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Research Article

On Inverse Moments for a Class of Nonnegative Random Variables

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Using exponential inequalities, Wu et al. (2009) and Wang et al. (2010) obtained asymptotic approximations of inverse moments for nonnegative independent random variables and nonnegative negatively orthant dependent random variables, respectively. In this paper, we improve and extend their results to nonnegative random variables satisfying a Rosenthal-type inequality.

1. Introduction

Let $\{Z_n, n \geq 1\}$ be a sequence of nonnegative random variables with finite second moments. Let us denote

$$X_n = \frac{\sum_{i=1}^n Z_i}{\sigma_n}, \qquad \sigma_n^2 = \sum_{i=1}^n \operatorname{Var}(Z_i). \tag{1.1}$$

We will establish that, under suitable conditions, the inverse moment can be approximated by the inverse of the moment. More precisely, we will prove that

$$E(a + X_n)^{-\alpha} \sim (a + EX_n)^{-\alpha}, \tag{1.2}$$

where $a > 0, \alpha > 0$, and $c_n \sim d_n$ means that $c_n d_n^{-1} \to 1$ as $n \to \infty$. The left-hand side of (1.2) is the inverse moment and the right-hand side is the inverse of the moment. Generally, it is not easy to compute the inverse moment, but it is much easier to compute the inverse of the moment.

The inverse moments can be applied in many practical applications. For example, they appear in Stein estimation and Bayesian poststratification (see Wooff [1] and Pittenger [2]), evaluating risks of estimators and powers of test statistics (see Marciniak and Wesołowski [3] and Fujioka [4]), expected relaxation times of complex systems (see Jurlewicz and Weron [5]), and insurance and financial mathematics (see Ramsay [6]).

For nonnegative asymptotically normal random variables X_n, (1.2) was established in Theorem 2.1 of Garcia and Palacios [7]. Unfortunately, that theorem is not true under the suggested assumptions, as pointed out by Kaluszka and Okolewski [8]. Kaluszka and Okolewski [8] also proved (1.2) for $0 < \alpha < 3$ ($0 < \alpha < 4$ in the i.i.d. case) when $\{Z_n, n \geq 1\}$ is a sequence of nonnegative independent random variables satisfying $EX_n \to \infty$ and L_3 (Lyapunov's condition of order 3), that is, $\sum_{i=1}^{n} E|Z_i - EZ_i|^c/\sigma_n^c \to 0$ with $c = 3$. Hu et al. [9] generalized the result of Kaluszka and Okolewski [8] by considering L_c for some $2 < c \leq 3$ instead of L_3.

Recently, Wu et al. [10] obtained the following result by using the truncation method and Bernstein's inequality.

Theorem 1.1. *Let $\{Z_n, n \geq 1\}$ be a sequence of nonnegative independent random variables such that $EZ_n^2 < \infty$ and $EX_n \to \infty$, where X_n is defined by* (1.1)*. Furthermore, assume that*

$$\max_{1\leq i\leq n} \frac{EZ_i}{\sigma_n} = O(1), \tag{1.3}$$

$$\sigma_n^{-2} \sum_{i=1}^{n} EZ_i^2 I(Z_i > \eta\sigma_n) \longrightarrow 0 \quad \text{for some } \eta > 0. \tag{1.4}$$

Then (1.2) *holds for all real numbers $a > 0$ and $\alpha > 0$.*

For a sequence $\{Z_n, n \geq 1\}$ of nonnegative independent random variables with only rth moments for some $1 \leq r < 2$, Wu et al. [10] also obtained the following asymptotic approximation of the inverse moment:

$$E\left(a + X_n'\right)^{-\alpha} \sim \left(a + EX_n'\right)^{-\alpha} \tag{1.5}$$

for all real numbers $a > 0$ and $\alpha > 0$. Here X_n' is defined as

$$X_n' = \frac{\sum_{i=1}^{n} Z_i}{\widetilde{\sigma}_n}, \qquad \widetilde{\sigma}_n^2 = \sum_{i=1}^{n} \operatorname{Var}(Z_i I(Z_i \leq M_n)), \tag{1.6}$$

where $\{M_n, n \geq 1\}$ is a sequence of positive constants satisfying

$$M_n \to \infty, \quad M_n = O\left(n^{(2-\delta)/(4-r)}\right) \quad \text{for some } 0 < \delta < \frac{r}{2}. \tag{1.7}$$

Specifically, Wu et al. [10] proved the following result.

Theorem 1.2. *Let $\{Z_n, n \geq 1\}$ be a sequence of nonnegative independent random variables. Suppose that, for some $1 \leq r < 2$,*

(i) *$\{Z_n\}$ is uniformly integrable,*

(ii) $\sup_{n\geq 1} EZ_n^r < \infty$,

(iii) $n^{-1}\sum_{i=1}^{n} EZ_i \geq C$ *for some positive constant* $C > 0$,

(iv) $n^{-1/2}\widetilde{\sigma}_n \geq D$ *for some positive constant* $D > 0$,

where $\widetilde{\sigma}_n$ *is the same as in* (1.6) *for some positive constants* $\{M_n\}$ *satisfying* (1.7). *Then* (1.5) *holds for all real numbers* $a > 0$ *and* $\alpha > 0$.

Wang et al. [11] obtained some exponential inequalities for negatively orthant dependent (NOD) random variables. By using the exponential inequalities, they extended Theorem 1.1 for independent random variables to NOD random variables without condition (1.3).

The purpose of this work is to obtain asymptotic approximations of inverse moments for nonnegative random variables satisfying a Rosenthal-type inequality. For a sequence $\{Z_n, n \geq 1\}$ of independent random variables with $EZ_n = 0$ and $E|Z_n|^q < \infty$ for some $q > 2$, Rosenthal [12] proved that there exists a positive constant C_q depending only on q such that

$$E\left|\sum_{i=1}^{n} Z_i\right|^q \leq C_q\left\{\sum_{i=1}^{n} E|Z_i|^q + \left(\sum_{i=1}^{n} E|Z_i|^2\right)^{q/2}\right\}. \tag{1.8}$$

Note that the Rosenthal inequality holds for NOD random variables (see Asadian et al. [13]).

In this paper, we improve and extend Theorem 1.2 for independent random variables to random variables satisfying a Rosenthal type inequality. We also extend Wang et al. [11] result for NOD random variables to the more general case.

Throughout this paper, the symbol C denotes a positive constant which is not necessarily the same one in each appearance, and I_A denotes the indicator function of the event A.

2. Main Results

Throughout this section, we assume that $\{Z_n, n \geq 1\}$ is a sequence of nonnegative random variables satisfying a Rosenthal type inequality (see (2.1)).

The following theorem gives sufficient conditions under which the inverse moment is asymptotically approximated by the inverse of the moment.

Theorem 2.1. *Let* $\{Z_n, n \geq 1\}$ *be a sequence of nonnegative random variables. Let* $\mu_n' = EX_n'$ *and* $\widetilde{\mu}_n = \widetilde{\sigma}_n^{-1}\sum_{i=1}^{n} EZ_i I(Z_i \leq M_n)$, *where* X_n' *and* $\widetilde{\sigma}_n$ *are defined by* (1.6), *and* $\{M_n, n \geq 1\}$ *is a sequence of positive real numbers. Suppose that the following conditions hold:*

(i) *for any* $q > 2$, *there exists a positive constant* C_q *depending only on* q *such that*

$$E\left|\sum_{i=1}^{n}\left(Z_{ni}' - EZ_{ni}'\right)\right|^q \leq C_q\left\{\sum_{i=1}^{n} E\left|Z_{ni}' - EZ_{ni}'\right|^q + \left(\sum_{i=1}^{n}\operatorname{Var}\left(Z_{ni}'\right)\right)^{q/2}\right\}, \tag{2.1}$$

where $Z_{ni}' = Z_i I(Z_i \leq M_n) + M_n I(Z_i > M_n)$,

(ii) $\mu_n' \to \infty$ *as* $n \to \infty$,

(iii) $\tilde{\mu}_n/\mu_n' \to 1$ *as* $n \to \infty$,

(iv) $M_n/(\tilde{\sigma}_n\tilde{\mu}_n^s) = O(1)$ *for some* $0 < s < 1$.

Then (1.5) *holds for all real numbers* $a > 0$ *and* $\alpha > 0$.

Proof. Let us decompose X_n' as

$$X_n' = U_n' + V_n', \tag{2.2}$$

where $U_n' = \tilde{\sigma}_n^{-1}\sum_{i=1}^n Z_{ni}'$ and $V_n' = \tilde{\sigma}_n^{-1}\sum_{i=1}^n (Z_i - M_n)I(Z_i > M_n)$. Denote $u_n' = EU_n'$. Since $Z_i I(Z_i \le M_n) \le Z_{ni}' \le Z_i$, we have that $\tilde{\mu}_n \le u_n' \le \mu_n'$. It follows by (ii) and (iii) that

$$\frac{u_n'}{\mu_n'} \longrightarrow 1, \quad \frac{u_n'}{\tilde{\mu}_n} \longrightarrow 1, \quad u_n' \longrightarrow \infty. \tag{2.3}$$

Now, applying Jensen's inequality to the convex function $f(x) = (a+x)^{-\alpha}$ yields $E(a+X_n')^{-\alpha} \ge (a+EX_n')^{-\alpha}$. Therefore

$$\liminf_{n\to\infty}(a+EX_n')^{\alpha}E(a+X_n')^{-\alpha} \ge 1. \tag{2.4}$$

Hence it is enough to show that

$$\limsup_{n\to\infty}(a+EX_n')^{\alpha}E(a+X_n')^{-\alpha} \le 1. \tag{2.5}$$

Since $0 < s < 1$, we can take t such that $0 < s < t < 1$ and $2t > s+1$. Namely, $s < (s+1)/2 < t < 1$. Let us write

$$E(a+X_n')^{-\alpha} = Q_1' + Q_2', \tag{2.6}$$

where $Q_1' = E(a+X_n')^{-\alpha}I(U_n' \le u_n' - (u_n')^t)$ and $Q_2' = E(a+X_n')^{-\alpha}I(U_n' > u_n' - (u_n')^t)$. Since $X_n' \ge U_n'$, we get that

$$Q_2' \le E(a+X_n')^{-\alpha}I\left(X_n' > u_n' - (u_n')^t\right) \le \left(a + u_n' - (u_n')^t\right)^{-\alpha}, \tag{2.7}$$

which implies by (ii) and (2.3) that

$$\begin{aligned}\limsup_{n\to\infty}(a+EX_n')^{\alpha}Q_2' &\le \limsup_{n\to\infty}(a+EX_n')^{\alpha}\left(a+u_n'-(u_n')^t\right)^{-\alpha} \\ &= \limsup_{n\to\infty}\left(\frac{a/\mu_n'+1}{a/\mu_n' + u_n'/\mu_n' - (u_n')^t/\mu_n'}\right)^{\alpha} = 1.\end{aligned} \tag{2.8}$$

It remains to show that

$$\lim_{n\to\infty}\left(a+EX_n'\right)^{\alpha}Q_1'=0. \tag{2.9}$$

Observe by Markov's inequality and (i) that, for any $q>2$,

$$\begin{aligned}
Q_1' &\le a^{-\alpha}P\Big(U_n'\le u_n'-\left(u_n'\right)^t\Big)\\
&\le a^{-\alpha}P\Big(\left|U_n'-u_n'\right|\ge\left(u_n'\right)^t\Big)\\
&\le a^{-\alpha}\tilde{\sigma}_n^{-q}\left(u_n'\right)^{-tq}E\left|\sum_{i=1}^{n}\left(Z_{ni}'-EZ_{ni}'\right)\right|^q\\
&\le C_q a^{-\alpha}\tilde{\sigma}_n^{-q}\left(u_n'\right)^{-tq}\left\{\sum_{i=1}^{n}E\left|Z_{ni}'-EZ_{ni}'\right|^q+\left(\sum_{i=1}^{n}\operatorname{Var}\left(Z_{ni}'\right)\right)^{q/2}\right\}.
\end{aligned} \tag{2.10}$$

By the definition of Z_{ni}', we have that

$$\begin{aligned}
\left|Z_{ni}'-EZ_{ni}'\right| &\le \max\left\{Z_{ni}',EZ_{ni}'\right\}\le M_n,\\
\operatorname{Var}\left(Z_{ni}'\right) &= \operatorname{Var}(Z_iI(Z_i\le M_n))+\operatorname{Var}(M_nI(Z_i>M_n))\\
&\quad+2\operatorname{Cov}(Z_iI(Z_i\le M_n),M_nI(Z_i>M_n))\\
&= \operatorname{Var}(Z_iI(Z_i\le M_n))+\operatorname{Var}(M_nI(Z_i>M_n))\\
&\quad-2EZ_iI(Z_i\le M_n)M_nP(Z_i>M_n)\\
&\le \operatorname{Var}(Z_iI(Z_i\le M_n))+M_n^2P(Z_i>M_n)\\
&\le \operatorname{Var}(Z_iI(Z_i\le M_n))+M_nEZ_iI(Z_i>M_n).
\end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.10), we have that

$$\begin{aligned}
Q_1' &\le C_q a^{-\alpha}\tilde{\sigma}_n^{-q}\left(u_n'\right)^{-tq}\Bigg\{M_n^{q-2}\sum_{i=1}^{n}(\operatorname{Var}(Z_iI(Z_i\le M_n))+M_nEZ_iI(Z_i>M_n))\\
&\quad+\left(\sum_{i=1}^{n}\operatorname{Var}(Z_iI(Z_i\le M_n))+M_nEZ_iI(Z_i>M_n)\right)^{q/2}\Bigg\}\\
&\le C_q a^{-\alpha}\tilde{\sigma}_n^{-q}\left(u_n'\right)^{-tq}\Bigg\{M_n^{q-2}\tilde{\sigma}_n^2+M_n^{q-1}\sum_{i=1}^{n}EZ_iI(Z_i>M_n)\\
&\quad+2^{q/2-1}\left(\sum_{i=1}^{n}\operatorname{Var}(Z_iI(Z_i\le M_n))\right)^{q/2}+2^{q/2-1}\left(M_n\sum_{i=1}^{n}EZ_iI(Z_i>M_n)\right)^{q/2}\Bigg\}\\
&=: I_1+I_2+I_3+I_4.
\end{aligned} \tag{2.12}$$

For I_1, we have by (iv) that

$$I_1 = C_q a^{-\alpha}(u_n')^{-tq}\left(\frac{M_n}{\tilde{\sigma}_n\tilde{\mu}_n^s}\right)^{q-2}\tilde{\mu}_n^{s(q-2)} \le Ca^{-\alpha}(u_n')^{-tq}\tilde{\mu}_n^{s(q-2)}. \tag{2.13}$$

For I_2, we first note that

$$\frac{\tilde{\mu}_n}{\mu_n'} = 1 - \frac{\tilde{\sigma}_n^{-1}\sum_{i=1}^n EZ_iI(Z_i > M_n)}{\mu_n'}, \tag{2.14}$$

which entails by (iii) that

$$(\mu_n')^{-1}\tilde{\sigma}_n^{-1}\sum_{i=1}^n EZ_iI(Z_i > M_n) \longrightarrow 0. \tag{2.15}$$

It follows by (iv) that

$$\begin{aligned}
I_2 &= C_q a^{-\alpha}\tilde{\sigma}_n^{-q+1}(u_n')^{-tq}M_n^{q-1}\mu_n'(\mu_n')^{-1}\tilde{\sigma}_n^{-1}\sum_{i=1}^n EZ_iI(Z_i > M_n)\\
&\le Ca^{-\alpha}\tilde{\sigma}_n^{-q+1}(u_n')^{-tq}M_n^{q-1}\mu_n'\\
&= Ca^{-\alpha}(u_n')^{-tq}\left(\frac{M_n}{\tilde{\sigma}_n\tilde{\mu}_n^s}\right)^{q-1}\tilde{\mu}_n^{s(q-1)}\mu_n'\\
&\le Ca^{-\alpha}(u_n')^{-tq}\tilde{\mu}_n^{s(q-1)}\mu_n'.
\end{aligned} \tag{2.16}$$

For I_3, we have by the definition of $\tilde{\sigma}_n$ that

$$I_3 = C_q 2^{q/2-1}a^{-\alpha}(u_n')^{-tq}. \tag{2.17}$$

For I_4, we have by (2.15) and (iv) that

$$\begin{aligned}
I_4 &= C_q 2^{q/2-1}a^{-\alpha}\tilde{\sigma}_n^{-q/2}(u_n')^{-tq}M_n^{q/2}(\mu_n')^{q/2}\left((\mu_n')^{-1}\tilde{\sigma}_n^{-1}\sum_{i=1}^n EZ_iI(Z_i > M_n)\right)^{q/2}\\
&\le Ca^{-\alpha}\tilde{\sigma}_n^{-q/2}(u_n')^{-tq}M_n^{q/2}(\mu_n')^{q/2}\\
&= Ca^{-\alpha}(u_n')^{-tq}(\mu_n')^{q/2}\left(\frac{M_n}{\tilde{\sigma}_n\tilde{\mu}_n^s}\right)^{q/2}\tilde{\mu}_n^{sq/2}\\
&\le Ca^{-\alpha}(u_n')^{-tq}(\mu_n')^{q/2}\tilde{\mu}_n^{sq/2}.
\end{aligned} \tag{2.18}$$

Substituting (2.13) and (2.16)–(2.18) into (2.12), we get that

$$\begin{aligned}
&\limsup_{n\to\infty}(a+EX_n')^{\alpha}Q_1' \\
&\quad\le \limsup_{n\to\infty}(a+EX_n')^{\alpha}\Big\{Ca^{-\alpha}(u_n')^{-tq}\widetilde{\mu}_n^{s(q-2)}+Ca^{-\alpha}(u_n')^{-tq}\widetilde{\mu}_n^{s(q-1)}\mu_n' \\
&\qquad +C_q2^{q/2-1}a^{-\alpha}(u_n')^{-tq}+Ca^{-\alpha}(u_n')^{-tq}(\mu_n')^{q/2}\widetilde{\mu}_n^{sq/2}\Big\} \\
&\quad= \limsup_{n\to\infty}\left(\frac{a+\mu_n'}{\widetilde{\mu}_n}\right)^{\alpha}\Big\{Ca^{-\alpha}(u_n')^{-tq}\widetilde{\mu}_n^{s(q-2)+\alpha}+Ca^{-\alpha}(u_n')^{-tq}\widetilde{\mu}_n^{s(q-1)+\alpha}\mu_n' \\
&\qquad +C_q2^{q/2-1}a^{-\alpha}(u_n')^{-tq}\widetilde{\mu}_n^{\alpha}+Ca^{-\alpha}(u_n')^{-tq}(\mu_n')^{q/2}\widetilde{\mu}_n^{sq/2+\alpha}\Big\}.
\end{aligned} \tag{2.19}$$

Since $0<s<(s+1)/2<t<1$, we can take $q>2$ large enough such that $tq>\max\{s(q-1)+\alpha+1, sq/2+q/2+\alpha\}$. Then we have by (2.3) that

$$\begin{aligned}
&(u_n')^{-tq}\widetilde{\mu}_n^{s(q-2)+\alpha}=(u_n')^{-s(q-2)-\alpha}\widetilde{\mu}_n^{s(q-2)+\alpha}(u_n')^{-tq+s(q-2)+\alpha}\longrightarrow 0, \\
&(u_n')^{-tq}\widetilde{\mu}_n^{s(q-1)+\alpha}\mu_n'=(u_n')^{-s(q-1)-\alpha}\widetilde{\mu}_n^{s(q-1)+\alpha}\mu_n'(u_n')^{-1}(u_n')^{-tq+s(q-1)+\alpha+1}\longrightarrow 0, \\
&(u_n')^{-tq}\widetilde{\mu}_n^{\alpha}=(u_n')^{-\alpha}\widetilde{\mu}_n^{\alpha}(u_n')^{-tq+\alpha}\longrightarrow 0, \\
&(u_n')^{-tq}(\mu_n')^{q/2}\widetilde{\mu}_n^{sq/2+\alpha}=(u_n')^{-sq/2-\alpha}\widetilde{\mu}_n^{sq/2+\alpha}(\mu_n')^{q/2}(u_n')^{-q/2}(u_n')^{-tq+sq/2+q/2+\alpha}\longrightarrow 0.
\end{aligned} \tag{2.20}$$

Hence all the terms in the second brace of (2.19) converge to 0 as $n\to\infty$. Moreover, we have by (ii) and (iii) that

$$\left(\frac{a+\mu_n'}{\widetilde{\mu}_n}\right)^{\alpha}=\left(\frac{(a/\mu_n')+1}{\widetilde{\mu}_n/\mu_n'}\right)^{\alpha}\longrightarrow 1. \tag{2.21}$$

Therefore $\limsup_{n\to\infty}(a+EX_n')^{\alpha}Q_1'=0$ and so (2.9) is proved. □

Remark 2.2. In (2.1), $\{Z_{ni}', 1\le i\le n\}$ are monotone transformations of $\{Z_i, 1\le i\le n\}$. If $\{Z_n, n\ge 1\}$ is a sequence of independent random variables, then (2.1) is clearly satisfied from the Rosenthal inequality (1.8). There are many sequences of dependent random variables satisfying (2.1) for all $q>2$. Examples include sequences of NOD random variables (see Asadian et al. [13]), ϕ-mixing identically distributed random variables satisfying $\sum_{n=1}^{\infty}\phi^{1/2}(2^n)<\infty$ (see Shao [14]), ρ-mixing identically distributed random variables satisfying $\sum_{n=1}^{\infty}\rho^{2/q}(2^n)<\infty$ (see Shao [15]), negatively associated random variables (see Shao [16]), and ρ^*-mixing random variables (see Utev and Peligrad [17]).

We can extend Theorem 1.1 for independent random variables to the more general random variables by using Theorem 2.1. To do this, the following lemma is needed.

Lemma 2.3. *Let $\{Y_n, n \geq 1\}$ be a sequence of nonnegative random variables with $EY_n \to \infty$. Let $\{b_n, n \geq 1\}$ be a sequence of positive real numbers satisfying $b_n \to b$, where $b > 0$. Assume that*

$$E(a + Y_n)^{-\alpha} \sim (a + EY_n)^{-\alpha} \quad \forall a > 0, \ \alpha > 0. \tag{2.22}$$

Then $E(b_n + Y_n)^{-\alpha} \sim (b + EY_n)^{-\alpha}$.

Proof. Take $\epsilon > 0$ such that $0 < \epsilon < b$. Since $b_n \to b$, there exists a positive integer N such that $0 < b - \epsilon < b_n < b + \epsilon$ if $n > N$. We have by (2.22) that, for $n > N$,

$$E(b_n + Y_n)^{-\alpha} \leq E(b - \epsilon + Y_n)^{-\alpha} \sim (b - \epsilon + EY_n)^{-\alpha}. \tag{2.23}$$

It follows that

$$\begin{aligned} &\limsup_{n\to\infty}(b + EY_n)^{\alpha} E(b_n + Y_n)^{-\alpha} \\ &\quad\leq \limsup_{n\to\infty}(b + EY_n)^{\alpha} E(b - \epsilon + Y_n)^{-\alpha} \\ &\quad= \limsup_{n\to\infty}\frac{(b + EY_n)^{\alpha}}{(b - \epsilon + EY_n)^{\alpha}}(b - \epsilon + EY_n)^{\alpha} E(b - \epsilon + Y_n)^{-\alpha} \\ &\quad= \limsup_{n\to\infty}\left(\frac{b/EY_n + 1}{(b - \epsilon)/EY_n + 1}\right)^{\alpha}(b - \epsilon + EY_n)^{\alpha} E(b - \epsilon + Y_n)^{-\alpha} = 1. \end{aligned} \tag{2.24}$$

Similar to the above case, we get that, for $n > N$,

$$E(b_n + Y_n)^{-\alpha} \geq E(b + \epsilon + Y_n)^{-\alpha} \sim (b + \epsilon + EY_n)^{-\alpha}, \tag{2.25}$$

$$\begin{aligned} &\liminf_{n\to\infty}(b + EY_n)^{\alpha} E(b_n + Y_n)^{-\alpha} \\ &\quad\geq \liminf_{n\to\infty}(b + EY_n)^{\alpha} E(b + \epsilon + Y_n)^{-\alpha} \\ &\quad= \liminf_{n\to\infty}\left(\frac{b/EY_n + 1}{(b + \epsilon)/EY_n + 1}\right)^{\alpha}(b + \epsilon + EY_n)^{\alpha} E(b + \epsilon + Y_n)^{-\alpha} = 1. \end{aligned} \tag{2.26}$$

Hence the result is proved by (2.24) and (2.26). □

By using Theorem 2.1, we can obtain the following theorem which improves and extends Theorem 1.1 for independent random variables to the more general random variables satisfying the Rosenthal-type inequality (2.1).

Theorem 2.4. *Let $\{Z_n, n \geq 1\}$ be a sequence of nonnegative random variables with $EZ_n^2 < \infty$. Let X_n and σ_n be defined by* (1.1). *Assume that the Rosenthal-type inequality* (2.1) *with $M_n = \eta\sigma_n$ holds for all $q > 2$, where $\eta > 0$ is the same as in (ii). Furthermore, assume that*

(i) *$EX_n \to \infty$ as $n \to \infty$,*

(ii) *$\sigma_n^{-2}\sum_{i=1}^{n} EZ_i^2 I(Z_i > \eta\sigma_n) \to 0$ for some $\eta > 0$.*

Then (1.2) *holds for all real numbers $a > 0$ and $\alpha > 0$.*

Proof. Let $\mu_n' = EX_n'$ and $\tilde{\mu}_n = \tilde{\sigma}_n^{-1} \sum_{i=1}^n EZ_i I(Z_i \le M_n)$, where X_n' and $\tilde{\sigma}_n$ are defined by (1.6). Note that

$$\begin{aligned}
\sigma_n^2 &= \sum_{i=1}^n \mathrm{Var}(Z_i I(Z_i \le M_n) + Z_i I(Z_i > M_n)) \\
&= \sum_{i=1}^n \mathrm{Var}(Z_i I(Z_i \le M_n)) + \mathrm{Var}(Z_i I(Z_i > M_n)) + 2\mathrm{Cov}(Z_i I(Z_i \le M_n), Z_i I(Z_i > M_n)) \\
&= \tilde{\sigma}_n^2 + \sum_{i=1}^n \mathrm{Var}(Z_i I(Z_i > M_n)) - 2\sum_{i=1}^n EZ_i I(Z_i \le M_n) EZ_i I(Z_i > M_n),
\end{aligned} \tag{2.27}$$

which implies that

$$1 = \frac{\tilde{\sigma}_n^2}{\sigma_n^2} + \frac{\sum_{i=1}^n \mathrm{Var}(Z_i I(Z_i > M_n))}{\sigma_n^2} - 2\frac{\sum_{i=1}^n EZ_i I(Z_i \le M_n) EZ_i I(Z_i > M_n)}{\sigma_n^2}. \tag{2.28}$$

But, we have by (ii) that

$$\begin{aligned}
\sigma_n^{-2}\sum_{i=1}^n \mathrm{Var}(Z_i I(Z_i > M_n)) &\le \sigma_n^{-2}\sum_{i=1}^n EZ_i^2 I(Z_i > M_n) \longrightarrow 0, \\
\sigma_n^{-2}\sum_{i=1}^n EZ_i I(Z_i \le M_n) EZ_i I(Z_i > M_n) &\le \sigma_n^{-2} M_n \sum_{i=1}^n EZ_i I(Z_i > M_n) \\
&\le \sigma_n^{-2}\sum_{i=1}^n EZ_i^2 I(Z_i > M_n) \longrightarrow 0.
\end{aligned} \tag{2.29}$$

Substituting (2.29) into (2.28), we have that

$$\sigma_n^{-1}\tilde{\sigma}_n \longrightarrow 1 \quad \text{as } n \longrightarrow \infty. \tag{2.30}$$

Now we will apply Theorem 2.1 to the random variable X_n'. By (2.30) and (i), we get that

$$\mu_n' = \tilde{\sigma}_n^{-1}\sum_{i=1}^n EZ_i = \tilde{\sigma}_n^{-1}\sigma_n EX_n \longrightarrow \infty. \tag{2.31}$$

We also get that

$$\tilde{\mu}_n(\mu_n')^{-1} = \frac{\tilde{\sigma}_n^{-1}\sum_{i=1}^n EZ_iI(Z_i \le M_n)}{\tilde{\sigma}_n^{-1}\sum_{i=1}^n EZ_i} = \frac{\sigma_n^{-1}\sum_{i=1}^n EZ_iI(Z_i \le M_n)}{\sigma_n^{-1}\sum_{i=1}^n EZ_i} = 1 - \frac{\sigma_n^{-1}\sum_{i=1}^n EZ_iI(Z_i > M_n)}{EX_n} \longrightarrow 1, \tag{2.32}$$

since $EX_n \to \infty$ by (i) and $\sigma_n^{-1}\sum_{i=1}^n EZ_iI(Z_i > M_n) \le \sigma_n^{-1}M_n^{-1}\sum_{i=1}^n EZ_i^2I(Z_i > M_n) = \eta^{-1}\sigma_n^{-2}\sum_{i=1}^n EZ_i^2I(Z_i > M_n) \to 0$ by (ii). From (2.31) and (2.32), $\tilde{\mu}_n \to \infty$ and so we have by (2.30) that, for any $s > 0$,

$$\frac{M_n}{\tilde{\sigma}_n\tilde{\mu}_n^s} = \frac{\eta\sigma_n}{\tilde{\sigma}_n\tilde{\mu}_n^s} \longrightarrow 0. \tag{2.33}$$

Hence all conditions of Theorem 2.1 are satisfied. By Theorem 2.1,

$$E\left(a + \tilde{\sigma}_n^{-1}\sum_{i=1}^n Z_i\right)^{-\alpha} \sim \left(a + \tilde{\sigma}_n^{-1}\sum_{i=1}^n EZ_i\right)^{-\alpha} \quad \forall a > 0,\ \alpha > 0. \tag{2.34}$$

Note that the norming constants in (2.34) are different from those in X_n.

To complete the proof, we will use Lemma 2.3. Since $\sigma_n^{-1}\tilde{\sigma}_n \to 1$, we have by Lemma 2.3 that

$$E\left(a\sigma_n\tilde{\sigma}_n^{-1} + \tilde{\sigma}_n^{-1}\sum_{i=1}^n Z_i\right)^{-\alpha} \sim \left(a + \tilde{\sigma}_n^{-1}\sum_{i=1}^n EZ_i\right)^{-\alpha}. \tag{2.35}$$

Namely,

$$E\left(a + \sigma_n^{-1}\sum_{i=1}^n Z_i\right)^{-\alpha} \sim \left(a\tilde{\sigma}_n\sigma_n^{-1} + \sigma_n^{-1}\sum_{i=1}^n EZ_i\right)^{-\alpha}. \tag{2.36}$$

By (i) and (2.30),

$$\left(a\tilde{\sigma}_n\sigma_n^{-1} + \sigma_n^{-1}\sum_{i=1}^n EZ_i\right)^{-\alpha} \sim \left(a + \sigma_n^{-1}\sum_{i=1}^n EZ_i\right)^{-\alpha}. \tag{2.37}$$

Combining (2.36) with (2.37) gives the desired result. □

Remark 2.5. Wang et al. [11] extended Wu et al. [10] result (see Theorem 1.1) to NOD random variables without condition (1.3). As observed in Remark 2.2, (2.1) holds for not only independent random variables but also NOD random variables. Hence Theorem 2.4 improves and extends the results of Wu et al. [10] and Wang et al. [11] to the more general random variables.

Theorem 2.6. *Let* $\{Z_n, n \geq 1\}$ *be a sequence of nonnegative random variables. Let* $\mu_n' = EX_n'$ *and* $\widetilde{\mu}_n = \widetilde{\sigma}_n^{-1} \sum_{i=1}^n EZ_i I(Z_i \leq M_n)$, *where* X_n' *and* $\widetilde{\sigma}_n$ *are defined by* (1.6), *and* $\{M_n, n \geq 1\}$ *is a sequence of positive real numbers satisfying*

$$M_n \longrightarrow \infty, \quad M_n = O(n^t) \text{ for some } 0 < t < 1. \tag{2.38}$$

Assume that the Rosenthal-type inequality (2.1) *holds for all* $q > 2$. *Furthermore, assume that*

(i) $\{Z_n\}$ *is uniformly integrable,*

(ii) $n^{-1} \sum_{i=1}^n EZ_i \geq C$ *for some positive constant* $C > 0$,

(iii) $n^{-1/2}\widetilde{\sigma}_n \geq D$ *for some positive constant* $D > 0$.

Then (1.5) *holds for all real numbers* $a > 0$ *and* $\alpha > 0$.

Proof. We first note by (i) and (ii) that

$$\begin{gathered} Cn \leq \sum_{i=1}^n EZ_i \leq n \sup_{i\geq 1} EZ_i = O(n), \\ n^{-1}\sum_{i=1}^n EZ_i I(Z_i > M_n) \leq \sup_{i\geq 1} EZ_i I(Z_i > M_n) = o(1). \end{gathered} \tag{2.39}$$

We next estimate $\widetilde{\sigma}_n$. By (2.39),

$$\widetilde{\sigma}_n^2 \leq \sum_{i=1}^n EZ_i^2 I(Z_i \leq M_n) \leq M_n \sum_{i=1}^n EZ_i = M_n O(n). \tag{2.40}$$

Combining (2.40) with (iii) gives

$$D^2 n \leq \widetilde{\sigma}_n^2 \leq C_1 n M_n \quad \text{for some constant } C_1 > 0. \tag{2.41}$$

Now we will apply Theorem 2.1 to the random variable X_n'. By (ii), (2.41), and (2.38), we get that

$$\mu_n' = \frac{\sum_{i=1}^n EZ_i}{\widetilde{\sigma}_n} \geq \frac{Cn}{\widetilde{\sigma}_n} \geq \frac{Cn}{\sqrt{C_1 n M_n}} = \frac{Cn^{1/2}}{\sqrt{C_1 O(n^{t/2})}} \longrightarrow \infty. \tag{2.42}$$

We also get by (ii) and (2.39) that

$$\begin{aligned}\frac{\tilde{\mu}_n}{\mu'_n} &= \frac{\tilde{\sigma}_n^{-1}\sum_{i=1}^n EZ_iI(Z_i \le M_n)}{\tilde{\sigma}_n^{-1}\sum_{i=1}^n EZ_i} \\ &= \frac{n^{-1}\sum_{i=1}^n EZ_iI(Z_i \le M_n)}{n^{-1}\sum_{i=1}^n EZ_i} \\ &= 1 - \frac{n^{-1}\sum_{i=1}^n EZ_iI(Z_i > M_n)}{n^{-1}\sum_{i=1}^n EZ_i} \longrightarrow 1.\end{aligned} \tag{2.43}$$

Since $0 < t < 1$, we can take s such that $\max\{2t-1,0\} < s < 1$. Then we have by (ii), (iii), (2.38), and (2.43) that

$$\begin{aligned}\frac{M_n}{\tilde{\sigma}_n\tilde{\mu}_n^s} &= \frac{M_n}{\tilde{\sigma}_n(\mu'_n)^s(\mu'_n)^{-s}\tilde{\mu}_n^s} \\ &\le \frac{M_n}{\tilde{\sigma}_n(Cn\tilde{\sigma}_n^{-1})^s(\mu'_n)^{-s}\tilde{\mu}_n^s}\quad(\text{by(ii)}) \\ &\le \frac{O(n^t)}{C^sD^{1-s}n^{s+(1-s)/2}(\mu'_n)^{-s}\tilde{\mu}_n^s} \longrightarrow 0,\end{aligned} \tag{2.44}$$

since $s + (1-s)/2 > t$ and $(\mu'_n)^{-1}\tilde{\mu}_n \to 1$. Hence all conditions of Theorem 2.1 are satisfied. The result follows from Theorem 2.1. □

Remark 2.7. The conditions of Theorem 2.6 are much weaker than those of Theorem 1.2 in the following three directions.

(i) If $\{Z_n, n \ge 1\}$ is a sequence of independent random variables, then (2.1) is satisfied from the Rosenthal inequality. Hence (2.1) is weaker than independence condition.

(ii) If $\{M_n, n \ge 1\}$ satisfies (1.7), then it also satisfies (2.38) by the fact that $(2-\delta)/(4-r) < (2-\delta)/2 < 1$. Hence (2.38) is weaker than (1.7).

(iii) The condition $\sup_{n\ge1} EZ_n^r < \infty$ in Theorem 1.2 is not needed in Theorem 2.6. Therefore Theorem 2.6 improves and extends Wu et al. [10] result (see Theorem 1.2) to the more general random variables.

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