Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 504982, 12 pages
doi:10.1155/2010/504982

Research Article

On Lyapunov-Type Inequalities for Two-Dimensional Nonlinear Partial Systems

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Received 27 February 2010; Accepted 23 June 2010

Academic Editor: Alberto Cabada

We establish a new Laypunov-type inequality for two nonlinear systems of partial differential equations and the discrete analogue is also established. As application, boundness of the two-dimensional Emden-Fowler-type equation is proved.

1. Introduction

In a celebrated paper of 1893, Liapunov [1] proved the following well-known inequality: if y is a nontrivial solution of

$$y'' + q(t)y = 0, \tag{1.1}$$

on an interval containing the points a and b $(a < b)$ such that $y(a) = y(b) = 0$, then

$$4 < (b-a)\int_a^b |q(s)|\,ds. \tag{1.2}$$

Since the appearance of Liapunov's fundamental paper [1], considerable attention has been given to various extensions and improvements of the Lyapunov-type inequality from different viewpoints [2–7]. In particular, the Lyapunov-type inequalities for the following nonlinear system of differential equations were given in [8]

$$x'(t) = \alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t),$$
$$u'(t) = -\beta_2(t)|x(t)|^{\beta-2}x(t) - \alpha_1(t)u(t). \tag{1.3}$$

In this paper, we obtain new Lyapunov-type inequalities for the two-dimensional nonlinear system and discrete nonlinear system, respectively.

2. The Lyapunov-Type Integral Inequality for the Two-Dimensional Nonlinear System

$$\frac{\partial^2 x(s,t)}{\partial s\partial t} = \alpha_1(s,t)x(s,t) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t),$$
$$\frac{\partial^2 u(s,t)}{\partial s\partial t} = -\beta_2(s,t)|x(s,t)|^{\beta-2}x(s,t) - \alpha_1(s,t)u(s,t). \tag{2.1}$$

We shall assume the existence of nontrivial solution $(x(s,t), u(s,t))$ of the system (2.1), and furthermore, (2.1) satisfies the following assumptions (i), (ii), and (iii):

(i) $\gamma > 1$, $\beta > 1$ are real constants;

(ii) $\beta_1(s,t)$, $\beta_2(s,t) : [s_0,\infty) \times [t_0,\infty) \subset \mathbb{R}^2 \to \mathbb{R}$ are continuous functions such that $\beta_1(s,t) > 0$ for $(s,t) \in [s_0,\infty) \times [t_0,\infty)$;

(iii) $\alpha_1(s,t) : [s_0,\infty) \times [t_0,\infty) \to \mathbb{R}$ is a continuous function.

Theorem 2.1. *Let the hypotheses* (*i*)–(*iii*) *hold. If the nonlinear system* (2.1) *has a real solution* $(x(s,t), u(s,t))$ *such that* $x(a,t) = x(b,t) = x(s,c) = x(s,d) = 0$ *for* $(s,t) \in [a,b] \times [c,d]$, *and* $(\partial u(s,t)/\partial s)(\partial x(s,t)/\partial t) + (\partial u(s,t)/\partial t)(\partial x(s,t)/\partial s)$ *and* $x(s,t)$ *is not identically zero on* $[a,b] \times [c,d]$, *where* $a,b,c,d \in \mathbb{R}$ *with* $a < b$, $c < d$, *then*

$$2 \le \int_a^b \int_c^d |\alpha_1(s,t)| dt\, ds + M^{\beta/\alpha-1} \left(\int_a^b \int_c^d \beta_1(s,t) dt\, ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_2^+(s,t) dt\, ds \right)^{1/\alpha}, \tag{2.2}$$

where $(1/\alpha) + (1/\gamma) = 1$, $M = \max_{\substack{a<s<b \\ c<t<d}} |x(s,t)|$, *and* $\beta_2^+(s,t) = \max_{\substack{a<s<b \\ c<t<d}} \{\beta_2(s,t), 0\}$ *is the nonnegative part of* $\beta_2(s,t)$.

Proof. Since $x(a,t) = x(b,t) = x(s,c) = x(s,d) = 0$ and $x(s,t)$ is not identically zero on $[a,b] \times [c,d]$, we can choose $(\tau,\sigma) \in (a,b) \times (c,d)$ such that $|x(\tau,\sigma)| = \max_{\substack{a<s<b \\ c<t<d}} |x(s,t)| > 0$.

Let $M = |x(\tau,\sigma)| > 0$. Integrating the first equation of system (2.1) over t from c to σ and over s from a to τ, respectively, we obtain

$$\int_a^\tau \int_c^\sigma \frac{\partial^2 x(s,t)}{\partial s \partial t} dt\, ds = \int_a^\tau \int_c^\sigma \left(\alpha_1(s,t)x(s,t) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t)\right) dt\, ds. \tag{2.3}$$

On the other hand, we have

$$\begin{aligned}
\int_a^\tau \int_c^\sigma \frac{\partial^2 x(s,t)}{\partial s \partial t} dt\, ds &= \int_a^\tau \int_c^\sigma \frac{\partial}{\partial t}\left(\frac{\partial x(s,t)}{\partial s}\right) dt\, ds \\
&= \int_a^\tau \left[\int_c^\sigma \frac{\partial x(s,t)}{\partial s}\bigg|_t dt\right] ds \\
&= \int_a^\tau \frac{\partial x(s,\sigma)}{\partial s} ds - \int_a^\tau \frac{\partial x(s,c)}{\partial s} ds \\
&= x(\tau,\sigma) - x(a,\sigma) - x(\tau,c) + x(a,c) \\
&= x(\tau,\sigma).
\end{aligned} \tag{2.4}$$

Hence,

$$x(\tau,\sigma) = \int_a^\tau \int_c^\sigma \left(\alpha_1(s,t)x(s,t) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t)\right) dt\, ds, \tag{2.5}$$

and similarly, we have

$$x(\tau,\sigma) = \int_\tau^b \int_\sigma^d \left(\alpha_1(s,t)x(s,t) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t)\right) dt\, ds. \tag{2.6}$$

Employing the triangle inequality gives

$$|x(\tau,\sigma)| \le \int_a^\tau \int_c^\sigma |\alpha_1(s,t)||x(s,t)| dt\, ds + \int_a^\tau \int_c^\sigma \beta_1(s,t)|u(s,t)|^{\gamma-1} dt\, ds, \tag{2.7}$$

$$|x(\tau,\sigma)| \le \int_\tau^b \int_\sigma^d |\alpha_1(s,t)||x(s,t)| dt\, ds + \int_\tau^b \int_\sigma^d \beta_1(s,t)|u(s,t)|^{\gamma-1} dt\, ds. \tag{2.8}$$

Summing (2.7) and (2.8), we obtain

$$2|x(\tau,\sigma)| \le \int_a^b \int_c^d |\alpha_1(s,t)||x(s,t)| dt\, ds + \int_a^b \int_c^d \beta_1(s,t)|u(s,t)|^{\gamma-1} dt\, ds. \tag{2.9}$$

By using Hölder inequality on the second integral of the right side of (2.9) with indices α and γ, we have

$$\begin{aligned}
&\int_a^b\int_c^d \beta_1(s,t)|u(s,t)|^{\gamma-1}dt\,ds \\
&\quad= \int_a^b\int_c^d \beta_1(s,t)^{1/\gamma}\beta_1(s,t)^{1/\alpha}|u(s,t)|^{\gamma-1}dt\,ds \\
&\quad\le \left(\int_a^b\int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma}\left(\int_a^b\int_c^d \beta_1(s,t)|u(s,t)|^{\alpha(\gamma-1)}dt\,ds\right)^{1/\alpha} \\
&\quad= \left(\int_a^b\int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma}\left(\int_a^b\int_c^d \beta_1(s,t)|u(s,t)|^{\gamma}dt\,ds\right)^{1/\alpha},
\end{aligned} \tag{2.10}$$

where $(1/\alpha)+(1/\gamma)=1$.

Therefore, we obtain from (2.9)

$$\begin{aligned}
2|x(\tau,\sigma)| \le{}& \int_a^b\int_c^d |\alpha_1(s,t)||x(s,t)|dt\,ds \\
&+ \left(\int_a^b\int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma}\left(\int_a^b\int_c^d \beta_1(s,t)|u(s,t)|^{\gamma}dt\,ds\right)^{1/\alpha}.
\end{aligned} \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}
\frac{\partial^2}{\partial s\partial t}(x(s,t)u(s,t)) &= \frac{\partial}{\partial t}\left(\frac{\partial x(s,t)}{\partial s}\cdot u(s,t) + x(s,t)\cdot\frac{\partial u(s,t)}{\partial s}\right) \\
&= \frac{\partial^2 x(s,t)}{\partial s\partial t}\cdot u(s,t) + \frac{\partial x(s,t)}{\partial s}\cdot\frac{\partial u(s,t)}{\partial t} \\
&\quad+ \frac{\partial x(s,t)}{\partial t}\cdot\frac{\partial u(s,t)}{\partial s} + x(s,t)\cdot\frac{\partial^2 u(s,t)}{\partial s\partial t}.
\end{aligned} \tag{2.12}$$

Multiplying the first equation of (2.1) by $u(s,t)$ and the second one by $x(s,t)$, adding the result, and noting $(\partial u(s,t)/\partial s)(\partial x(s,t)/\partial t)+(\partial u(s,t)/\partial t)(\partial x(s,t)/\partial s)=0$, we have

$$\frac{\partial^2}{\partial s\partial t}[x(s,t)u(s,t)] = \beta_1(s,t)|u(s,t)|^{\gamma} - \beta_2(s,t)|x(s,t)|^{\beta}. \tag{2.13}$$

Integrating the left side of (2.13) over t from c to d and over s from a to b, respectively, we get

$$\begin{aligned}
&\int_a^b \int_c^d \frac{\partial^2}{\partial s \partial t}[x(s,t)u(s,t)]dt\,ds \\
&\quad = \int_a^b \int_c^d \frac{\partial}{\partial t}\left[\frac{\partial(x(s,t)u(s,t))}{\partial s}\right]dt\,ds \\
&\quad = \int_a^b \left[\int_c^d \frac{\partial(x(s,t)u(s,t))}{\partial s}\Big|_t dt\right]ds \\
&\quad = \int_a^b \frac{\partial(x(s,d)u(s,d))}{\partial s}ds - \int_a^b \frac{\partial(x(s,c)u(s,c))}{\partial s}ds \\
&\quad = x(b,d)u(b,d) - x(a,d)u(a,d) - x(b,c)u(b,c) + x(a,c)u(a,c).
\end{aligned} \tag{2.14}$$

Now integrating both sides of (2.13) over t from c to d and over s from a to b, respectively, and noting $x(a,t) = x(b,t) = 0$, we get

$$\int_a^b \int_c^d \beta_1(s,t)|u(s,t)|^{\gamma} dt\,ds = \int_a^b \int_c^d \beta_2(s,t)|x(s,t)|^{\beta} dt\,ds. \tag{2.15}$$

Substituting equality (2.15) by (2.11), we have

$$\begin{aligned}
2|x(\tau,\sigma)| \le &\int_a^b \int_c^d |\alpha_1(s,t)||x(s,t)|dt\,ds \\
&+ \left(\int_a^b \int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_2(s,t)|x(s,t)|^{\beta} dt\,ds\right)^{1/\alpha}.
\end{aligned} \tag{2.16}$$

Noticing that $M = |x(\tau,\sigma)| = \max_{\substack{a<s<b \\ c<t<d}} |x(s,t)| > 0$ and $\beta_2^+(s,t) = \max_{\substack{a<s<b \\ c<t<d}} \{\beta_2(s,t), 0\}$, we obtain

$$2 \le \int_a^b \int_c^d |\alpha_1(s,t)|dt\,ds + M^{\beta/\alpha-1}\left(\int_a^b \int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_2^+(s,t)dt\,ds\right)^{1/\alpha}. \tag{2.17}$$

The proof is complete. $\square$

Remark 2.2. Let $x(s,t), u(s,t), \alpha_1(s,t)$, and $\beta_1(s,t)$ change to $x(t), u(t), \alpha_1(t)$, and $\beta_1(t)$ in (2.2), and with suitable changes, (2.2) changes to the following result:

$$2 \leq \int_a^b |\alpha_1(t)|dt + M^{\beta/\alpha - 1}\left(\int_a^b \beta_1(t)dt\right)^{1/\gamma}\left(\int_a^b \beta_2^+(t)dt\right)^{1/\alpha}. \tag{2.18}$$

This is just a new Lyapunov-type inequality which was given by Tiryaki et al. [8].

3. The Lyapunov-Type Discrete Inequality for the Two-Dimensional Nonlinear System

$$\begin{aligned}\Delta_1\Delta_2 x(s,t) &= \alpha_1(s,t)x(s+1,t+1) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t),\\ \Delta_1\Delta_2 u(s,t) &= -\beta_2(s,t)|x(s+1,t+1)|^{\beta-2}x(s+1,t+1) - \alpha_1(s,t)u(s,t),\end{aligned} \tag{3.1}$$

where $s, t \in \mathbb{Z}$, Δ_1 denotes the forward difference operator for s, that is, $\Delta_1 x(s,t) = x(s+1,t) - x(s,t)$, and Δ_2 denotes the forward difference operator for t, that is, $\Delta_2 x(s,t) = x(s,t+1) - x(s,t)$. We shall assume the existence of nontrivial solution $(x(s,t), u(s,t))$ of the system (3.1), and furthermore, (3.1) satisfies the following assumptions (i), (ii), and (iii):

(i) $\gamma > 1, \beta > 1$ are real constants;

(ii) $\beta_1(s,t), \beta_2(s,t)$ are real-valued functions such that $\beta_1(s,t) > 0$ for all $s, t \in \mathbb{Z}$;

(iii) $\alpha_1(s,t)$ is a real-valued function for all $s, t \in \mathbb{Z}$.

Theorem 3.1. *Let the hypotheses (i)–(iii) hold. Assume $n_1, m_1, n_2, m_2 \in \mathbb{Z}$ and $n_1 < m_1 - 2, n_2 < m_2 - 2$. If the nonlinear system (3.1) has a real solution $(x(s,t), u(s,t))$ such that $x(n_1,t) = x(m_1,t) = x(s,n_2) = x(s,m_2) = 0$ for all $(s,t) \in [n_1,m_1] \times [n_2,m_2]$, and $\Delta_1 x(s,t+1) \cdot \Delta_2 u(s,t) + \Delta_2 x(s+1,t) \cdot \Delta_1 u(s,t) = 0$ and $x(s,t)$ is not identically zero on $[n_1,m_1] \times [n_2,m_2]$, then*

$$2 \leq \sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}|\alpha_1(s,t)| + M^{\beta/\alpha-1}\left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^+(s,t)\right)^{1/\alpha}, \tag{3.2}$$

where $(1/\alpha) + (1/\gamma) = 1$, $M = |x(\tau,\sigma)| = \max_{\substack{n_1+1<s<m_1-1\\ n_2+1<t<m_2-1}}|x(s,t)|$, and $\beta_2^+(s,t) = \max_{\substack{n_1+1<s<m_1-1\\ n_2+1<t<m_2-1}}\{\beta_2(s,t), 0\}$.

Proof. Let $(x(s,t), u(s,t))$ be nontrivial real solution of system (3.1) such that $x(n_1,t) = x(m_1,t) = x(s,n_2) = x(s,m_2) = 0$ and $x(s,t)$ is not identically zero on $[n_1,m_1] \times [n_2,m_2]$.

Then multiplying the first equation of (3.1) by $u(s,t)$ and the second one by $x(s+1,t+1)$, adding the result, and noting $\Delta_1 x(s,t+1)\cdot\Delta_2 u(s,t)+\Delta_2 x(s+1,t)\cdot\Delta_1 u(s,t)=0$, and

$$\begin{aligned}
&\Delta_1\Delta_2[x(s,t)u(s,t)]\\
&\quad=\Delta_2((x(s+1,t)-x(s,t))u(s,t)+x(s+1,t)(u(s+1,t)-u(s,t)))\\
&\quad=\Delta_2((x(s+1,t)-x(s,t))u(s,t))+\Delta_2(x(s+1,t)(u(s+1,t)-u(s,t)))\\
&\quad=(x(s+1,t+1)-x(s,t+1)-(x(s+1,t)-x(s,t)))u(s,t)\\
&\qquad+(x(s+1,t+1)-x(s,t+1))(u(s,t+1)-u(s,t))\\
&\qquad+(x(s+1,t+1)-x(s+1,t))(u(s+1,t)-u(s,t))\\
&\qquad+x(s+1,t+1)(u(s+1,t+1)-u(s,t+1)-(u(s+1,t)-u(s,t)))\\
&\quad=(\Delta_1\Delta_2 x(s,t))u(s,t)+\Delta_1 x(s,t+1)\Delta_2 u(s,t)\\
&\qquad+\Delta_2 x(s+1,t)\Delta_1 u(s,t)+x(s+1,t+1)(\Delta_1\Delta_2 u(s,t)),
\end{aligned}\tag{3.3}$$

we have

$$\Delta_1\Delta_2[x(s,t)u(s,t)]=\beta_1(s,t)|u(s,t)|^{\gamma}-\beta_2(s,t)|x(s+1,t+1)|^{\beta}.\tag{3.4}$$

Summing the left side of (3.4) over t from n_2 to m_2-1 and over s from n_1 to m_1-1, respectively, we have

$$\begin{aligned}
&\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\Delta_1\Delta_2(x(s,t)u(s,t))\\
&\quad=\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}(x(s+1,t+1)u(s+1,t+1)-x(s+1,t)u(s+1,t)\\
&\qquad\qquad-(x(s,t+1)u(s,t+1)-x(s,t)u(s,t)))\\
&\quad=\sum_{s=n_1}^{m_1-1}(x(s+1,m_2)u(s+1,m_2)-x(s,m_2)u(s,m_2)\\
&\qquad\quad-(x(s+1,n_2)u(s+1,n_2)-x(s,n_2)u(s,n_2)))\\
&\quad=x(m_1,m_2)u(m_1,m_2)-x(n_1,m_2)u(n_1,m_2)-x(m_1,n_2)u(m_1,n_2)\\
&\qquad+x(n_1,n_2)u(n_1,n_2).
\end{aligned}\tag{3.5}$$

Summing both sides of (3.4) over t from n_2 to m_2-1 and over s from n_1 to m_1-1, respectively, and noting $x(n_1,t)=x(m_1,t)=0$, we obtain

$$\sum_{s=n_1}^{m_1-1}\sum_{t=\mathrm{m}_1}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma}=\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_2(s,t)|x(s+1,t+1)|^{\beta}.\tag{3.6}$$

Noticing that $x(m_1,t)=x(s,m_2)=0$ and $\beta_2^+(s,t)=\max_{\substack{n_1+1<s<m_1-1\\ n_2+1<t<m_2-1}}\{\beta_2(s,t),0\}$, we have

$$\begin{aligned}\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma}&=\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2(s,t)|x(s+1,t+1)|^{\beta}\\&\le\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^+(s,t)|x(s+1,t+1)|^{\beta}.\end{aligned}\tag{3.7}$$

Choose $(\tau,\sigma)\in[n_1+1,m_1-1]\times[n_2+1,m_2-1]$ such that $M=|x(\tau,\sigma)|=\max_{\substack{n_1+1<s<m_1-1\\ n_2+1<t<m_2-1}}|x(s,t)|$. Hence $M=|x(\tau,\sigma)|>0$. Summing the first equation of (3.1) over t from n_2 to $\sigma-1$ and over s from n_1 to $\tau-1$, respectively, we obtain

$$\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\Delta_1\Delta_2x(s,t)=\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\alpha_1(s,t)x(s+1,t+1)+\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t).\tag{3.8}$$

Considering the left side of (3.8) and noting $x(n_1,t)=x(s,n_2)=0$ for all $(s,t)\in[n_1,m_1]\times[n_2,m_2]$, we have

$$\begin{aligned}\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\Delta_1\Delta_2x(s,t)&=\sum_{s=n_1}^{\tau-1}\left(\sum_{t=n_2}^{\sigma-1}(x(s+1,t+1)-x(s+1,t)-(x(s,t+1)-x(s,t)))\right)\\&=\sum_{s=n_1}^{\tau-1}(x(s+1,\sigma)-x(s,\sigma)-(x(s+1,n_2)-x(s,n_2)))\\&=x(\tau,\sigma)-x(n_1,\sigma)-x(\tau,n_2)+x(n_1,n_2)\\&=x(\tau,\sigma).\end{aligned}\tag{3.9}$$

Hence,

$$x(\tau,\sigma)=\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\alpha_1(s,t)x(s+1,t+1)+\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t),\tag{3.10}$$

and similarly, we have

$$x(\tau,\sigma)=\sum_{s=\tau}^{m_1-2}\sum_{t=\sigma}^{m_2-2}\alpha_1(s,t)x(s+1,t+1)+\sum_{s=\tau}^{m_1-1}\sum_{t=\sigma}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t).\tag{3.11}$$

Employing the triangle inequality gives

$$|x(\tau,\sigma)| \leq \sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}|\alpha_1(s,t)||x(s+1,t+1)| + \sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\beta_1(s,t)|u(s,t)|^{\gamma-1}, \tag{3.12}$$

$$|x(\tau,\sigma)| \leq \sum_{s=\tau}^{m_1-2}\sum_{t=\sigma}^{m_2-2}|\alpha_1(s,t)||x(s+1,t+1)| + \sum_{s=\tau}^{m_1-1}\sum_{t=\sigma}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-1}. \tag{3.13}$$

Summing (3.12) and (3.13), we obtain

$$2|x(\tau,\sigma)| \leq \sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}|\alpha_1(s,t)||x(s+1,t+1)| + \sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-1}. \tag{3.14}$$

On the other hand, using Hölder inequality on the second sum of the right side of (3.14) with indices α and γ, we have

$$\begin{aligned}\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-1} &= \sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)^{1/\gamma}\beta_1(s,t)^{1/\alpha}|u(s,t)|^{\gamma-1} \\ &\leq \left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\alpha(\gamma-1)}\right)^{1/\alpha} \\ &= \left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma}\right)^{1/\alpha},\end{aligned} \tag{3.15}$$

where $(1/\alpha)+(1/\gamma)=1$. Therefore, from (3.7) and (3.10), we obtain

$$\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-1} \leq \left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^{+}(s,t)|x(s+1,t+1)|^{\beta}\right)^{1/\alpha}. \tag{3.16}$$

Substituting (3.16) to (3.14), we have

$$\begin{aligned}2|x(\tau,\sigma)| &\leq \sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}|\alpha_1(s,t)||x(s+1,t+1)| \\ &\quad+ \left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^{+}(s,t)|x(s+1,t+1)|^{\beta}\right)^{1/\alpha}.\end{aligned} \tag{3.17}$$

Noticing that $M = |x(\tau,\sigma)| = \max_{\substack{n_1+1<s<m_1-1\\ n_2+1<t<m_2-1}} |x(s,t)| > 0$, we get

$$2 \leq \sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}|\alpha_1(s,t)| + M^{\beta/\alpha-1}\left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^{+}(s,t)\right)^{1/\alpha}. \tag{3.18}$$

This completes the proof. □

Remark 3.2. Let $x(s,t), u(s,t), \alpha_1(s,t)$, and $\beta_1(s,t)$ change to $x(t), u(t), \alpha_1(t)$, and $\beta_1(t)$ in (3.2) and with suitable changes, (3.2) changes to the following result:

$$2 \leq \sum_{t=n}^{m-2}|\alpha_1(t)| + M^{\beta/\alpha-1}\left(\sum_{t=n}^{m-1}\beta_1(t)\right)^{1/\gamma}\left(\sum_{t=n}^{m-2}\beta_2^{+}(t)\right)^{1/\alpha}. \tag{3.19}$$

This is just a new Lyapunov-type inequality which was given by Ünal et al. [2].

4. An application

Two-dimensional Emden-Fowler-type equation

$$\frac{\partial}{\partial s\partial t}\left(r(s,t)\left|\frac{\partial x(x,t)}{\partial s\partial t}\right|^{\alpha-2}\frac{\partial x(x,t)}{\partial s\partial t}\right) + q(s,t)|x(s,t)|^{\beta-2}x(s,t) = 0, \tag{4.1}$$

where $\alpha > 1$ is a constant, $r(s,t)$ and $q(s,t)$ are real functions, and $r(s,t) > 0$ for all $(s,t) \in \mathbb{R}\times\mathbb{R}$.

Consider the following special case of system (2.1), which is an equivalent system for the two-dimensional Emden-Fowler-type equation (4.1)

$$\begin{aligned}\frac{\partial x^2(s,t)}{\partial s\partial t} &= \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t),\\ \frac{\partial u^2(s,t)}{\partial s\partial t} &= -\beta_2(s,t)|x(s,t)|^{\beta-2}x(s,t),\end{aligned} \tag{4.2}$$

where $\beta_1(s,t) = r(s,t)^{1-\gamma}$ and $\beta_2(s,t) = q(s,t)$.

Obviously Theorem 2.1 for the two-dimensional nonlinear system (2.1) with $\alpha_1(s,t) \equiv 0$ is satisfied for system (4.2). Therefore, we have

$$2 \leq M^{\beta/\alpha-1}\left(\int_a^b\int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma}\left(\int_a^b\int_c^d \beta_2^{+}(s,t)dt\,ds\right)^{1/\alpha}. \tag{4.3}$$

A nontrivial solution $(x(s,t), u(s,t))$ of system (4.2) defined on $[s_0, \infty) \times [t_0, \infty)$ is said to be *proper* if and only if

$$\sup\{|x(s,t)| + |u(s,t)| : a \leq s < \infty, c \leq t < \infty\} > 0, \tag{4.4}$$

for any $a \geq s_0, c \geq t_0$.A proper solution $(x(s,t), u(s,t))$ of system (4.2) is called *weakly oscillatory* if and only if at least one component has a sequence of zeros tending to $+\infty$.

Theorem 4.1. *If* $|x(\tau,\sigma)| = \max\{|x(s,t)| : a < s < b, c < t < d\}$*, where* $a > s_0$*,* $c > t_0$ *and* $s_0, t_0, a, b, c, d \in \mathbb{R}$*,* $u(\tau,t)$ *is bounded on* $[t_0,\infty)$ *and* $u(s,\sigma)$ *is bounded on* $[s_0,\infty)$*,*

$$\int^{\infty}\int^{\infty} \beta_1(s,t)dt\,ds < \infty, \qquad \int^{\infty}\int^{\infty} \left|\beta_2(s,t)\right|dt\,ds < \infty, \tag{4.5}$$

then every weakly oscillatory proper solution of (4.2) *is bounded on* $I = [s_0,\infty) \times [t_0,\infty)$*.*

Proof. Let $(x(s,t), u(s,t))$ be any nontrivial weakly oscillatory proper solution of nonlinear system (4.2) on $I = [s_0,\infty) \times [t_0,\infty)$ such that $x(s,t)$ has a sequence of zeros tending to $+\infty$. Suppose to the contrary that $\limsup |x(s,t)| = \infty$; then given any positive number M_0, we can find positive numbers S_0 and T_0 such that $|x(s,t)| > M_0$ for all $s > S_0, t > T_0$. Since $x(s,t)$ is an oscillatory solution, there exist $(a,b) \times (c,d) \in \mathbb{R} \times \mathbb{R}$ with $a > S_0, c > T_0$ such that $x(a,t) = x(b,t) = x(s,c) = x(s,d) = 0$ and $|x(s,t)| > 0$ on $(a,b) \times (c,d)$. Choose (τ,σ) in $(a,b) \times (c,d)$ such that $M = |x(\tau,\sigma)| = \max\{|x(s,t)| : a < s < b, c < t < d\} > M_0$; in view of (4.5), we can choose S_0 and T_0 large enough such that for every $a \geq S_0$, $c \geq T_0$,

$$\int_a^{\infty}\int_c^{\infty} \beta_1(s,t)dt\,ds < M^{-(\beta-\alpha)/(\alpha-1)}, \qquad \int_a^{\infty}\int_c^{\infty} \left|\beta_2(s,t)\right|dt\,ds < 1. \tag{4.6}$$

Taking αth power of both sides of (4.3) and combining (4.6), we obtain

$$\begin{aligned} 2^{\alpha} &\leq M^{\beta-\alpha}\left(\int_a^b\int_c^d \beta_1(s,t)dt\,ds\right)^{\alpha-1}\left(\int_a^b\int_c^d \beta_2^{+}(s,t)dt\,ds\right) \\ &\leq M^{\beta-\alpha}\left(\int_a^{\infty}\int_c^{\infty} \beta_1(s,t)dt\,ds\right)^{\alpha-1}\left(\int_a^{\infty}\int_c^{\infty} \left|\beta_2(s,t)\right|dt\,ds\right) \\ &< M^{\beta-\alpha}M^{-\beta+\alpha} = 1, \end{aligned} \tag{4.7}$$

where $\alpha > 1$ and $\beta_2^{+}(s,t) \leq |\beta_2(s,t)|$.

This contradiction shows that $|x(s,t)|$ is bounded on $I = [s_0,\infty) \times [t_0,\infty)$. Therefore, there exists a positive constant K such that $|x(s,t)| \leq K$ for all $(s,t) \in I$.

On the other hand, integrating the second equation of system (4.2) over t from σ to t and over s from σ to s, respectively, we obtain

$$u(s,t) - u(\tau,t) - u(s,\sigma) + u(\tau,\sigma) = \int_{\sigma}^{s}\int_{\tau}^{t} -\beta_2(s,t)|x(s,t)|^{\beta-2}x(s,t)dt\,ds. \tag{4.8}$$

Notice that $u(\tau,t)$ is bounded on $[t_0,\infty)$, $u(s,\sigma)$ is bounded on $[s_0,\infty)$, and in view of triangle inequality, we have

$$\begin{aligned} |u(s,t)| &\leq |u(\tau,t)+u(s,\sigma)-C| + \int_\sigma^s \int_\tau^t \left|\beta_2(s,t)\right| |x(s,t)|^{\beta-1} dt\, ds \\ &\leq |u(\tau,t)+u(s,\sigma)-C| + K^{\beta-1} \int_\sigma^\infty \int_\tau^\infty \left|\beta_2(s,t)\right| dt\, ds, \end{aligned} \tag{4.9}$$

where $C = u(\tau,\sigma)$ is a constant.

Equation (4.9) implies that $|u(s,t)|$ is bounded on $I = [s_0,\infty) \times [t_0,\infty)$ since $\int_\tau^\infty \int_\sigma^\infty |\beta_2(s,t)| dt\, ds < \infty$. It follows from

$$\limsup\{|x(s,t)| + |u(s,t)|\} \leq \limsup|x(s,t)| + \limsup|u(s,t)| \tag{4.10}$$

that $\limsup\{|x(s,t)| + |u(s,t)|\}$ is bounden on $I = [s_0,\infty) \times [t_0,\infty)$.

This completes the proof. □

Acknowledgments

This research is supported by National Natural Sciences Foundation of China (10971205). It is also partially supported by the Research Grants Council of the Hong Kong SAR, China (Project no. HKU7016/07P) and an HKU Seed Grant for Basic Research.

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