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Research Article

Improvement and Reversion of Slater's Inequality and Related Results

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We use an inequality given by Matić and Pečarić (2000) and obtain improvement and reverse of Slater's and related inequalities.

1. Introduction

In 1981 Slater has proved an interesting companion inequality to Jensen's inequality [1].

Theorem 1.1. *Suppose that $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is increasing convex function on interval I, for $x_1, x_2, \ldots, x_n \in I^\circ$ (where I° is the interior of the interval I) and for $p_1, p_2, \ldots, p_n \geq 0$ with $P_n = \sum_{i=1}^n p_i > 0$, if $\sum_{i=1}^n p_i \phi'_+(x_i) > 0$, then*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \phi(x_i) \leq \phi\left(\frac{\sum_{i=1}^n p_i \phi'_+(x_i) x_i}{\sum_{i=1}^n p_i \phi'_+(x_i)}\right). \tag{1.1}$$

When ϕ is strictly convex on I, inequality (1.1) *becomes equality if and only if $x_i = c$ for some $c \in I^\circ$ and for all i with $p_i > 0$.*

It was noted in [2] that by using the same proof the following generalization of Slater's inequality (1981) can be given.

Theorem 1.2. *Suppose that $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex function on interval I, for $x_1, x_2, \ldots, x_n \in I^\circ$ (where I° is the interior of the interval I) and for $p_1, p_2, \ldots, p_n \geq 0$ with $P_n = \sum_{i=1}^n p_i > 0$. Let*

$$\sum_{i=1}^{n} p_i \phi'_+(x_i) \neq 0, \qquad \frac{\sum_{i=1}^n p_i \phi'_+(x_i) x_i}{\sum_{i=1}^n p_i \phi'_+(x_i)} \in I^\circ, \tag{1.2}$$

then inequality (1.1) *holds.*

When ϕ is strictly convex on I, inequality (1.1) *becomes equality if and only if $x_i = c$ for some $c \in I^\circ$ and for all i with $p_i > 0$.*

Remark 1.3. For multidimensional version of Theorem 1.2 see [3].

Another companion inequality to Jensen's inequality is a converse proved by Dragomir and Goh in [4].

Theorem 1.4. *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable convex function defined on interval I. If $x_i \in I, i = 1, 2, \ldots, n$ $(n \geq 2)$ are arbitrary members and $p_i \geq 0$ $(i = 1, 2, \ldots, n)$ with $P_n = \sum_{i=1}^n p_i > 0$, and let*

$$\overline{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i, \qquad \overline{y} = \frac{1}{P_n} \sum_{i=1}^{n} p_i \phi(x_i). \tag{1.3}$$

Then the inequalities

$$0 \leq \overline{y} - \phi(\overline{x}) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \phi'(x_i)(x_i - \overline{x}) \tag{1.4}$$

hold.

In the case when ϕ is strictly convex, one has equalities in (1.4) *if and only if there is some $c \in I$ such that $x_i = c$ holds for all i with $p_i > 0$.*

Matić and Pečarić in [5] proved more general inequality from which (1.1) and (1.4) can be obtained as special cases.

Theorem 1.5. *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable convex function defined on interval I and let x_i, p_i, P_n, $\overline{x}$, and $\overline{y}$ be stated as in Theorem 1.4. If $d \in I$ is arbitrary chosen number, then one has*

$$\overline{y} \leq \phi(d) + \frac{1}{P_n} \sum_{i=1}^{n} p_i (x_i - d) \phi'(x_i). \tag{1.5}$$

Also, when ϕ is strictly convex, one has equality in (1.5) *if and only if $x_i = d$ holds for all i with $p_i > 0$.*

Remark 1.6. If ϕ, x_i, p_i, P_n, and $\overline{x}$ are stated as in Theorem 1.4 and we let $\sum_{i=1}^n p_i \phi'(x_i) \neq 0$, also if $\overline{\overline{x}} = \sum_{i=1}^n p_i x_i \phi'(x_i) / \sum_{i=1}^n p_i \phi'(x_i) \in I$, then by setting $d = \overline{\overline{x}}$ in (1.5), we get Slater's inequality (1.1) and similarly by setting $d = \overline{x}$ in (1.5), we get (1.4).

The following refinement of (1.4) is also valid [5].

Theorem 1.7. *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be strictly convex differentiable function defined on interval I and let x_i, p_i, P_n, $\overline{x}$, and $\overline{y}$ be stated as in Theorem 1.4 and $\overline{d} = (\phi')^{-1}((1/P_n)\sum_{i=1}^{n} p_i\phi'(x_i))$, then the inequalities*

$$\overline{y} \leq \phi\left(\overline{d}\right) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\phi'(x_i)\left(x_i - \overline{d}\right), \tag{1.6}$$

$$0 \leq \overline{y} - \phi(\overline{x}) \leq \phi\left(\overline{d}\right) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\phi'(x_i)\left(x_i - \overline{d}\right) - \phi(\overline{x}) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i\phi'(x_i)(x_i - \overline{x}) \tag{1.7}$$

hold.

The equalities hold in (1.6) *and in* (1.7) *if and only if* $x_1 = x_2 = \cdots = x_n$.

Remark 1.8. In [6] Dragomir has also proved Theorem 1.7.

In this paper, we use an inequality given in [5] and derive two mean value theorems, exponential convexity, log-convexity, and Cauchy means. As applications, such results are also deduce for related inequality. We use some log-convexity criterion and prove improvement and reverse of Slater's and related inequalities. We also prove some determinantal inequalities.

2. Mean Value Theorems

Theorem 2.1. *Let $\phi \in C^2(I)$, where I is closed interval in $\mathbb{R}$, and let $P_n = \sum_{i=1}^{n} p_i$, $p_i > 0$, $x_i, d \in I$ with $x_i \neq d$ $(i = 1, 2, \ldots, n)$ and $\overline{y} = (1/P_n)\sum_{i=1}^{n} p_i\phi(x_i)$. Then there exists $\xi \in I$ such that*

$$\phi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d)\phi'(x_i) - \overline{y} = \frac{\phi''(\xi)}{2P_n}\sum_{i=1}^{n} p_i(x_i - d)^2. \tag{2.1}$$

Proof. Since $\phi''(x)$ is continuous on I, $m \leq \phi''(x) \leq M$ for $x \in I$, where $m = \min_{x\in I}\phi''(x)$ and $M = \max_{x\in I}\phi''(x)$.

Consider the functions ϕ_1, ϕ_2 defined as

$$\begin{aligned} \phi_1(x) &= \frac{Mx^2}{2} - \phi(x), \\ \phi_2(x) &= \phi(x) - \frac{mx^2}{2}. \end{aligned} \tag{2.2}$$

Since

$$\begin{aligned} \phi_1''(x) &= M - \phi''(x) \geq 0, \\ \phi_2''(x) &= \phi''(x) - m \geq 0, \end{aligned} \tag{2.3}$$

$\phi_i(x)$ for $i = 1, 2$ are convex.

Now by applying ϕ_1 for ϕ in inequality (1.5), we have

$$\frac{Md^2}{2} - \phi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d)\left(Mx_i - \phi'(x_i)\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\left(\frac{Mx_i^2}{2} - \phi(x_i)\right) \geq 0. \tag{2.4}$$

From (2.4) we get

$$\phi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d)\phi'(x_i) - \overline{y} \leq \frac{M}{2P_n}\sum_{i=1}^{n} p_i(x_i - d)^2, \tag{2.5}$$

and similarly by applying ϕ_2 for ϕ in (1.5), we get

$$\phi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d)\phi'(x_i) - \overline{y} \geq \frac{m}{2P_n}\sum_{i=1}^{n} p_i(x_i - d)^2. \tag{2.6}$$

Since

$$\sum_{i=1}^{n} p_i(x_i - d)^2 > 0 \quad \text{as } x_i \neq d,\ p_i > 0\ (i = 1, 2, \ldots, n), \tag{2.7}$$

by combining (2.5) and (2.6), we have

$$m \leq \frac{2P_n\left[\phi(d) + (1/P_n)\sum_{i=1}^{n} p_i(x_i - d)\phi'(x_i) - \overline{y}\right]}{\sum_{i=1}^{n} p_i(x_i - d)^2} \leq M. \tag{2.8}$$

Now using the fact that for $m \leq \rho \leq M$ there exists $\xi \in I$ such that $\phi''(\xi) = \rho$, we get (2.1). □

Corollary 2.2. *Let $\phi \in C^2(I)$, where I is closed interval in $\mathbb{R}$, and let x_i, $\overline{x}$, $\overline{y}$, and P_n be stated as in Theorem 1.4 with $p_i > 0$ and $x_i \neq \overline{x}$ $(i = 1, 2, \ldots, n)$. Then there exists $\xi \in I$ such that*

$$\phi(\overline{x}) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - \overline{x})\phi'(x_i) - \overline{y} = \frac{\phi''(\xi)}{2P_n}\sum_{i=1}^{n} p_i(x_i - \overline{x})^2. \tag{2.9}$$

Proof. By setting $d = \overline{x}$ in Theorem 2.1, we get (2.9). □

Theorem 2.3. *Let $\phi, \psi \in C^2(I)$, where I is closed interval in $\mathbb{R}$, and let $P_n = \sum_{i=1}^{n} p_i$, $p_i > 0$ and $x_i, d \in I$ with $x_i \neq d$ $(i = 1, 2, \ldots, n)$. Then there exists $\xi \in I$ such that*

$$\frac{\phi''(\xi)}{\psi''(\xi)} = \frac{\phi(d) + (1/P_n)\sum_{i=1}^{n} p_i(x_i - d)\phi'(x_i) - (1/P_n)\sum_{i=1}^{n} p_i\phi(x_i)}{\psi(d) + (1/P_n)\sum_{i=1}^{n} p_i(x_i - d)\psi'(x_i) - (1/P_n)\sum_{i=1}^{n} p_i\psi(x_i)}, \tag{2.10}$$

provided that the denominators are nonzero.

Proof. Let the function $k \in C^2(I)$ be defined by

$$k = c_1\phi - c_2\psi, \tag{2.11}$$

where c_1 and c_2 are defined as

$$\begin{aligned} c_1 &= \psi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d)\psi'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\psi(x_i), \\ c_2 &= \phi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d)\phi'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\phi(x_i). \end{aligned} \tag{2.12}$$

Then, using Theorem 2.1 with $\phi = k$, we have

$$0 = \left(\frac{c_1\phi''(\xi)}{2P_n} - \frac{c_2\psi''(\xi)}{2P_n}\right)\sum_{i=1}^{n} p_i(x_i - d)^2, \tag{2.13}$$

because $k(d) + (1/P_n)\sum_{i=1}^{n} p_i(x_i - d)k'(d) - (1/P_n)\sum_{i=1}^{n} p_i k(x_i) = 0$.

Since $(1/P_n)\sum_{i=1}^{n} p_i(x_i - d)^2 > 0$ as $x_i \neq d$ and $p_i > 0$ $(i = 1, 2, \ldots, n)$, therefore, (2.13) gives us

$$\frac{c_2}{c_1} = \frac{\phi''(\xi)}{\psi''(\xi)}. \tag{2.14}$$

After putting the values of c_1 and c_2, we get (2.10). □

Corollary 2.4. *Let $\phi, \psi \in C^2(I)$, where I is closed interval in $\mathbb{R}$, and $P_n = \sum_{i=1}^{n} p_i$, $p_i > 0$ and let $x_i \in I$, $\overline{x} = (1/P_n)\sum_{i=1}^{n} p_i x_i$ with $x_i \neq \overline{x}$ $(i = 1, 2, \ldots, n)$. Then there exists $\xi \in I$ such that*

$$\frac{\phi''(\xi)}{\psi''(\xi)} = \frac{\phi(\overline{x}) + (1/P_n)\sum_{i=1}^{n} p_i(x_i - \overline{x})\phi'(x_i) - (1/P_n)\sum_{i=1}^{n} p_i\phi(x_i)}{\psi(\overline{x}) + (1/P_n)\sum_{i=1}^{n} p_i(x_i - \overline{x})\psi'(x_i) - (1/P_n)\sum_{i=1}^{n} p_i\psi(x_i)}, \tag{2.15}$$

provided that the denominators are nonzero.

Proof. By setting $d = \overline{x}$ in Theorem 2.3, we get (2.15). □

Corollary 2.5. *Let $x_i, d \in I$ with $x_i \neq d$ and $P_n = \sum_{i=1}^{n} p_i$, $p_i > 0$ $(i = 1, 2, \ldots, n)$. Then for $u, v \in \mathbb{R} \setminus \{0, 1\}$, $u \neq v$, there exists $\xi \in I$, where I is positive closed interval, such that*

$$\xi^{u-v} = \frac{v(v-1)\left[d^u + (u/P_n)\sum_{i=1}^{n} p_i(x_i - d)x_i^{u-1} - (1/P_n)\sum_{i=1}^{n} p_i x_i^u\right]}{u(u-1)\left[d^v + (v/P_n)\sum_{i=1}^{n} p_i(x_i - d)x_i^{v-1} - (1/P_n)\sum_{i=1}^{n} p_i x_i^v\right]}. \tag{2.16}$$

Proof. By setting $\phi(x) = x^u$ and $\psi(x) = x^v$, $x \in I$, in Theorem 2.3, we get (2.16). □

Corollary 2.6. *Let $x_i \in I$, $P_n = \sum_{i=1}^{n} p_i$, $p_i > 0$ $(i = 1,2,\ldots,n)$, and $\overline{x} = (1/P_n)\sum_{i=1}^{n} p_i x_i$ with $x_i \neq \overline{x}$. Then for $u, v \in \mathbb{R} \setminus \{0,1\}$, $u \neq v$, there exists $\xi \in I$, where I is positive closed interval, such that*

$$\xi^{u-v} = \frac{v(v-1)\left[\overline{x}^{u} + (u/P_n)\sum_{i=1}^{n} p_i(x_i - \overline{x})x_i^{u-1} - (1/P_n)\sum_{i=1}^{n} p_i x_i^{u}\right]}{u(u-1)\left[\overline{x}^{v} + (v/P_n)\sum_{i=1}^{n} p_i(x_i - \overline{x})x_i^{v-1} - (1/P_n)\sum_{i=1}^{n} p_i x_i^{v}\right]}. \tag{2.17}$$

Proof. By setting $\phi(x) = x^u$ and $\psi(x) = x^v$, $x \in I$, in (2.15), we get (2.17). □

Remark 2.7. Note that we can consider the interval $I = [m_x, M_x]$, where $m_x = \min_i\{x_i, d\}$, $M_x = \max_i\{x_i, d\}$.

Since the function $\xi \to \xi^{u-v}$ with $u \neq v$ is invertible, then from (2.16) we have

$$m_x \leq \left\{\frac{v(v-1)\left[d^{u} + (u/P_n)\sum_{i=1}^{n} p_i(x_i - d)x_i^{u-1} - (1/P_n)\sum_{i=1}^{n} p_i x_i^{u}\right]}{u(u-1)\left[d^{v} + (v/P_n)\sum_{i=1}^{n} p_i(x_i - d)x_i^{v-1} - (1/P_n)\sum_{i=1}^{n} p_i x_i^{v}\right]}\right\}^{1/(u-v)} \leq M_x. \tag{2.18}$$

We will say that the expression in the middle is a mean of x_i, d.

From (2.17) we have

$$\min_i\{x_i\} \leq \left\{\frac{v(v-1)\left[\overline{x}^{u} + (u/P_n)\sum_{i=1}^{n} (x_i - \overline{x})x_i^{u-1} - (1/P_n)\sum_{i=1}^{n} p_i x_i^{u}\right]}{u(u-1)\left[\overline{x}^{v} + (v/P_n)\sum_{i=1}^{n} p_i(x_i - \overline{x})x_i^{v-1} - (1/P_n)\sum_{i=1}^{n} p_i x_i^{v}\right]}\right\}^{1/(u-v)} \leq \max_i\{x_i\}. \tag{2.19}$$

The expression in the middle of (2.19) is a mean of x_i.

In fact similar results can also be given for (2.10) and (2.15). Namely, suppose that ϕ''/ψ'' has inverse function, then from (2.10) and (2.15) we have

$$\begin{aligned}
\xi &= \left(\frac{\phi''}{\psi''}\right)^{-1}\left(\frac{\phi(d) + (1/P_n)\sum_{i=1}^{n} p_i(x_i - d)\phi'(x_i) - (1/P_n)\sum_{i=1}^{n} p_i\phi(x_i)}{\psi(d) + (1/P_n)\sum_{i=1}^{n} p_i(x_i - d)\psi\prime(x_i) - (1/P_n)\sum_{i=1}^{n} p_i\psi(x_i)}\right), \\
\xi &= \left(\frac{\phi''}{\psi''}\right)^{-1}\left(\frac{\phi(\overline{x}) + (1/P_n)\sum_{i=1}^{n} p_i(x_i - \overline{x})\phi'(x_i) - (1/P_n)\sum_{i=1}^{n} p_i\phi(x_i)}{\psi(\overline{x}) + (1/P_n)\sum_{i=1}^{n} p_i(x_i - \overline{x})\psi'(x_i) - (1/P_n)\sum_{i=1}^{n} p_i\psi(x_i)}\right).
\end{aligned} \tag{2.20}$$

So, we have that the expression on the right-hand side of (2.20) is also means.

3. Improvements and Related Results

Definition 3.1 (see [7, page 2]). A function $\phi : I \to \mathbb{R}$ is convex if

$$\phi(s_1)(s_3 - s_2) + \phi(s_2)(s_1 - s_3) + \phi(s_3)(s_2 - s_1) \geq 0 \tag{3.1}$$

holds for every $s_1 < s_2 < s_3$, $s_1, s_2, s_3 \in I$.

Lemma 3.2 (see [8]). *Let one define the function*

$$\varphi_t(x) = \begin{cases} \dfrac{x^t}{t(t-1)}, & t \neq 0, 1, \\ -\log x, & t = 0, \\ x \log x, & t = 1. \end{cases} \tag{3.2}$$

Then $\varphi_t''(x) = x^{t-2}$, that is, φ_t is convex for $x > 0$.

Definition 3.3 (see [9]). A function $\phi : I \to \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{k,l=1}^{n} a_k a_l \phi(x_k + x_l) \geq 0, \tag{3.3}$$

for all $n \in \mathbb{N}$, $a_k \in \mathbb{R}$, and $x_k \in I$, $k = 1, 2, \ldots, n$ such that $x_k + x_l \in I$, $1 \leq k, l \leq n$, or equivalently

$$\sum_{k,l=1}^{n} a_k a_l \phi\left(\frac{x_k + x_l}{2}\right) \geq 0. \tag{3.4}$$

Corollary 3.4 (see [9]). *If ϕ is exponentially convex function, then*

$$\det\left[\phi\left(\frac{x_k + x_l}{2}\right)\right]_{k,l=1}^{n} \geq 0 \tag{3.5}$$

for every $n \in \mathbb{N}$ $x_k \in I$, $k = 1, 2, \ldots, n$.

Corollary 3.5 (see [9]). *If $\phi : I \to (0, \infty)$ is exponentially convex function, then ϕ is a log-convex function that is*

$$\phi(\lambda x + (1-\lambda)y) \leq \phi^{\lambda}(x)\phi^{1-\lambda}(y), \quad \forall x, y \in I, \ \lambda \in [0,1]. \tag{3.6}$$

Theorem 3.6. *Let $x_i, p_i, d \in \mathbb{R}^+$ $(i = 1, 2, \ldots, n)$, $P_n = \sum_{i=1}^{n} p_i$. Consider Γ_t to be defined by*

$$\Gamma_t = \varphi_t(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d)\varphi_t'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi_t(x_i). \tag{3.7}$$

Then

(i) *for every $m \in \mathbb{N}$ and for every $s_k \in \mathbb{R}$, $k \in \{1, 2, 3, \ldots, m\}$, the matrix $[\Gamma_{(s_k+s_l)/2}]_{k,l=1}^{m}$ is a positive semidefinite matrix; particularly*

$$\det\left[\Gamma_{(s_k+s_l)/2}\right]_{k,l=1}^{m} \geq 0; \tag{3.8}$$

(ii) *the function $t \to \Gamma_t$ is exponentially convex;*

(iii) *if* $\Gamma_t > 0$, *then the function* $t \to \Gamma_t$ *is log-convex, that is, for* $-\infty < r < s < t < \infty$, *one has*

$$(\Gamma_s)^{t-r} \le (\Gamma_r)^{t-s}(\Gamma_t)^{s-r}. \tag{3.9}$$

Proof. (i) Let us consider the function defined by

$$\mu(x) = \sum_{k,l=1}^{m} a_k a_l \varphi_{s_{kl}}(x), \tag{3.10}$$

where $s_{kl} = (s_k + s_l)/2$, $a_k \in \mathbb{R}$ for all $k \in \{1,2,3,\dots,m\}$, $x > 0$

Then we have

$$\mu''(x) = \sum_{k,l=1}^{m} a_k a_l x^{s_{kl}-2} = \left(\sum_{k=1}^{m} a_k x^{(s_k-2)/2}\right)^2 \ge 0. \tag{3.11}$$

Therefore, $\mu(x)$ is convex function for $x > 0$. Using $\mu(x)$ in inequality (1.5), we get

$$\sum_{k,l=1}^{m} a_k a_l \Gamma_{s_{kl}} \ge 0, \tag{3.12}$$

so the matrix $[\Gamma_{(s_k+s_l)/2}]_{k,l=1}^{m}$ is positive semi-definite.

(ii) Since $\lim_{t\to 0}\Gamma_t = \Gamma_0$ and $\lim_{t\to 1}\Gamma_t = \Gamma_1$, so Γ_t is continuous for all $t \in \mathbb{R}$, $x > 0$, and we have exponentially convexity of the function $t \to \Gamma_t$.

(iii) Let $\Gamma_t > 0$, then by Corollary 3.5 we have that Γ_t is log-convex, that is, $t \to \log\Gamma_t$ is convex, and by (3.1) for $-\infty < r < s < t < \infty$ and taking $\phi(t) = \log\Gamma_t$, we get

$$(t-s)\log\Gamma_r + (r-t)\log\Gamma_s + (s-r)\log\Gamma_t \ge 0, \tag{3.13}$$

which is equivalent to (3.9). □

Corollary 3.7. *Let* $x_i, p_i \in \mathbb{R}^+$ $(i = 1,2,\dots,n)$, $P_n = \sum_{i=1}^{n} p_i$ *and* $\overline{x} = (1/P_n)\sum_{i=1}^{n} p_i x_i$. *Consider* $\widetilde{\Gamma}_t$ *to be defined by*

$$\widetilde{\Gamma}_t = \varphi_t(\overline{x}) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - \overline{x})\varphi_t'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi_t(x_i). \tag{3.14}$$

Then

(i) *for every* $m \in \mathbb{N}$ *and for every* $s_k \in \mathbb{R}$, $k \in \{1,2,3,\dots,m\}$, *the matrix* $[\widetilde{\Gamma}_{(s_k+s_l)/2}]_{k,l=1}^{m}$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\widetilde{\Gamma}_{(s_k+s_l)/2}\right]_{k,l=1}^{m} \ge 0, \tag{3.15}$$

(ii) *the function* $t \to \widetilde{\Gamma}_t$ *is exponentially convex;*

(iii) *if* $\widetilde{\Gamma}_t > 0$, *then the function* $t \to \widetilde{\Gamma}_t$ *is log-convex, that is, for* $-\infty < r < s < t < \infty$, *one has*

$$\left(\widetilde{\Gamma}_s\right)^{t-r} \leq \left(\widetilde{\Gamma}_r\right)^{t-s} \left(\widetilde{\Gamma}_t\right)^{s-r}. \tag{3.16}$$

Proof. To get the required results, set $d = \overline{x}$ in Theorem 3.6. □

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be positive n-tuple and $p_1, p_2, \ldots, p_n$ positive real numbers, and let $P_n = \sum_{i=1}^{n} p_i$. Let $M_t(\mathbf{x})$ denote the power mean of order t $(t \in \mathbb{R})$, defined by

$$M_t(\mathbf{x}) = \begin{cases} \left(\dfrac{1}{P_n}\displaystyle\sum_{i=1}^{n} p_i x_i^t\right)^{1/t}, & t \neq 0, \\ \left(\displaystyle\prod_{i=1}^{n} x_i^{p_i}\right)^{1/P_n}, & t = 0. \end{cases} \tag{3.17}$$

Let us note that $M_1(\mathbf{x}) = \overline{x}$.

By (2.18) we can give the following definition of Cauchy means.

Let $x_i, d \in I$ with $x_i \neq d$, I is positive closed interval, and $P_n = \sum_{i=1}^{n} p_i$, $p_i > 0$ $(i = 1, 2, \ldots, n)$,

$$M_{u,v} = \left(\frac{\Gamma_u}{\Gamma_v}\right)^{1/(u-v)} \tag{3.18}$$

for $-\infty < u \neq v < +\infty$ are means of x_i, d. Moreover we can extend these means to the other cases.

So by limit we have

$$\begin{aligned} M_{u,u} &= \exp\left(\frac{P_n d^u \log d + (u-1)\sum_{i=1}^{n} p_i x_i^u \log x_i + P_n M_u^u(\mathbf{x}) - d\left(u\sum_{i=1}^{n} p_i x_i^{u-1} \log x_i + P_n M_{u-1}^{u-1}(\mathbf{x})\right)}{P_n\left[d^u + (u-1)M_u^u(\mathbf{x}) - du M_{u-1}^{u-1}(\mathbf{x})\right]} \right. \\ &\qquad \left. - \frac{2u-1}{u(u-1)}\right), \quad u \neq 0, 1, \\ M_{0,0} &= \exp\left(\frac{P_n \log^2 d - P_n M_2^2(\log \mathbf{x}) + 2P_n \log M_0(\mathbf{x}) - 2d\sum_{i=1}^{n} p_i x_i^{-1} \log x_i}{2P_n\left[\log d - \log M_0(\mathbf{x}) + 1 - dM_{-1}^{-1}(\mathbf{x})\right]} + 1\right), \\ M_{1,1} &= \exp\left(\frac{P_n d \log^2 d + 2\sum_{i=1}^{n} p_i x_i \log x_i - dP_n\left(M_2^2(\log \mathbf{x}) - 2\log M_0(\mathbf{x})\right)}{2\left[P_n d(\log d - 1) + P_n \overline{x} - dP_n \log M_0(\mathbf{x})\right]} - 1\right), \end{aligned} \tag{3.19}$$

where $\log \mathbf{x} = (\log x_1, \log x_2, \ldots, \log x_n)$.

Theorem 3.8. *Let $t, s, u, v \in \mathbb{R}$ such that $t \leq u, s \leq v$, then the following inequality is valid:*

$$M_{t,s} \leq M_{u,v}. \tag{3.20}$$

Proof. For convex function ϕ it holds that ([7, page 2])

$$\frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1} \leq \frac{\phi(y_2) - \phi(y_1)}{y_2 - y_1} \tag{3.21}$$

with $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$. Since by Theorem 3.6, Γ_t is log-convex, we can set in (3.21): $\phi(x) = \log \Gamma_x$, $x_1 = t$, $x_2 = s$, $y_1 = u$, and $y_2 = v$, then we get

$$\frac{\log \Gamma_s - \log \Gamma_t}{s - t} \leq \frac{\log \Gamma_v - \log \Gamma_u}{v - u}. \tag{3.22}$$

From (3.22) we get (3.20) for $s \neq t$ and $u \neq v$.

For $s = t$ and $u = v$ we have limiting case. □

Similarly by (2.19) we can give the following definition of Cauchy type means.

Let $x_i \in I$ with $x_i \neq \overline{x}$, I is positive closed interval, and $P_n = \sum_{i=1}^{n} p_i$, $p_i > 0$ $(i = 1, 2, \ldots, n)$,

$$\widetilde{M}_{u,v} = \left(\frac{\widetilde{\Gamma}_u}{\widetilde{\Gamma}_v} \right)^{1/(u-v)} \tag{3.23}$$

for $-\infty < u \neq v < +\infty$ are means of x_i. Moreover we can extend these means to the other cases.

So by limit we have

$$\begin{aligned}
&\widetilde{M}_{u,u} \\
&= \exp\Bigg(\frac{P_n \overline{x}^u \log \overline{x} + (u-1) \sum_{i=1}^{n} p_i x_i^u \log x_i + P_n M_u^u(\mathbf{x}) - \overline{x}\left(u \sum_{i=1}^{n} p_i x_i^{u-1} \log x_i + P_n M_{u-1}^{u-1}(\mathbf{x})\right)}{P_n\left[\overline{x}^u + (u-1) M_u^u(\mathbf{x}) - \overline{x} u M_{u-1}^{u-1}(\mathrm{x})\right]} \\
&\qquad - \frac{2u-1}{u(u-1)} \Bigg), \quad u \neq 0, 1, \\
&\widetilde{M}_{0,0} = \exp\left(\frac{P_n \log^2 \overline{x} - P_n M_2^2(\log \mathbf{x}) + 2P_n \log M_0(\mathbf{x}) - 2\overline{x} \sum_{i=1}^{n} p_i x_i^{-1} \log x_i}{2P_n\left[\log \overline{x} - \log M_0(\mathbf{x}) + 1 - \overline{x} M_{-1}^{-1}(\mathbf{x})\right]} + 1 \right), \\
&\widetilde{M}_{1,1} = \exp\left(\frac{P_n \overline{x} \log^2 \overline{x} + 2 \sum_{i=1}^{n} p_i x_i \log x_i - \overline{x} P_n\left(M_2^2(\log \mathbf{x}) + 2 \log M_0(\mathbf{x})\right)}{2\left[P_n \overline{x}(\log \overline{x} - 1) + P_n \overline{x} - \overline{x} P_n \log M_0(\mathbf{x})\right]} - 1 \right),
\end{aligned} \tag{3.24}$$

where $\log \mathbf{x} = (\log x_1, \log x_2, \ldots, \log x_n)$.

Theorem 3.9. *Let $t, s, u, v \in \mathbb{R}$ such that $t \leq u, s \leq v$, then the following inequality is valid:*

$$\widetilde{M}_{t,s} \leq \widetilde{M}_{u,v}. \tag{3.25}$$

Proof. The proof is similar to the proof of Theorem 3.8. □

Let $M_t(\mathbf{x})$ be stated as above, define d_t as

$$d_t = \frac{\sum_{i=1}^n p_i x_i \varphi_t'(x_i)}{\sum_{i=1}^n p_i \varphi_t'(x_i)} = \begin{cases} \dfrac{M_t^t(\mathbf{x})}{M_{t-1}^{t-1}(\mathbf{x})}, & t \neq 0, 1, \\ M_{-1}(\mathbf{x}), & t = 0, \\ \dfrac{P_n \overline{x} + \sum_{i=1}^n p_i x_i \log x_i}{P_n\left(1 + \log M_0(\mathbf{x})\right)}, & t = 1. \end{cases} \tag{3.26}$$

The following improvement and reverse of Slater's inequality are valid.

Theorem 3.10. *Let $x_i,\ p_i,\ d_t \in \mathbb{R}^+$ $(i = 1, 2, \ldots, n)$, $P_n = \sum_{i=1}^n p_i$. Let F_t be defined by*

$$F_t = \varphi_t(d_t) - \frac{1}{P_n}\sum_{i=1}^n p_i \varphi_t(x_i). \tag{3.27}$$

Then

(i)

$$F_t \geq [H(s;t)]^{(t-r)/(s-r)}[H(r;t)]^{(s-t)/(s-r)}, \tag{3.28}$$

for $-\infty < r < s < t < \infty$ and $-\infty < t < r < s < \infty$.

(ii)

$$F_t \leq [H(s;t)]^{(t-r)/(s-r)}[H(r;t)]^{(s-t)/(s-r)}, \tag{3.29}$$

for $-\infty < r < t < s < \infty$.

where,

$$H(s;t) = \varphi_s(d_t) + \frac{1}{P_n}\sum_{i=1}^n p_i (x_i - d_t)\varphi_s'(x_i) - \frac{1}{P_n}\sum_{i=1}^n p_i \varphi_s(x_i). \tag{3.30}$$

Proof. (i) By setting $d = d_t$ in (3.7), Γ_t becomes F_t, and for $-\infty < r < s < t < \infty$, setting $d = d_t$ in (3.9), we get

$$\begin{aligned}&\left(\varphi_s(d_t) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d_t)\varphi_s'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi_s(x_i)\right)^{t-r}\\ &\quad\le \left(\varphi_r(d_t) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d_t)\varphi_r'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi_r(x_i)\right)^{t-s}(F_t)^{s-r},\end{aligned}\tag{3.31}$$

that is,

$$\begin{aligned}(F_t)^{s-r} \ge &\left(\varphi_s(d_t) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d_t)\varphi_s'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi_s(x_i)\right)^{t-r}\\ &\times\left(\varphi_r(d_t) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d_t)\varphi_r'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi_r(x_i)\right)^{s-t}.\end{aligned}\tag{3.32}$$

From (3.32) we get (3.28), and similarly for $-\infty < t < r < s < \infty$ (3.9) becomes

$$(\Gamma_r)^{s-t} \le (\Gamma_t)^{s-r}(\Gamma_s)^{r-t};\tag{3.33}$$

by the same process we can get (3.28).

(ii) For $-\infty < r < t < s < \infty$ (3.9) becomes

$$(\Gamma_s)^{t-r} \le (\Gamma_r)^{t-s}(\Gamma_t)^{s-r};\tag{3.34}$$

setting $d = d_t$ in (3.34), we get (3.29). □

Theorem 3.11. *Let $x_i, p_i, d_t \in \mathbb{R}^+$ $(i = 1, 2, \ldots, n)$, $P_n = \sum_{i=1}^{n} p_i$.*

Then for every $m \in \mathbb{N}$ and for every $s_k \in \mathbb{R}, k \in \{1, 2, 3, \ldots, m\}$, the matrices $[H((s_k + s_l)/2, s_1)]_{k,l=1}^{m}$, $[H((s_k + s_l)/2, (s_1 + s_2)/2)]_{k,l=1}^{m}$ are positive semi-definite matrices. Particularly

$$\det\left[H\left(\frac{s_k + s_l}{2}, s_1\right)\right]_{k,l=1}^{m} \ge 0,\tag{3.35}$$

$$\det\left[H\left(\frac{s_k + s_l}{2}, \frac{s_1 + s_2}{2}\right)\right]_{k,l=1}^{m} \ge 0,\tag{3.36}$$

where $H(s, t)$ is defined by (3.30)*.*

Proof. By setting $d = d_{s_1}$ and $d = d_{(s_1+s_2)/2}$ in Theorem 3.6(i), we get the required results. □

Remark 3.12. We note that $H(t, t) = F_t$. So by setting $m = 2$ in (3.35), we have special case of (3.28) for $t = s_1$, $s = s_2$, and $r = (s_1 + s_2)/2$ if $s_1 < s_2$ and for $t = s_1$, $r = s_2$, and $s = (s_1 + s_2)/2$ if $s_2 < s_1$. Similarly by setting $m = 2$ in (3.36), we have special case of (3.29) for $r = s_1$, $s = s_2$, $t = (s_1 + s_2)/2$ if $s_1 < s_2$ and for $r = s_2$, $s = s_1$, $t = (s_1 + s_2)/2$ if $s_2 < s_1$.

Let $M_t(\mathbf{x})$ be stated as above, define $\overline{d}_t$ as

$$\overline{d}_t = (\varphi_t')^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi_t'(x_i)\right) = M_{t-1}(\mathbf{x}), \quad t \in \mathbb{R}. \tag{3.37}$$

The following improvement and reverse of inequality (1.6) are also valid.

Theorem 3.13. *Let $x_i, p_i, \overline{d_t} \in \mathbb{R}^+$ for all $i = 1, 2, \ldots, n$, $P_n = \sum_{i=1}^{n} p_i$. Let G_t be defined by*

$$G_t = \varphi_t\left(\overline{d_t}\right) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\left(x_i - \overline{d_t}\right)\varphi_t'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi_t(x_i). \tag{3.38}$$

Then

(i)

$$G_t \geq [K(s;t)]^{(t-r)/(s-r)}[K(r;t)]^{(s-t)/(s-r)}, \tag{3.39}$$

for $-\infty < r < s < t < \infty$ and $-\infty < t < r < s < \infty$.

(ii)

$$G_t \leq [K(s;t)]^{(t-r)/(s-r)}[K(r;t)]^{(s-t)/(s-r)}, \tag{3.40}$$

for $-\infty < r < t < s < \infty$,

where

$$K(s;t) = \varphi_s\left(\overline{d}_t\right) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\left(x_i - \overline{d}_t\right)\varphi_s'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi_s(x_i). \tag{3.41}$$

Proof. (i) By setting $d = \overline{d}_t$ in (3.9), we get (3.39) for $-\infty < r < s < t < \infty$, and similarly we can get (3.39) for the case $-\infty < t < r < s < \infty$.

(ii) For $-\infty < r < t < s < \infty$ (3.9) becomes

$$(\Gamma_s)^{t-r} \leq (\Gamma_r)^{t-s}(\Gamma_t)^{s-r}; \tag{3.42}$$

setting $d = \overline{d}_t$ in (3.42), we get (3.40). □

Theorem 3.14. *Let* $x_i, p_i, \overline{d}_t \in \mathbb{R}^+$ $(i = 1, 2, \ldots, n)$, $P_n = \sum_{i=1}^{n} p_i$.

Then for every $m \in \mathbb{N}$ *and for every* $s_k \in \mathbb{R}$, $k \in \{1, 2, 3, \ldots, m\}$, *the matrices* $[K((s_k + s_l)/2, s_1)]_{k,l=1}^{m}$, $[K((s_k + s_l)/2, (s_1 + s_2)/2)]_{k,l=1}^{m}$ *are positive semi-definite matrices. Particularly*

$$\det\left[K\left(\frac{s_k + s_l}{2}, s_1\right)\right]_{k,l=1}^{m} \geq 0, \tag{3.43}$$

$$\det\left[K\left(\frac{s_k + s_l}{2}, \frac{s_1 + s_2}{2}\right)\right]_{k,l=1}^{m} \geq 0, \tag{3.44}$$

where $K(s, t)$ *is defined by* (3.41).

Proof. By setting $d = \overline{d}_{s_1}$ and $d = \overline{d}_{(s_1+s_2)/2}$ in Theorem 3.6(i), we get the required results. □

Remark 3.15. We note that $K(t, t) = G_t$. So by setting $m = 2$ in (3.43), we have special case of (3.39) for $t = s_1$, $s = s_2$, $r = (s_1 + s_2)/2$ if $s_1 < s_2$ and for $t = s_1$, $r = s_2$, and $s = (s_1 + s_2)/2$ if $s_2 < s_1$. Similarly by setting $m = 2$ in (3.44), we have special case of (3.40) for $r = s_1$, $s = s_2$, and $t = (s_1 + s_2)/2$ if $s_1 < s_2$ and for $r = s_2$, $s = s_1$, and $t = (s_1 + s_2)/2$ if $s_2 < s_1$.

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