \frac{n\sqrt{n}\|A\|_F}{\operatorname{tr} A} \leq \kappa_F(A). \tag{2.52}$$

All these bounds can be combined with the results which are previously obtained to produce practical bounds for $\kappa_F(A)$. In particular, combining the results given by Theorems 2.1, 2.3, and 2.5 and other results, we present the following practical new bound:

$$\kappa_F(A) \geq \max\left(2\frac{\operatorname{tr} A}{(\det A)^{1/n}} - n, 2n\frac{\operatorname{tr} A^{3/2}}{\operatorname{tr} A(\det A)^{1/2n}} - n, \frac{n\sqrt{n}\|A\|_F}{\operatorname{tr} A}, 1 + \frac{2s}{m - s/p}\right). \tag{2.53}$$

Example 2.6.

$$A = \begin{bmatrix} 4 & 1 & 0 & 2 \\ 1 & 5 & 1 & 2 \\ 0 & 1 & 6 & 3 \\ 2 & 2 & 3 & 8 \end{bmatrix}. \tag{2.54}$$

Here $\operatorname{tr} A = 23$, $\|A\|_F = \sqrt{179}$, $\det A = 581$, and have $n = 4$. Then, we obtain that $2(\operatorname{tr} A/(\det A)^{1/n}) - n = 5.369444$, $2n(\operatorname{tr} A^{3/2}/\operatorname{tr} A(\det A)^{1/2n}) - n = 5.741241$, $n\sqrt{n}\|A\|_F/\operatorname{tr} A = 4.653596$, and $1 + (2s/(m - s/p)) = 2.810649$. Since $\kappa_F(A) = 6.882583$, in this example, the best lower bound is the second lower bound given by Theorem 2.3.

Acknowledgments

The authors thank very much the associate editors and reviewers for their insightful comments and kind suggestions that led to improving the presentation. This study was supported by the Coordinatorship of Selçuk University's Scientific Research Projects.

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