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Research Article

Hyers-Ulam Stability of Differential Equation $y'' + 2xy' - 2ny = 0$

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We solve the inhomogeneous differential equation of the form $y'' + 2xy' - 2ny = \sum_{m=0}^{\infty} a_m x^m$, where n is a nonnegative integer, and apply this result to the proof of a local Hyers-Ulam stability of the differential equation $y'' + 2xy' - 2ny = 0$ in a special class of analytic functions.

1. Introduction

Assume that X and Y are a topological vector space and a normed space, respectively, and that I is an open subset of X. If for any function $f : I \to Y$ satisfying the differential inequality

$$\left\| a_n(x) y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) + h(x) \right\| \leq \varepsilon \tag{1.1}$$

for all $x \in I$ and for some $\varepsilon \geq 0$, there exists a solution $f_0 : I \to Y$ of the differential equation

$$a_n(x) y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) + h(x) = 0 \tag{1.2}$$

such that $\|f(x) - f_0(x)\| \leq K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ is an expression of ε only, then we say that the above differential equation satisfies the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain I is not the whole space X). We may apply this terminology for other differential equations. For more detailed definition of the Hyers-Ulam stability, refer to [1–6].

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [7, 8]). Here, we will introduce a result of Alsina and Ger (see [9]): If a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality

$|y'(x) - y(x)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(x) = y(x)$ such that $|f(x) - f_0(x)| \leq 3\varepsilon$ for any $x \in I$.

This result of Alsina and Ger has been generalized by Takahasi et al.: They proved in [10] that the Hyers-Ulam stability holds true for the Banach space-valued differential equation $y'(x) = \lambda y(x)$ (see also [11]).

Using the conventional power series method, the author in [12] investigated the general solution of the inhomogeneous Legendre differential equation of the form

$$\left(1 - x^2\right)y''(x) - 2xy'(x) + p(p+1)y(x) = \sum_{m=0}^{\infty} a_m x^m \tag{1.3}$$

under some specific conditions, where p is a real number and the convergence radius of the power series is positive. Moreover, he applied this result to prove that every analytic function can be approximated in a neighborhood of 0 by the Legendre function with an error bound expressed by $C(x^2/(1-x^2))$ (see [13–15]).

Let us consider the error function and the complementary error function defined by

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \qquad \operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = 1 - \operatorname{erf} x, \tag{1.4}$$

respectively. We recursively define the integrals of the error function as follows:

$$\mathbf{i}^{-1}\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} e^{-x^2}, \qquad \mathbf{i}^{0}\operatorname{erfc} x = \operatorname{erfc} x, \qquad \mathbf{i}^{m}\operatorname{erfc} x = \int_x^{\infty} \mathbf{i}^{m-1}\operatorname{erfc} t\, dt \tag{1.5}$$

for any $m \in \mathbb{N}_0$. Suppose that we are given a nonnegative integer n, and we introduce a differential equation

$$y''(x) + 2xy'(x) - 2ny(x) = 0, \tag{1.6}$$

whose general solution is given by

$$y(x) = A\mathbf{i}^n\operatorname{erfc} x + B\mathbf{i}^n\operatorname{erfc}(-x) \tag{1.7}$$

(see [16, §7.2.2]).

In Section 2 of this paper, using power series method, we will investigate the general solution of the inhomogeneous differential equation:

$$y''(x) + 2xy'(x) - 2ny(x) = \sum_{m=0}^{\infty} a_m x^m, \tag{1.8}$$

where the radius of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ is $\rho > 0$, whose value is in general permitted to have infinity. Moreover, using the idea from [12–14], we will prove the Hyers-Ulam stability of the differential equation (1.6) in a class of special analytic functions (see the class C_K in Section 3).

In this paper, $\mathbb{N}_0$ denotes the set of all nonnegative integers.

2. General Solution of (1.8)

In the following theorem, we solve the inhomogeneous differential equation (1.8).

Theorem 2.1. *Assume that n is a nonnegative integer, the radius of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ is $\rho > 0$, and that there exists a real number $\mu \geq 0$ with*

$$
\begin{aligned}
|a_{2m}| &\leq \mu^2 m(2m+2)(2m+1)|\alpha_{2m+2}| \quad (\text{if } n \text{ is odd}), \\
|a_{2m+1}| &\leq \mu^2 m(2m+3)(2m+2)\left|\beta_{2m+3}\right| \quad (\text{if } n \text{ is even})
\end{aligned}
\tag{2.1}
$$

for all sufficiently large integers m, where

$$
\begin{aligned}
\alpha_{2m} &= \frac{2^{m-1}}{(2m)!} \sum_{k=0}^{m-2} \frac{(2k)!}{2^k} a_{2k} \prod_{i=k+1}^{m-1} (n-2i), \\
\beta_{2m+1} &= \frac{2^{m-1}}{(2m+1)!} \sum_{k=0}^{m-2} \frac{(2k+1)!}{2^k} a_{2k+1} \prod_{i=k+1}^{m-1} [n-(2i+1)]
\end{aligned}
\tag{2.2}
$$

for any $m \in \{2, 3, \ldots\}$. Let us define $\rho_0 = \min\{\rho, 1/\mu\}$ and $1/0 = \infty$. Every solution $y : (-\rho_0, \rho_0) \to \mathbb{C}$ of the inhomogeneous differential equation (1.8) *can be represented by*

$$
y(x) = y_h(x) + \sum_{m=2}^{\infty} \frac{a_{m-2}}{m(m-1)} x^m + \sum_{m=2}^{\infty} \alpha_{2m} x^{2m} + \sum_{m=2}^{\infty} \beta_{2m+1} x^{2m+1},
\tag{2.3}
$$

where $y_h(x)$ is a solution of the homogeneous differential equation (1.6).

Proof. Assume that a function $y : (-\rho_0, \rho_0) \to \mathbb{C}$ is given by (2.3). We first prove that the function $y_p(x)$, defined by $y(x) - y_h(x)$, satisfies the inhomogeneous differential equation (1.8). Since

$$
\begin{aligned}
y_p'(x) &= \sum_{m=2}^{\infty} \frac{a_{m-2}}{m-1} x^{m-1} + \sum_{m=2}^{\infty} 2m\alpha_{2m} x^{2m-1} + \sum_{m=2}^{\infty} (2m+1)\beta_{2m+1} x^{2m}, \\
y_p''(x) &= \sum_{m=0}^{\infty} a_m x^m + \sum_{m=1}^{\infty} (2m+2)(2m+1)\alpha_{2m+2} x^{2m} \\
&\quad + \sum_{m=1}^{\infty} (2m+3)(2m+2)\beta_{2m+3} x^{2m+1},
\end{aligned}
\tag{2.4}
$$

we have

$$
\begin{aligned}
y_p''(x) + 2xy_p'(x) - 2ny_p(x) &= \sum_{m=0}^{\infty} a_m x^m + \sum_{m=1}^{\infty} (2m+2)(2m+1)\alpha_{2m+2}x^{2m} \\
&\quad + \sum_{m=1}^{\infty} (2m+3)(2m+2)\beta_{2m+3}x^{2m+1} + \sum_{m=2}^{\infty} \frac{2a_{m-2}}{m-1}x^m \\
&\quad + \sum_{m=2}^{\infty} 4m\alpha_{2m}x^{2m} + \sum_{m=2}^{\infty} 2(2m+1)\beta_{2m+1}x^{2m+1} \\
&\quad - \sum_{m=2}^{\infty} \frac{2na_{m-2}}{m(m-1)}x^m - \sum_{m=2}^{\infty} 2n\alpha_{2m}x^{2m} - \sum_{m=2}^{\infty} 2n\beta_{2m+1}x^{2m+1} \\
&= \sum_{m=0}^{\infty} a_m x^m + 12\alpha_4 x^2 + \sum_{m=2}^{\infty} (2m+2)(2m+1)\alpha_{2m+2}x^{2m} \\
&\quad + 20\beta_5 x^3 + \sum_{m=2}^{\infty} (2m+3)(2m+2)\beta_{2m+3}x^{2m+1} \\
&\quad + \sum_{m=2}^{\infty} \frac{2(m-n)}{m(m-1)}a_{m-2}x^m - \sum_{m=2}^{\infty} 2(n-2m)\alpha_{2m}x^{2m} \\
&\quad - \sum_{m=2}^{\infty} 2[n-(2m+1)]\beta_{2m+1}x^{2m+1}
\end{aligned}
\tag{2.5}
$$

for all $x \in (-\rho_0, \rho_0)$.

It is not difficult to see that

$$
\begin{aligned}
(2m+2)(2m+1)\alpha_{2m+2} &= 2(n-2m)\alpha_{2m} + \frac{n-2m}{m(2m-1)}a_{2m-2}, \\
(2m+3)(2m+2)\beta_{2m+3} &= 2[n-(2m+1)]\beta_{2m+1} + \frac{n-(2m+1)}{(2m+1)m}a_{2m-1}
\end{aligned}
\tag{2.6}
$$

for any $m \in \mathbb{N}$. Hence, we obtain

$$
\begin{aligned}
y_p''(x) + 2xy_p'(x) - 2ny_p(x) &= \sum_{m=0}^{\infty} a_m x^m + 12\alpha_4 x^2 + \sum_{m=2}^{\infty} \frac{n-2m}{m(2m-1)}a_{2m-2}x^{2m} + 20\beta_5 x^3 \\
&\quad + \sum_{m=2}^{\infty} \frac{n-(2m+1)}{m(2m+1)}a_{2m-1}x^{2m+1} + \sum_{m=2}^{\infty} \frac{2(m-n)}{m(m-1)}a_{m-2}x^m \\
&= \sum_{m=0}^{\infty} a_m x^m,
\end{aligned}
\tag{2.7}
$$

which proves that $y_p(x)$ is a particular solution of the inhomogeneous equation (1.8).

We now apply the ratio test to the power series expression of $y_p(x)$. If n is an odd integer not less than 0, then $\beta_{n+2} = \beta_{n+4} = \beta_{n+6} = \cdots = 0$. Hence, the power series $\sum_{m=2}^{\infty} \beta_{2m+1}x^{2m+1}$ is a polynomial. And it follows from the first conditions of (2.1) and (2.6) that

$$\begin{aligned}
\lim_{m\to\infty}\left|\frac{\alpha_{2m+2}}{\alpha_{2m}}\right| &= \lim_{m\to\infty}\left|\frac{n-2m}{(m+1)(2m+1)} + \frac{n-2m}{m(2m+2)(2m+1)(2m-1)}\frac{a_{2m-2}}{\alpha_{2m}}\right| \\
&= \lim_{m\to\infty}\frac{|n-2m|}{m(2m+2)(2m+1)(2m-1)}\left|\frac{a_{2m-2}}{\alpha_{2m}}\right| \\
&\le \lim_{m\to\infty}\frac{|n-2m|}{m(2m+2)(2m+1)(2m-1)}\mu^2(m-1)2m(2m-1) \\
&= \mu^2.
\end{aligned} \tag{2.8}$$

If $n \ge 0$ is an even integer, then we have $\alpha_{n+2} = \alpha_{n+4} = \alpha_{n+6} = \cdots = 0$. Thus, for each even integer $n \ge 0$, the power series $\sum_{m=2}^{\infty} \alpha_{2m}x^{2m}$ is a polynomial. By the second conditions in (2.1) and (2.6), we get

$$\begin{aligned}
\lim_{m\to\infty}\left|\frac{\beta_{2m+3}}{\beta_{2m+1}}\right| &= \lim_{m\to\infty}\left|\frac{n-(2m+1)}{(2m+3)(m+1)} + \frac{n-(2m+1)}{(2m+3)(2m+2)(2m+1)m}\frac{a_{2m-1}}{\beta_{2m+1}}\right| \\
&= \lim_{m\to\infty}\frac{|n-(2m+1)|}{(2m+3)(2m+2)(2m+1)m}\left|\frac{a_{2m-1}}{\beta_{2m+1}}\right| \\
&\le \lim_{m\to\infty}\frac{|n-(2m+1)|}{(2m+3)(2m+2)(2m+1)m}\mu^2(m-1)(2m+1)2m \\
&= \mu^2.
\end{aligned} \tag{2.9}$$

Therefore, the power series expression of $y_p(x)$ converges for all $x \in (-\rho_0, \rho_0)$.

Moreover, the convergence region of the power series for $y_p(x)$ is the same as those of power series for $y_p'(x)$ and $y_p''(x)$. In this paper, the convergence region will denote the maximum open set where the relevant power series converges. Hence, the power series expression for $y_p''(x) + 2xy_p'(x) - 2ny_p(x)$ has the same convergence region as that of $y_p(x)$. This implies that $y_p(x)$ is well defined on $(-\rho_0, \rho_0)$ and so does for $y(x)$ in (2.3) because $y_h(x)$ converges for all $x \in \mathbb{R}$ under our hypotheses.

Since every solution to (1.8) can be expressed as a sum of a solution $y_h(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the inhomogeneous equation, every solution of (1.8) is certainly in the form of (2.3). □

Remark 2.2. We might have thought that the conditions presented in (2.1) were too strong. However, we can show that some familiar sequences $\{a_m\}$ satisfy the conditions in (2.1). For example, let $n = 0$ and $a_0 = a_1 = 1$, $a_{2m} = a_{2m+1} = 1/(m-1)!$ for all $m \in \mathbb{N}$ and choose an arbitrary $\mu > 0$. Then, by some manipulations, we can show that the coefficients sequence

$\{a_m\}$ satisfies the second condition of (2.1) for all sufficiently large integers m as we see in the following:

$$\begin{aligned}\mu^2 m(2m+3)(2m+2)\left|\beta_{2m+3}\right| &= \frac{\mu^2}{(m-1)!}\left|1+\sum_{k=1}^{m-1}(-1)^k k\right| \\ &\geq \frac{1}{(m-1)!} \\ &= |a_{2m+1}|.\end{aligned} \tag{2.10}$$

3. Hyers-Ulam Stability of (1.6)

In this section, let n be a nonnegative integer and let ρ be a constant with $0 < \rho \leq \infty$. For a given $K \geq 0$, let us denote by $\mathcal{C}_K$ the set of all functions $y : (-\rho, \rho) \to \mathbb{C}$ with the properties (a) and (b):

(a) $y(x)$ is represented by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least ρ;

(b) it holds true that $\sum_{m=0}^{\infty} |a_m x^m| \leq K|\sum_{m=0}^{\infty} a_m x^m|$ for all $x \in (-\rho, \rho)$, where $a_m = (m+2)(m+1)b_{m+2} + 2(m-n)b_m$ for each $m \in \mathbb{N}_0$.

It should be remarked that the power series $\sum_{m=0}^{\infty} a_m x^m$ in (b) has the same radius of convergence as that of $\sum_{m=0}^{\infty} b_m x^m$ given in (a).

In the following theorem, we prove that if an analytic function satisfies some given conditions, then it can be approximated by a combination of integrals of the error function (see the last part of Section 1 or [16, §7.2.2]).

Theorem 3.1. *Let n be a nonnegative integer. For given constants K and ρ with $K \geq 0$ and $0 < \rho \leq \infty$, suppose that $y : (-\rho, \rho) \to \mathbb{C}$ is a function which belongs to $\mathcal{C}_K$. Assume that there exist constants $\mu, \nu \geq 0$ satisfying*

$$\nu^2|\alpha_{2m+2}| \leq |a_{2m}| \leq \mu^2 m(2m+2)(2m+1)|\alpha_{2m+2}|, \tag{3.1}$$

$$\nu^2\left|\beta_{2m+3}\right| \leq |a_{2m+1}| \leq \mu^2 m(2m+3)(2m+2)\left|\beta_{2m+3}\right| \tag{3.2}$$

for all $m \in \mathbb{N}$. (See the definitions of α_{2m} and β_{2m+1} given in Theorem 2.1. Indeed, it is sufficient for the second inequalities in (3.1) *and* (3.2) *to hold true for all sufficiently large integers m.) Let us define $\rho_0 = \min\{\rho, 1/\mu\}$, where $1/0 = \infty$. If the function y satisfies the differential inequality*

$$\left|y''(x) + 2xy'(x) - 2ny(x)\right| \leq \varepsilon \tag{3.3}$$

for all $x \in (-\rho_0, \rho_0)$ *and for some* $\varepsilon \geq 0$, *then there exists a solution* $y_h : \mathbb{R} \to \mathbb{C}$ *of the differential equation* (1.6) *such that*

$$\left|y(x) - y_h(x)\right| \leq \left(\frac{1}{2} + \frac{1}{\nu^2}\right) K\varepsilon x^2 \tag{3.4}$$

for any $x \in (-\rho_0, \rho_0)$.

Proof. Since $y \in \mathcal{C}_K$, it follows from (a) and (b) that

$$\begin{aligned} y''(x) + 2xy'(x) - 2ny(x) &= \sum_{m=0}^{\infty} [(m+2)(m+1)b_{m+2} + 2(m-n)b_m]x^m \\ &= \sum_{m=0}^{\infty} a_m x^m \end{aligned} \tag{3.5}$$

for all $x \in (-\rho, \rho)$. It follows from the last equality and (3.3) that

$$\left|\sum_{m=0}^{\infty} a_m x^m\right| \leq \varepsilon \tag{3.6}$$

for any $x \in (-\rho_0, \rho_0)$. This inequality, together with (b), yields that

$$\sum_{m=0}^{\infty} |a_m x^m| \leq K \left|\sum_{m=0}^{\infty} a_m x^m\right| \leq K\varepsilon \tag{3.7}$$

for each $x \in (-\rho_0, \rho_0)$.

By Abel's formula (see [17, Theorem 6.30]), we have

$$\begin{aligned} \sum_{m=0}^{p} |a_m x^m| \frac{1}{(m+2)(m+1)} &= \sum_{m=0}^{p} |a_m x^m| \frac{1}{(p+2)(p+1)} \\ &\quad - \sum_{m=0}^{p-1} \left(\sum_{i=0}^{m} \left|a_i x^i\right|\right) \left[\frac{1}{(m+3)(m+2)} - \frac{1}{(m+2)(m+1)}\right] \\ &= \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} |a_m x^m| \\ &\quad + \sum_{m=0}^{p-1} \left(\sum_{i=0}^{m} \left|a_i x^i\right|\right) \frac{2}{(m+1)(m+2)(m+3)} \\ &\leq \sum_{m=0}^{\infty} K\varepsilon \frac{2}{(m+1)(m+2)(m+3)} \\ &= \frac{K}{2}\varepsilon \end{aligned} \tag{3.8}$$

for any $x \in (-\rho_0, \rho_0)$ and $p \in \mathbb{N}$, since

$$\sum_{m=0}^{\infty} \frac{2}{(m+1)(m+2)(m+3)} = \sum_{m=0}^{\infty} \left\{ \left(\frac{1}{m+1} - \frac{1}{m+2} \right) + \left(\frac{1}{m+3} - \frac{1}{m+2} \right) \right\} = \frac{1}{2}. \tag{3.9}$$

Hence, it follows from (3.7) and (3.8) that

$$\sum_{m=2}^{\infty} \left| \frac{a_{m-2}}{m(m-1)} x^m \right| = x^2 \sum_{m=0}^{\infty} |a_m x^m| \frac{1}{(m+2)(m+1)} \leq \frac{K}{2} \varepsilon x^2 \tag{3.10}$$

for each $x \in (-\rho_0, \rho_0)$.

Finally, it follows from Theorem 2.1, (3.1), (3.2), (3.7), and (3.10) that there exists a solution function $y_h : \mathbb{R} \to \mathbb{C}$ of the homogeneous differential equation (1.6) such that

$$\begin{aligned} |y(x) - y_h(x)| &\leq \sum_{m=2}^{\infty} \left| \frac{a_{m-2}}{m(m-1)} x^m \right| + \sum_{m=2}^{\infty} |\alpha_{2m}| \left| x^{2m} \right| + \sum_{m=2}^{\infty} |\beta_{2m+1}| \left| x^{2m+1} \right| \\ &\leq \frac{K}{2} \varepsilon x^2 + \frac{1}{\nu^2} \sum_{m=2}^{\infty} |a_{2m-2}| \left| x^{2m} \right| + \frac{1}{\nu^2} \sum_{m=2}^{\infty} |a_{2m-1}| \left| x^{2m+1} \right| \\ &\leq \left(\frac{1}{2} + \frac{1}{\nu^2} \right) K \varepsilon x^2 \end{aligned} \tag{3.11}$$

for all $x \in (-\rho_0, \rho_0)$. □

If ρ is finite, then the local Hyers-Ulam stability of the differential equation (1.6) immediately follows from Theorem 3.1.

Corollary 3.2. *Let n be a nonnegative integer. For given constants K and ρ with $K \geq 0$ and $0 < \rho < \infty$, suppose that $y : (-\rho, \rho) \to \mathbb{C}$ is a function which belongs to $\mathcal{C}_K$. Assume that there exist constants $\mu, \nu \geq 0$ satisfying the conditions in* (3.1) *and* (3.2) *for all $m \in \mathbb{N}$.* (*It is sufficient for the second inequalities in* (3.1) *and* (3.2) *to hold true for all sufficiently large integers m.*) *Let us define $\rho_0 = \min\{\rho, 1/\mu\}$ and $1/0 = \infty$. If the function y satisfies the differential inequality* (3.3) *for all $x \in (-\rho_0, \rho_0)$ and for some $\varepsilon \geq 0$, then there exists a solution $y_h : \mathbb{R} \to \mathbb{C}$ of the differential equation* (1.6) *such that*

$$|y(x) - y_h(x)| \leq \left(\frac{1}{2} + \frac{1}{\nu^2} \right) K \rho^2 \varepsilon \tag{3.12}$$

for any $x \in (-\rho_0, \rho_0)$.

We now deal with an asymptotic behavior of functions in $\mathcal{C}_K$ under the additional conditions (3.1) and (3.2).

Corollary 3.3. *Let n be a nonnegative integer. For given constants K, ρ, and ρ_1 with $K \geq 0$ and $0 < \rho_1 < \rho \leq \infty$, suppose that $y : (-\rho, \rho) \to \mathbb{C}$ is a function belonging to $\mathcal{C}_K$. Assume that there exist constants $\mu \geq 0$ and $\nu > 0$ satisfying the conditions in* (3.1) *and* (3.2) *for any $m \in \mathbb{N}$.* (*It is sufficient* *for the second inequalities in* (3.1) *and* (3.2) *to hold true for all sufficiently large integers* m.) *Then there exists a solution* $y_h : \mathbb{R} \to \mathbb{C}$ *of the differential equation* (1.6) *such that*

$$\left|y(x) - y_h(x)\right| = O\left(x^2\right) \tag{3.13}$$

as $x \to 0$.

Proof. Since $y \in \mathcal{C}_K$, it follows from the first 4 lines of the proof of Theorem 3.1 that

$$y''(x) + 2xy'(x) - 2ny(x) = \sum_{m=0}^{\infty} a_m x^m \tag{3.14}$$

for all $x \in (-\rho, \rho)$. As was remarked in the first part of Section 3, the radius of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ is same as that of $\sum_{m=0}^{\infty} b_m x^m (= y(x))$, that is, it is at least ρ. Since $0 < \rho_1 < \rho$, if we set $\rho_0 = \min\{\rho_1, 1/\mu\}$, then there exists a constant $\delta > 0$ such that

$$\left|y''(x) + 2xy'(x) - 2ny(x)\right| = \left|\sum_{m=0}^{\infty} a_m x^m\right| \le \delta \tag{3.15}$$

for any $x \in (-\rho_0, \rho_0)$.

According to Theorem 3.1, there exists a solution $y_h : \mathbb{R} \to \mathbb{C}$ of the differential equation (1.6) satisfying

$$\left|y(x) - y_h(x)\right| \le \left(\frac{1}{2} + \frac{1}{\nu^2}\right) K\delta x^2 \tag{3.16}$$

for any $x \in (-\rho_0, \rho_0)$. Hence, we have

$$\left|y(x) - y_h(x)\right| = O\left(x^2\right) \tag{3.17}$$

as $x \to 0$. □

4. An Example

The conditions in (3.1) and (3.2) may seem too strong to construct some examples for the coefficients a_m's. In this section, however, we will show that the sequence $\{a_m\}$ given in Remark 2.2 satisfies these conditions: let $n = 0$ and $a_0 = a_1 = 1$, $a_{2m} = a_{2m+1} = 1/(m-1)!$ for all $m \in \mathbb{N}$ and choose some constants $\mu > 0$ and $\nu = \sqrt{2}$. The second inequality in (3.2) has been verified in Remark 2.2.

The first inequality in (3.2) is also true for all $m \in \mathbb{N}$ as we see in the following:

$$\begin{aligned}
\nu^2 \left|\beta_{2m+3}\right| &= 2\left|\frac{(-1)^m}{(2m+3)(2m+2)} \frac{1}{m!}\sum_{k=0}^{m-1}(-1)^k k! a_{2k+1}\right| \\
&\leq \frac{2}{(2m+3)(2m+2)m}|a_{2m+1}|\left|1+\sum_{k=1}^{m-1}(-1)^k k\right| \\
&\leq \frac{2}{(2m+3)(2m+2)m}|a_{2m+1}|\left[\frac{m+1}{2}\right] \\
&\leq \frac{m+1}{(2m+3)(2m+2)m}|a_{2m+1}| \\
&\leq |a_{2m+1}|,
\end{aligned} \tag{4.1}$$

where $[x]$ denotes the largest integer not exceeding x.

It is not difficult to show that

$$\frac{1}{4k} \leq \frac{(2k+2)!}{4^{k+1}(k+1)!k!} - \frac{(2k)!}{4^k k!(k-1)!} \leq \frac{2k-1}{4k}, \tag{4.2}$$

$$\frac{1}{2} \leq \frac{(2k)!}{4^k k!(k-1)!} \leq k - \frac{1}{2} \tag{4.3}$$

for all $k \in \mathbb{N}$.

By using (4.2), we will now prove that

$$\left|1+\sum_{k=1}^{m-1}\frac{(-1)^k}{4^k}\frac{(2k)!}{k!(k-1)!}\right| \longrightarrow \infty \tag{4.4}$$

as $m \to \infty$: if $m = 2\ell$ for some $\ell \in \mathbb{N}$, then

$$\begin{aligned}
1+\sum_{k=1}^{m-1}\frac{(-1)^k}{4^k}\frac{(2k)!}{k!(k-1)!} &= \frac{1}{2}+\sum_{i=1}^{\ell-1}\left[\frac{1}{4^{2i}}\frac{(4i)!}{(2i)!(2i-1)!} - \frac{1}{4^{2i+1}}\frac{(4i+2)!}{(2i+1)!(2i)!}\right] \\
&\leq \frac{1}{2}+\sum_{i=1}^{\ell-1}\left(-\frac{1}{8i}\right) \\
&\longrightarrow -\infty \quad \text{as } m \longrightarrow \infty.
\end{aligned} \tag{4.5}$$

If $m = 2\ell + 1$ for some $\ell \in \mathbb{N}$, then

$$\begin{aligned} 1+\sum_{k=1}^{m-1}\frac{(-1)^k}{4^k}\frac{(2k)!}{k!(k-1)!} &= 1+\sum_{i=1}^{\ell}\left[\frac{1}{4^{2i}}\frac{(4i)!}{(2i)!(2i-1)!}-\frac{1}{4^{2i-1}}\frac{(4i-2)!}{(2i-1)!(2i-2)!}\right] \\ &\geq 1+\sum_{i=1}^{\ell}\frac{1}{8i-4} \\ &\longrightarrow \infty \quad \text{as } m \longrightarrow \infty. \end{aligned} \tag{4.6}$$

It then follows from (4.3) and (4.4) that

$$\begin{aligned} \mu^2 m(2m+2)(2m+1)|\alpha_{2m+2}| &= \mu^2\frac{4^m m!}{2(2m-1)!}\left|\sum_{k=0}^{m-1}\frac{(-1)^k}{4^k}\frac{(2k)!}{k!}a_{2k}\right| \\ &= \mu^2 m\frac{4^m m!(m-1)!}{(2m)!}|a_{2m}|\left|1+\sum_{k=1}^{m-1}\frac{(-1)^k}{4^k}\frac{(2k)!}{k!(k-1)!}\right| \\ &\geq \mu^2 m\frac{1}{m}|a_{2m}|\left|1+\sum_{k=1}^{m-1}\frac{(-1)^k}{4^k}\frac{(2k)!}{k!(k-1)!}\right| \\ &\geq |a_{2m}| \end{aligned} \tag{4.7}$$

for all sufficiently large integers m, which proves that the sequence $\{a_m\}$ satisfies the second inequality in (3.1).

Finally, we will show that the sequence $\{a_m\}$ satisfies the first inequality in (3.1). It follows from (4.3) that

$$\begin{aligned} \nu^2|\alpha_{2m+2}| &= 2\frac{4^m m!}{(2m+2)!}\left|1+\sum_{k=1}^{m-1}\frac{(-1)^k}{4^k}\frac{(2k)!}{k!(k-1)!}\right| \\ &= 2\frac{4^m m!(m-1)!}{(2m+2)!}|a_{2m}|\left|1+\sum_{k=1}^{m-1}\frac{(-1)^k}{4^k}\frac{(2k)!}{k!(k-1)!}\right| \\ &= \frac{2}{(2m+2)(2m+1)}\frac{4^m m!(m-1)!}{(2m)!}|a_{2m}|\left|1+\sum_{k=1}^{m-1}\frac{(-1)^k}{4^k}\frac{(2k)!}{k!(k-1)!}\right| \\ &\leq \frac{4}{(2m+2)(2m+1)}|a_{2m}|\left|1+\sum_{k=1}^{m-1}\frac{(-1)^k}{4^k}\frac{(2k)!}{k!(k-1)!}\right| \\ &\leq \frac{4}{(2m+2)(2m+1)}|a_{2m}|\left|1+\sum_{k=1}^{m-1}\frac{(2k)!}{4^k k!(k-1)!}\right| \end{aligned}$$

$$
\begin{aligned}
&\le \frac{4}{(2m+2)(2m+1)}|a_{2m}|\left|1+\sum_{k=1}^{m-1}\left(k-\frac{1}{2}\right)\right| \\
&= \frac{2m^2-4m+6}{(2m+2)(2m+1)}|a_{2m}| \\
&< |a_{2m}|
\end{aligned}
\tag{4.8}
$$

for each $m \in \mathbb{N}$.

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