

Research Article

Some Identities on the Generalized q -Bernoulli Numbers and Polynomials Associated with q -Volkenborn Integrals

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We give some interesting equation of p -adic q -integrals on \mathbb{Z}_p . From those p -adic q -integrals, we present a systemic study of some families of extended Carlitz type q -Bernoulli numbers and polynomials in p -adic number field.

1. Introduction

Let p be a fixed prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p , \mathbb{C} , and \mathbb{C}_p will, respectively, denote the ring of p -adic rational integer, the field of p -adic rational numbers, the complex number field, and the completion of algebraic closure of \mathbb{Q}_p . Let \mathbb{N} be the set of natural numbers and $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$.

Let v_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q -extension, q is considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we normally assume that $|q| < 1$, and if $q \in \mathbb{C}_p$, we normally assume that $|1 - q|_p < 1$. We use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}. \quad (1.1)$$

The q -factorial is defined as

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q, \quad (1.2)$$

and the Gaussian q -binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!} = \frac{[n]_q [n-1]_q \cdots [n-k+1]_q}{[k]_q!}, \quad (1.3)$$

(see [1]). Note that

$$\lim_{q \rightarrow 1} \binom{n}{k}_q = \binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!}. \quad (1.4)$$

From (1.3), we easily see that

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n+1-k} \binom{n}{k-1}_q + \binom{n}{k}_q, \quad (1.5)$$

(see [2, 3]). For a fixed positive integer f , $(f, p) = 1$, let

$$X = X_f = \lim_{\overline{N}} \left(\frac{\mathbb{Z}}{fp^N \mathbb{Z}} \right), \quad X_1 = \mathbb{Z}_p, \quad (1.6)$$

$$X^* = \bigcup_{\substack{0 < a < fp \\ (a,p)=1}} (a + fp\mathbb{Z}_p), \quad a + fp^N \mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{fp^N}\},$$

where $a \in \mathbb{Z}$ and $0 \leq a < fp^N$ (see [1–14]).

We say that f is a uniformly differential function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$ if the difference quotients

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y} \quad (1.7)$$

have a limit $l = f'(a)$ as $(x, y) \rightarrow (a, a)$. For $f \in UD(\mathbb{Z}_p)$, let us begin with the expression

$$\frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x = \sum_{0 \leq x < p^N} f(x) \mu_q(x + p^N \mathbb{Z}_p), \quad (1.8)$$

representing a q -analogue of the Riemann sums for f , (see [1–3, 11–18]). The integral of f

on \mathbb{Z}_p is defined as the limit ($N \rightarrow \infty$) of the sums (if exists). The p -adic q -integral (= q -Volkenborn integral) of $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_X f(x) d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{0 \leq x < p^N} f(x) q^x, \tag{1.9}$$

(see [12]). Carlitz's q -Bernoulli numbers $\beta_{k,q}$ can be defined recursively by $\beta_{0,q} = 1$ and by the rule that

$$q(q\beta^* + 1)^k - \beta_{k,q}^* = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{1.10}$$

with the usual convention of replacing $(\beta^*)^i$ by $\beta_{i,q}^*$, (see [1–13]). It is well known that

$$\begin{aligned} \beta_{n,q}^* &= \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \int_X [x]_q^n d\mu_q(x), \quad n \in \mathbb{Z}_+, \\ \beta_{n,q}^*(x) &= \int_{\mathbb{Z}_p} [y+x]_q^n d\mu_q(y) = \int_X [y+x]_q^n d\mu_q(y), \quad n \in \mathbb{Z}_+, \end{aligned} \tag{1.11}$$

(see [1]), where $\beta_{n,q}^*(x)$ are called the n th Carlitz's q -Bernoulli polynomials (see [1, 12, 13]).

Let χ be the Dirichlet's character with conductor $f \in \mathbb{N}$, then the generalized Carlitz's q -Bernoulli numbers attached to χ are defined as follows:

$$\beta_{n,\chi,q}^* = \int_X \chi(x) [x]_q^n d\mu_q(x), \tag{1.12}$$

(see [13]). Recently, many authors have studied in the different several areas related to q -theory (see [1–13]). In this paper, we present a systemic study of some families of multiple Carlitz's type q -Bernoulli numbers and polynomials by using the integral equations of p -adic q -integrals on \mathbb{Z}_p . First, we derive some interesting equations of p -adic q -integrals on \mathbb{Z}_p . From these equations, we give some interesting formulae for the higher-order Carlitz's type q -Bernoulli numbers and polynomials in the p -adic number field.

2. On the Generalized Higher-Order q -Bernoulli Numbers and Polynomials

In this section, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. We first consider the q -extension of Bernoulli polynomials as follows:

$$\sum_{n=0}^{\infty} \beta_{n,q}^*(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} q^{-y} e^{[x+y]_q t} d\mu_q(y) = -t \sum_{m=0}^{\infty} e^{[x+m]_q t} q^{x+m}. \tag{2.1}$$

From (2.1), we note that

$$\begin{aligned}
 \beta_{n,q}(x) &= \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-q^x)^l \frac{l}{[l]_q} \\
 &= \frac{1}{(1-q)^{n-1}} \sum_{l=0}^n \binom{n}{l} (-q^x)^l \left(\frac{l}{1-q^l} \right) \\
 &= \frac{n}{(1-q)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} q^{(l+1)x} \left(\frac{1}{1-q^{l+1}} \right) (-1)^{l+1} \\
 &= \frac{-n}{(1-q)^{n-1}} \sum_{m=0}^{\infty} q^{m+x} \sum_{l=0}^{n-1} \binom{n-1}{l} q^{l(x+m)} \\
 &= -n \sum_{m=0}^{\infty} q^{m+x} [x+m]_q^{n-1}.
 \end{aligned} \tag{2.2}$$

Note that

$$\lim_{q \rightarrow 1} \beta_{n,q}(x) = -n \sum_{m=0}^{\infty} (x+m)^{n-1} = B_n(x), \tag{2.3}$$

where $B_n(x)$ are called the n th ordinary Bernoulli polynomials. In the special case, $x = 0$, $\beta_{n,q}(0) = \beta_{n,q}$ are called the n th q -Bernoulli numbers.

By (2.2), we have the following lemma.

Lemma 2.1. For $n \geq 0$, one has

$$\begin{aligned}
 \beta_{n,q}(x) &= \int_{\mathbb{Z}_p} q^{-y} [x+y]_q^n d\mu_q(y) = -n \sum_{m=0}^{\infty} q^{m+x} [x+m]_q^{n-1} \\
 &= \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-q^x)^l \frac{l}{[l]_q}.
 \end{aligned} \tag{2.4}$$

Now, one considers the q -Bernoulli polynomials of order $r \in \mathbb{N}$ as follows:

$$\sum_{n=0}^{\infty} \beta_{n,q}^{(r)}(x) \frac{t^n}{n!} = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} q^{-(x_1+\cdots+x_r)} e^{[x+x_1+\cdots+x_r]_q} d\mu_q(x_1) \cdots d\mu_q(x_r). \tag{2.5}$$

By (2.5), one sees that

$$\begin{aligned} \beta_{n,q}^{(r)}(x) &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} q^{-(x_1+\cdots+x_r)} [x+x_1+\cdots+x_r]_q^n d\mu_q(x_1) \cdots d\mu_q(x_r) \\ &= \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{xl} \left(\frac{l}{[l]_q} \right)^r. \end{aligned} \tag{2.6}$$

In the special case, $x = 0$, the sequence $\beta_{n,q}^{(r)}(0) = \beta_{n,q}^{(r)}$ is referred to as the q -extension of Bernoulli numbers of order r . For $f \in \mathbb{N}$, one has

$$\begin{aligned} \beta_{n,q}^{(r)}(x) &= \underbrace{\int_X \cdots \int_X}_{r \text{ times}} q^{-(x_1+\cdots+x_r)} [x+x_1+\cdots+x_r]_q^n d\mu_q(x_1) \cdots d\mu_q(x_r) \\ &= \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \sum_{a_1, \dots, a_r=0}^{f-1} q^{l(x+a_1+\cdots+a_r)} \frac{l^r}{[lf]_q^r} \\ &= [f]_q^{n-r} \sum_{a_1, \dots, a_r=0}^{f-1} \beta_{n,q^f}^{(r)} \left(\frac{a_1 + \cdots + a_r + x}{f} \right). \end{aligned} \tag{2.7}$$

By (2.5) and (2.7), one obtains the following theorem.

Theorem 2.2. For $r \in \mathbb{Z}_+, f \in \mathbb{N}$, one has

$$\begin{aligned} \beta_{n,q}^{(r)}(x) &= \frac{1}{(1-q)^n} \sum_{l=0}^n \sum_{a_1, \dots, a_r=0}^{f-1} \binom{n}{l} (-1)^l q^{l(a_1+\cdots+a_r+x)} \frac{l^r}{[lf]_q^r} \\ &= [f]_q^{n-r} \sum_{a_1, \dots, a_r=0}^{f-1} \beta_{n,q^f}^{(r)} \left(\frac{a_1 + \cdots + a_r + x}{f} \right). \end{aligned} \tag{2.8}$$

Let χ be the primitive Dirichlet's character with conductor $f \in \mathbb{N}$, then the generalized q -Bernoulli polynomials attached to χ are defined by

$$\sum_{n=0}^{\infty} \beta_{n,\chi,q}(x) \frac{t^n}{n!} = \int_X \chi(y) q^{-y} e^{[x+y]_q t} d\mu_q(y). \tag{2.9}$$

From (2.9), one derives

$$\begin{aligned}
 \beta_{n,\chi,q}(x) &= \int_X \chi(y) q^{-y} [x+y]_q^n d\mu_q(y) \\
 &= \sum_{a=0}^{f-1} \chi(a) \lim_{N \rightarrow \infty} \frac{1}{[fp^N]_q} \sum_{y=0}^{fp^N-1} [a+x+fy]_q^n \\
 &= \frac{1}{(1-q)^n} \sum_{a=0}^{f-1} \chi(a) \sum_{l=0}^n \binom{n}{l} (-1)^l q^{l(x+a)} \frac{l}{[lf]_q} \\
 &= \sum_{a=0}^{f-1} \chi(a) \sum_{m=0}^{\infty} (-n[x+a+mf]_q^{n-1}) \\
 &= -n \sum_{m=0}^{\infty} \chi(m) [x+m]_q^{n-1}.
 \end{aligned} \tag{2.10}$$

By (2.9) and (2.10), one can give the generating function for the generalized q -Bernoulli polynomials attached to χ as follows:

$$F_{\chi,q}(x,t) = -t \sum_{m=0}^{\infty} \chi(m) e^{[x+m]_q t} = \sum_{n=0}^{\infty} \beta_{n,\chi,q}(x) \frac{t^n}{n!}. \tag{2.11}$$

From (1.3), (2.10), and (2.11), one notes that

$$\begin{aligned}
 \beta_{n,\chi,q}(x) &= \frac{1}{[f]_q} \sum_{a=0}^{f-1} \chi(a) \int_{\mathbb{Z}_p} q^{-fy} [a+x+fy]_q^n d\mu_{q^f}(y) \\
 &= [f]_q^{n-1} \sum_{a=0}^{f-1} \chi(a) \beta_{n,q^f} \left(\frac{a+x}{f} \right).
 \end{aligned} \tag{2.12}$$

In the special case, $x = 0$, the sequence $\beta_{n,\chi,q}(0) = \beta_{n,\chi,q}$ are called the n th generalized q -Bernoulli numbers attached to χ .

Let one consider the higher-order q -Bernoulli polynomials attached to χ as follows:

$$\underbrace{\int_X \cdots \int_X}_{r \text{ times}} \left(\prod_{i=1}^r \chi(x_i) \right) e^{[x+x_1+\cdots+x_r]_q t} q^{-(x_1+\cdots+x_r)} d\mu_q(x_1) \cdots d\mu_q(x_r) = \sum_{n=0}^{\infty} \beta_{n,\chi,q}^{(r)}(x) \frac{t^n}{n!}, \tag{2.13}$$

where $\beta_{n,\chi,q}^{(r)}(x)$ are called the n th generalized q -Bernoulli polynomials of order r attaches to χ .

By (2.13), one sees that

$$\begin{aligned} \beta_{n,\chi,q}^{(r)}(x) &= \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-q^x)^l \sum_{a_1, \dots, a_r=0}^{f-1} \left(\prod_{i=1}^r \chi(a_i) \right) q^{l \sum_{i=1}^r a_i} \frac{l^r}{[l]_q^r} \\ &= [f]_q^{n-r} \sum_{a_1, \dots, a_r=0}^{f-1} \left(\prod_{i=1}^r \chi(a_i) \right) \beta_{n,q^f}^{(r)} \left(\frac{x + a_1 + \dots + a_r}{f} \right). \end{aligned} \tag{2.14}$$

In the special case, $x = 0$, the sequence $\beta_{n,\chi,q}^{(r)}(0) = \beta_{n,\chi,q}^{(r)}$ are called the n th generalized q -Bernoulli numbers of order r attaches to χ .

By (2.13) and (2.14), one obtains the following theorem.

Theorem 2.3. Let χ be the primitive Dirichlet's character with conductor $f \in \mathbb{N}$. For $n \in \mathbb{Z}_+, r \in \mathbb{N}$, one has

$$\begin{aligned} \beta_{n,\chi,q}^{(r)}(x) &= \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-q^x)^l \sum_{a_1, \dots, a_r=0}^{f-1} \left(\prod_{i=1}^r \chi(a_i) \right) q^{l \sum_{i=1}^r a_i} \frac{l^r}{[l]_q^r} \\ &= [f]_q^{n-r} \sum_{a_1, \dots, a_r=0}^{f-1} \left(\prod_{i=1}^r \chi(a_i) \right) \beta_{n,q^f}^{(r)} \left(\frac{x + a_1 + \dots + a_r}{f} \right). \end{aligned} \tag{2.15}$$

For $h \in \mathbb{Z}$, and $r \in \mathbb{N}$, one introduces the extended higher-order q -Bernoulli polynomials as follows:

$$\beta_{n,q}^{(h,r)}(x) = \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{r \text{ times}} q^{\sum_{j=1}^r (h-j)x_j} [x + x_1 + \dots + x_r]_q^n d\mu_q(x_1) \dots d\mu_q(x_r). \tag{2.16}$$

From (2.16), one notes that

$$\beta_{n,q}^{(h,r)}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \frac{\binom{l+h-1}{r}_q}{\binom{l+h-1}{r}_q} \frac{r!}{[r]_q!}, \tag{2.17}$$

and

$$\beta_{n,q}^{(h,r)}(x) = [f]_q^{n-r} \sum_{a_1, \dots, a_r=0}^{f-1} q^{\sum_{j=1}^r (h-j)a_j} \beta_{n,q^f}^{(h,r)} \left(\frac{x + a_1 + \dots + a_r}{f} \right). \tag{2.18}$$

In the special case, $x = 0$, $\beta_{n,q}^{(h,r)}(0) = \beta_{n,q}^{(h,r)}$ are called the n th (h, q) -Bernoulli numbers of order r .

By (2.17), one obtains the following theorem.

Theorem 2.4. For $h \in \mathbb{Z}, r \in \mathbb{N}$, one has

$$\begin{aligned}\beta_{n,q}^{(h,r)}(x) &= \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-q^x)^l \frac{\binom{l+h-1}{r}_q}{\binom{l+h-1}{r}_q} \frac{r!}{[r]_q!}, \\ \beta_{n,q}^{(h,r)}(x) &= [f]_q^{n-r} \sum_{a_1, \dots, a_r=0}^{f-1} q^{\sum_{j=1}^r (h-j)a_j} \beta_{n,q^f}^{(h,r)} \left(\frac{x + a_1 + \dots + a_r}{f} \right).\end{aligned}\quad (2.19)$$

Let χ be the primitive Dirichlet's character with conductor $f \in \mathbb{N}$, then one considers the generalized (h, q) -Bernoulli polynomials attached to χ of order r as follows:

$$\beta_{n,\chi,q}^{(h,r)}(x) = \underbrace{\int_X \dots \int_X}_{r \text{ times}} q^{\sum_{j=1}^r (h-j-1)x_j} \left(\prod_{j=1}^r \chi(x_j) \right) [x + x_1 + \dots + x_r]_q^n d\mu_q(x_1) \dots d\mu_q(x_r). \quad (2.20)$$

By (2.20), one sees that

$$\beta_{n,\chi,q}^{(h,r)}(x) = [f]_q^{n-r} \sum_{a_1, \dots, a_r=0}^{f-1} q^{\sum_{j=1}^r (h-j)a_j} \left(\prod_{j=1}^r \chi(a_j) \right) \beta_{n,q^f}^{(h,r)} \left(\frac{x + a_1 + \dots + a_r}{f} \right). \quad (2.21)$$

In the special case, $x = 0$, $\beta_{n,\chi,q}^{(h,r)}(0) = \beta_{n,\chi,q}^{(h,r)}$ are called the n th generalized (h, q) -Bernoulli numbers attached to χ of order r .

From (2.20) and (2.21), one notes that

$$\beta_{n,\chi,q}^{(h,r)} = (q-1)\beta_{n+1,\chi,q}^{(h-1,r)} + \beta_{n,\chi,q}^{(h-1,r)}. \quad (2.22)$$

By (2.16), it is easy to show that

$$\begin{aligned}\beta_{n,\chi,q}^{(h,r)} &= \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{r \text{ times}} [x_1 + \dots + x_r]_q^n q^{\sum_{j=1}^r (h-j-1)x_j} d\mu_q(x_1) \dots d\mu_q(x_r) \\ &= \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} [x_1 + \dots + x_r]_q^n \left\{ [x_1 + \dots + x_r]_q (q-1) + 1 \right\} q^{\sum_{j=1}^r (h-j-2)x_j} d\mu_q(x_1) \dots d\mu_q(x_r).\end{aligned}\quad (2.23)$$

Thus, one has

$$\beta_{n,q}^{(h,r)} = (q-1)\beta_{n+1,q}^{(h-1,r)} + \beta_{n,q}^{(h-1,r)}. \quad (2.24)$$

From (2.16) and (2.23), one can also derive

$$\begin{aligned}
 & \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} q^{(n-2)x_1+(n-3)x_2+\cdots+(n-r-1)x_r} d\mu_q(x_1) \cdots d\mu_q(x_r) \\
 &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)} q^{n(x_1+\cdots+x_r)} q^{-x_1-2x_2-\cdots-rx_r} d\mu_q(x_1) \cdots d\mu_q(x_r) \\
 &= \sum_{l=0}^n \binom{n}{l} (q-1)^l \int_{\mathbb{Z}_p} [x_1 + \cdots + x_r]_q^l q^{-(x_1+\cdots+x_r)} q^{-x_1-2x_2-\cdots-rx_r} d\mu_q(x_1) \cdots d\mu_q(x_r) \\
 &= \sum_{l=0}^n \binom{n}{l} (q-1)^l \beta_{l,q}^{(0,r)}, \\
 & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{(n-2)x_1+(n-3)x_2+\cdots+(n-r-1)x_r} d\mu_q(x_1) \cdots d\mu_q(x_r) = \frac{\binom{n-1}{r}}{\binom{n-1}{r}_q} \frac{r!}{[r]_q!}.
 \end{aligned} \tag{2.25}$$

It is easy to see that

$$\begin{aligned}
 \sum_{j=0}^n \binom{n}{j} (q-1)^j \int_{\mathbb{Z}_p} [x]_q^j q^{(h-2)x} d\mu_q(x) &= \int_{\mathbb{Z}_p} \left((q-1)[x]_q + 1 \right)^n q^{(h-2)x} d\mu_q(x) \\
 &= \frac{n+h-1}{[n+h-1]_q}.
 \end{aligned} \tag{2.26}$$

By (2.23), (2.25), and (2.26), one obtains the following theorem.

Theorem 2.5. For $h \in \mathbb{Z}, r \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, one has

$$\begin{aligned}
 \beta_{n,q}^{(h,r)} &= (q-1)\beta_{n+1,q}^{(h-1,r)} + \beta_{n,q}^{(h-1,r)}, \\
 \sum_{l=0}^n \binom{n}{l} (q-1)^l \beta_{l,q}^{(0,r)} &= \frac{\binom{n-1}{r}}{\binom{n-1}{r}_q} \frac{r!}{[r]_q!}.
 \end{aligned} \tag{2.27}$$

Furthermore, one gets

$$\sum_{l=0}^n \binom{n}{l} (q-1)^l \beta_{l,q}^{(h,1)} = \frac{n+h-1}{[n+h-1]_q}. \tag{2.28}$$

Now, one considers the polynomials of $\beta_{n,q}^{(0,r)}(x)$ by

$$\begin{aligned}\beta_{n,q}^{(0,r)}(x) &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [x + x_1 + \cdots + x_r]_q^n q^{-2x_1 - 3x_2 - \cdots - (r-1)x_r} d\mu_q(x_1) \cdots d\mu_q(x_r) \\ &= \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \frac{\binom{l-1}{r}_q}{\binom{l-1}{r}_q} \frac{r!}{[r]_q!}.\end{aligned}\tag{2.29}$$

By (2.29), one obtains the following theorem.

Theorem 2.6. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has

$$(1-q)^n \beta_{n,q}^{(0,r)}(x) = \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \frac{\binom{l-1}{r}_q}{\binom{l-1}{r}_q} \frac{r!}{[r]_q!}.\tag{2.30}$$

By using multivariate p -adic q -integral on \mathbb{Z}_p , one sees that

$$\begin{aligned}q^{nx} \frac{\binom{n-1}{r}_q}{\binom{n-1}{r}_q} \frac{r!}{[r]_q!} &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} q^{nx + (n-2)x_1 + \cdots + (n-r-1)x_r} d\mu_q(x_1) \cdots d\mu_q(x_r) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left((q-1)[x + x_1 + \cdots + x_r]_q + 1 \right)^n q^{-2x_1 - \cdots - (r+1)x_r} d\mu_q(x_1) \cdots d\mu_q(x_r) \\ &= \sum_{l=0}^n \binom{n}{l} (q-1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_r]_q^l q^{-2x_1 - \cdots - (r+1)x_r} d\mu_q(x_1) \cdots d\mu_q(x_r) \\ &= \sum_{l=0}^n \binom{n}{l} (q-1)^l \beta_{l,q}^{(0,r)}(x).\end{aligned}\tag{2.31}$$

Therefore, one obtains the following corollary.

Corollary 2.7. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has

$$q^{nx} \frac{\binom{n-1}{r}_q}{\binom{n-1}{r}_q} \frac{r!}{[r]_q!} = \sum_{l=0}^n \binom{n}{l} (q-1)^l \beta_{l,q}^{(0,r)}(x).\tag{2.32}$$

It is easy to show that

$$\begin{aligned} & \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [x + x_1 + \cdots + x_r]_q^n q^{-2x_1 - \cdots - (r+1)x_r} d\mu_q(x_1) \cdots d\mu_q(x_r) \\ &= [f]_q^{n-r} \sum_{i_1, \dots, i_r=0}^{f-1} q^{-\sum_{l=1}^r i_l} \\ & \quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-f \sum_{l=1}^r (l+1)x_l} \left[\frac{x + \sum_{l=1}^r i_l}{f} + \sum_{l=1}^r x_l \right]_{q^f}^n d\mu_{q^f}(x_1) \cdots d\mu_{q^f}(x_r). \end{aligned} \tag{2.33}$$

From (2.33), one notes that

$$\beta_{n,q}^{(0,r)}(x) = [f]_q^{n-r} \sum_{i_1, \dots, i_r=0}^{f-1} q^{-i_1 - 2i_2 - \cdots - ri_r} \beta_{n,q^f}^{(0,r)}\left(\frac{x + i_1 + \cdots + i_r}{f}\right). \tag{2.34}$$

From the multivariate p -adic q -integral on \mathbb{Z}_p , one has

$$\begin{aligned} & \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [x + x_1 + \cdots + x_r]_q^n q^{-2x_1 - 3x_2 - \cdots - (r+1)x_r} d\mu_q(x_1) \cdots d\mu_q(x_r) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} ([x]_q + q^x [x_1 + \cdots + x_r]_q)^n q^{-2x_1 - 3x_2 - \cdots - (r+1)x_r} d\mu_q(x_1) \cdots d\mu_q(x_r) \\ &= \sum_{l=0}^n \binom{n}{l} [x]_q^{n-l} q^{lx} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_r]_q^l q^{-2x_1 - 3x_2 - \cdots - (r+1)x_r} d\mu_q(x_1) \cdots d\mu_q(x_r), \end{aligned} \tag{2.35}$$

$$\begin{aligned} & \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [x + y + x_1 + \cdots + x_r]_q^n q^{-2x_1 - 3x_2 - \cdots - (r+1)x_r} d\mu_q(x_1) \cdots d\mu_q(x_r) \\ &= \sum_{l=0}^n \binom{n}{l} [y]_q^{n-l} q^{ly} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_r]_q^l q^{-2x_1 - 3x_2 - \cdots - (r+1)x_r} d\mu_q(x_1) \cdots d\mu_q(x_r). \end{aligned} \tag{2.36}$$

By (2.35) and (2.36), one obtains the following corollary.

Corollary 2.8. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has

$$\begin{aligned}\beta_{n,q}^{(0,r)}(x) &= \sum_{l=0}^n \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q}^{(0,r)}, \\ \beta_{n,q}^{(0,r)}(x+y) &= \sum_{l=0}^n \binom{n}{l} [y]_q^{n-l} q^{ly} \beta_{l,q}^{(0,r)}(x).\end{aligned}\tag{2.37}$$

Now, one also considers the polynomial of $\beta_{n,q}^{(h,1)}(x)$. From the integral equation on \mathbb{Z}_p , one notes that

$$\begin{aligned}\beta_{n,q}^{(h,1)}(x) &= \int_{\mathbb{Z}_p} [x+x_1]_q^n q^{x_1(h-2)} d\mu_q(x_1) \\ &= \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \frac{l+h-1}{[l+h-1]_q}.\end{aligned}\tag{2.38}$$

By (2.38), one easily gets

$$\begin{aligned}\beta_{n,q}^{(h,1)}(x) &= \frac{1}{(1-q)^{n-1}} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{lx}}{1-q^{l+h-1}} + \frac{h-1}{(1-q)^{n-1}} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{lx}}{1-q^{l+h-1}} \\ &= \frac{-n}{(1-q)^{n-1}} \sum_{l=0}^{n-1} \frac{\binom{n-1}{l} (-1)^l q^x q^{lx}}{1-q^{l+h}} + \frac{h-1}{(1-q)^{n-1}} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{lx}}{1-q^{l+h-1}} \\ &= -n \sum_{m=0}^{\infty} q^{hm+x} [x+m]_q^{n-1} + (h-1)(1-q) \sum_{m=0}^{\infty} q^{(h-1)m} [x+m]_q^n.\end{aligned}\tag{2.39}$$

Thus, one obtains the following theorem.

Theorem 2.9. For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, one has

$$\beta_{n,q}^{(h,1)}(x) = -n \sum_{m=0}^{\infty} q^{hm+x} [x+m]_q^{n-1} + (h-1)(1-q) \sum_{m=0}^{\infty} q^{(h-1)m} [x+m]_q^n.\tag{2.40}$$

From the definition of p -adic q -integral on \mathbb{Z}_p , one notes that

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{(h-2)x_1} [x+x_1]_q^n d\mu_q(x_1) \\ = \frac{1}{[f]_q} \sum_{i=0}^{f-1} q^{(h-1)i} [i]_q^n \int_{\mathbb{Z}_p} \left[\frac{x+i}{f} + x_1 \right]_q^n q^{f(h-2)x_1} d\mu_{q^f}(x_1).\end{aligned}\tag{2.41}$$

Thus, one has

$$\beta_{n,q}^{(h,1)}(x) = \frac{1}{[f]_q} \sum_{i=0}^{f-1} q^{(h-1)i} [i]_q^n \beta_{n,q^f}^{(h,1)}\left(\frac{x+i}{f}\right). \quad (2.42)$$

By (2.38), one easily gets

$$\begin{aligned} & \int_{\mathbb{Z}_p} [x+x_1]_q^n q^{x_1(h-2)} d\mu_q(x_1) \\ &= q^{-x} \int_{\mathbb{Z}_p} [x+x_1]_q^n \{ [x+x_1]_q (q-1) + 1 \} q^{x_1(h-3)} d\mu_q(x_1). \end{aligned} \quad (2.43)$$

From (2.43), one has

$$\beta_{n,q}^{(h,1)}(x) = q^{-x} \left((q-1) \beta_{n+1,q}^{(h-1,1)}(x) + \beta_{n,q}^{(h-1,1)}(x) \right). \quad (2.44)$$

That is,

$$q^x \beta_{n,q}^{(h,1)}(x) = (q-1) \beta_{n+1,q}^{(h-1,1)}(x) + \beta_{n,q}^{(h-1,1)}(x). \quad (2.45)$$

By (2.38) and (2.43), one easily sees that

$$\begin{aligned} \int_{\mathbb{Z}_p} q^{(h-2)x_1} [x+x_1]_q^n d\mu_q(x_1) &= \int_{\mathbb{Z}_p} q^{(h-2)x_1} ([x]_q + q^x [x_1]_q)^n d\mu_q(x_1) \\ &= \sum_{l=0}^n \binom{n}{l} [x]_q^{n-l} q^{lx} \int_{\mathbb{Z}_p} q^{(h-2)x_1} [x_1]_q^l d\mu_q(x_1), \end{aligned} \quad (2.46)$$

and

$$\begin{aligned} & q^{h-1} \int_{\mathbb{Z}_p} q^{(h-2)x_1} [x_1+1+x]_q^n d\mu_q(x_1) - \int_{\mathbb{Z}_p} q^{(h-2)x_1} [x+x_1]_q^n d\mu_q(x_1) \\ &= q^x n [x]_q^{n-1} + h(q-1) [x]_q^n - (q-1) [x]_q^n. \end{aligned} \quad (2.47)$$

For $x=0$, this gives

$$q^{h-1} \int_{\mathbb{Z}_p} q^{(h-2)x_1} [x_1+1]_q^n d\mu_q(x_1) - \int_{\mathbb{Z}_p} q^{(h-2)x_1} [x_1]_q^n d\mu_q(x_1) = \begin{cases} 1, & \text{if } n=1, \\ 0, & \text{if } n>1, \end{cases} \quad (2.48)$$

and

$$\beta_{0,q}^{(h,1)} = \int_{\mathbb{Z}_p} q^{(h-2)x_1} d\mu_q(x_1) = \frac{h-1}{[h-1]_q}. \quad (2.49)$$

From (2.46) and (2.48), one can derive the recurrence relation for $\beta_{n,q}^{(h,1)}$ as follows:

$$q^{h-1}\beta_{n,q}^{(h,1)}(1) - \beta_{n,q}^{(h,1)} = \delta_{n,1}, \quad (2.50)$$

where $\delta_{n,1}$ is kronecker symbol.

By (2.46), (2.48), and (2.50), one obtains the following theorem.

Theorem 2.10. For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, one has

$$\begin{aligned} \beta_{n,q}^{(h,1)}(x) &= \sum_{l=0}^n \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q}^{(h,1)}, \\ q^{h-1}\beta_{n,q}^{(h,1)}(x+1) - \beta_{n,q}^{(h,1)} &= q^x n [x]_q^{n-1} + h(q-1)[x]_q^n - (q-1)[x]_q^n. \end{aligned} \quad (2.51)$$

Furthermore,

$$q^{h-2}(q-1)\beta_{n+1,q}^{(h-1,1)}(1) + q^{h-2}\beta_{n,q}^{(h-1,1)}(1) - \beta_{n,q}^{(h,1)} = \delta_{n,1}, \quad (2.52)$$

where $\delta_{n,1}$ is kronecker symbol.

From the definition of p -adic q -integral on \mathbb{Z}_p , one notes that

$$\begin{aligned} &\int_{\mathbb{Z}_p} q^{-(h-2)x_1} [1-x+x_1]_{q^{-1}}^n d\mu_{q^{-1}}(x_1) \\ &= (-1)^n q^{n+h-2} \int_{\mathbb{Z}_p} q^{(h-2)x_1} [x+x_1]_q^n d\mu_q(x_1). \end{aligned} \quad (2.53)$$

By (2.53), one sees that

$$\beta_{n,q^{-1}}^{(h,1)}(1-x) = (-1)^n q^{n+h-2} \beta_{n,q}^{(h,1)}(x). \quad (2.54)$$

Note that

$$B_n(1-x) = \lim_{q \rightarrow 1} \beta_{n,q^{-1}}^{(h,1)}(1-x) = \lim_{q \rightarrow 1} (-1)^n q^{n+h-2} \beta_{n,q}^{(h,1)}(x) = (-1)^n B_n(x), \quad (2.55)$$

where $B_n(x)$ are the n th ordinary Bernoulli polynomials.

In the special case, $x = 1$, one gets

$$\beta_{n,q^{-1}}^{(h,1)} = (-1)^n q^{n+h-2} \beta_{n,q}^{(h,1)}(1) = (-1)^n q^{n-1} \beta_{n,q}^{(h,1)}, \quad \text{if } n > 1. \quad (2.56)$$

It is not difficult to show that

$$\begin{aligned}
 & [f]_q^{n-1} \sum_{l=0}^{f-1} q^{l(h-1)} \int_{\mathbb{Z}_p} \left[x + \frac{l}{f} + x_1 \right]_q^n q^{f(h-2)x_1} d\mu_{q^f}(x_1) \\
 &= \int_{\mathbb{Z}_p} [fx + x_1]_q^n q^{(h-2)x_1} d\mu_q(x_1), \quad f \in \mathbb{N}.
 \end{aligned}
 \tag{2.57}$$

That is,

$$[f]_q^{n-1} \sum_{l=0}^{f-1} q^{l(h-1)} \beta_{n,q^f}^{(h,1)} \left(x + \frac{l}{f} \right) = \beta_{n,q}^{(h,1)}(fx).
 \tag{2.58}$$

Let one consider Barnes' type multiple q -Bernoulli polynomials. For $w_1, w_2, \dots, w_r \in \mathbb{Z}_p$, and $\delta_1, \delta_2, \dots, \delta_r \in \mathbb{Z}$, one defines Barnes' type multiple q -Bernoulli polynomials as follows:

$$\begin{aligned}
 & \beta_{n,q}^{(r)}(x \mid w_1, \dots, w_r : \delta_1, \dots, \delta_r) \\
 &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [w_1 x_1 + \cdots + w_r x_r + x]_q^n q^{\sum_{j=1}^r (\delta_j - 1)x_j} d\mu_q(x_1) \cdots d\mu_q(x_r).
 \end{aligned}
 \tag{2.59}$$

From (2.59), one can easily derive the following equation:

$$\begin{aligned}
 & \beta_{n,q}^{(r)}(x \mid w_1, \dots, w_r : \delta_1, \dots, \delta_r) \\
 &= \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \frac{(lw_1 + \delta_1)(lw_2 + \delta_2) \cdots (lw_r + \delta_r)}{[lw_1 + \delta_1]_q [lw_2 + \delta_2]_q \cdots [lw_r + \delta_r]_q}.
 \end{aligned}
 \tag{2.60}$$

Let $\delta_r = \delta_1 + r - 1$, then one has

$$\beta_{n,q}^{(r)} \left(x \mid \underbrace{w_1 \cdots w_1}_{r \text{ times}} : \delta_1, \delta_1 + 1, \dots, \delta_1 + r - 1 \right) = \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \frac{\binom{lw_1 + \delta_1 + r - 1}{r}_q}{\binom{lw_1 + \delta_1 + r - 1}{r}_q} \frac{r!}{[r]_q!}.
 \tag{2.61}$$

Therefore, one obtains the following theorem.

Theorem 2.11. For $w_1 \in \mathbb{Z}_p$, $r \in \mathbb{N}$, and $\delta_1 \in \mathbb{Z}$, one has

$$\beta_{n,q}^{(r)} \left(x \mid \underbrace{w_1 \cdots w_1}_{r \text{ times}} : \delta_1, \delta_1 + 1, \dots, \delta_1 + r - 1 \right) = \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \frac{\binom{l w_1 + \delta_1 + r - 1}{r}}{\binom{l w_1 + \delta_1 + r - 1}{r}_q} \frac{r!}{[r]_q!}. \quad (2.62)$$

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