Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 482717, 8 pages
doi:10.1155/2010/482717

*Research Article*

# Note on $q$-Nasybullin's Lemma Associated with the Modified $p$-Adic $q$-Euler Measure

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Received 1 December 2009; Accepted 14 March 2010

Academic Editor: N. Govil



We derive the modified $p$-adic $q$-measures related to $q$-Nasybullin's type lemma.

## 1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of rational integers, the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The $p$-adic absolute value in $\mathbb{C}_p$ is normalized in such a way that $|p|_p = 1/p$ (see [1–17]). For $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$, let $\overline{f} = [f, p]$ be the least common multiple of $f$ and $p$. We set

$$\begin{aligned} \mathbb{Z}_{\overline{f}} &= \frac{\lim_{\overleftarrow{n}} \mathbb{Z}}{\overline{f} p^n \mathbb{Z}}, \quad \text{for } n \geq 0, \\ \mathbb{Z}_{\overline{f}}^* &= \bigcup_{\substack{0<a<\overline{f}p \\ (a,p)=1}} \left( a + \overline{f}p\, \mathbb{Z}_p \right), \\ a + \overline{f} p^n \mathbb{Z}_p &= \left\{ x \in \mathbb{Z}_{\overline{f}} \mid\ x \equiv a \left( \operatorname{mod} \overline{f} p^n \right) \right\}, \end{aligned} \tag{1.1}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < \overline{f} p^n$.

When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ (see [1–6, 18–23]). As the definition of $q$-number, we use the following notations:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q} \tag{1.2}$$

(see [1–23]).

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the $p$-adic $q$-invariant integral on $\mathbb{Z}_p$ is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \tag{1.3}$$

(see [2, 3]).

The $q$-Euler numbers, $\varepsilon_{n,q}$, can be determined inductively by

$$\varepsilon_{0,q} = 1, \quad q(q\varepsilon+1)^n + \varepsilon_{n,q} = \begin{cases} [2]_q & \text{if } n=0, \\ 0 & \text{if } n>0, \end{cases} \tag{1.4}$$

with the usual convention of replacing $\varepsilon^i$ by $\varepsilon_{i,q}$ (see [11]). The modified $q$-Euler numbers $E_{n,q}$ of $\varepsilon_{n,q}$ are defined in [2] as follows:

$$E_{0,q} = \frac{[2]_q}{2}, \quad (qE+1)^n + E_{n,q} = \begin{cases} [2]_q & \text{if } n=0, \\ 0 & \text{if } n>0, \end{cases} \tag{1.5}$$

with the usual convention of replacing $E^i$ by $E_{i,q}$. For any positive integer $N$,

$$\mu_q\left(a+\overline{f}p^N\mathbb{Z}_p\right) = \frac{(-q)^a}{\left[\overline{f}p^N\right]_{-q}} \tag{1.6}$$

is known as a measure on $\mathbb{Z}_{\overline{f}}$ (see [9]). In [2], the Witt's type formulas for $E_{n,q}$ are given by

$$E_{n,q} = \int_{\mathbb{Z}_p} q^{-x}[x]_q^n\,d\mu_q(x) = [2]_q \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^l}. \tag{1.7}$$

The modified $q$-Euler polynomials are also defined by

$$E_{n,q}(x) = \left([x]_q + q^x E\right)^n = \sum_{l=0}^{n} \binom{n}{l} E_{l,q} q^{lx} [x]_q^{n-l}, \tag{1.8}$$

with the usual convention of replacing $E^n$ by $E_{n,q}$ (see [2]). Thus, we note that

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} q^{-t}[x+t]_q^n d\mu_q(t) = [2]_q \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{lx}}{1+q^l}. \tag{1.9}$$

Recently Govil and Gupta [22] have introduced a new type of q-integrated Meyer-König-Zeller-Durrmeyer (q-MKZD) operators, obtained moments for these operators, and estimated the convergence of these integrated q-MKZD operators. In this paper, we consider the q-extension which is in a direction different than that of Govil and Gupta [22].

Let $K$ be a field over $\mathbb{Q}_p$. Then we call a function $\mu$ a $K$-measure on $\mathbb{Z}_{\overline{f}}^*$ if $\mu$ is finitely additive function defined on open-closed subsets in $\mathbb{Z}_{\overline{f}}^*$, whose values are in the field $K$. Any open-closed subset in $\mathbb{Z}_{\overline{f}}^*$ is a disjoint union of some finite intervals $I_{a,n} = a + p^n \overline{f}\, \mathbb{Z}_p$ in $\mathbb{Z}_{\overline{f}}^*$, where $a \in \mathbb{Z}$is prime to $\overline{f}$, and therefore a $K$-measure $\mu$ is determined by its values on all intervals in $\mathbb{Z}_{\overline{f}}^*$. Let $Q^{(f)}$ denote the set of all rational numbers, whose denominator is a divisor of $\overline{f}p^n$ for some $n \geq 0$. In Section 2, we derive the modified $p$-adic $q$-measures related to $q$-Nasybullin's type lemma.

## 2. The Modified $p$-Adic $q$-Measure

Let $T$ be a $K$-valued function defined on $Q^{(f)}$ with the following property.

There exist two constants $A, B \in K$ such that

$$\begin{gathered} \sum_{k=0}^{p-1} T\left(\left[\frac{x+k}{p}\right]_{q^p}\right)(-1)^k = AT\left([x]_q\right) + BT\left([px]_{q^{1/p}}\right), \\ T\left([x+1]_q\right) = T\left([x]_q\right), \end{gathered} \tag{2.1}$$

for any number $x \in Q^{(f)}$. Suppose that $\rho$ is a root of the equation $y^2 = Ay + Bp$. Then we define

$$\mu(I_{a,n}) = \rho^{-n}(-1)^a T\left(\left[\frac{a}{p^n \overline{f}}\right]_{q^{p^n \overline{f}}}\right) + B\rho^{-(n+1)}(-1)^a T\left(\left[\frac{a}{p^{n-1}\overline{f}}\right]_{q^{p^{n-1}\overline{f}}}\right), \tag{2.2}$$

for any interval $I_{a,n}$. From (2.2), we note that

$$\begin{aligned}
&\sum_{k=0}^{p-1}\mu\left(I_{a+p^n\overline{f}k,n+1}\right)\\
&\quad=\rho^{-(n+1)}\sum_{k=0}^{p-1}T\left(\left[\frac{a+p^n\overline{f}k}{p^{n+1}\overline{f}}\right]_{q^{p^{n+1}\overline{f}}}\right)(-1)^{a+k}+B\rho^{-(n+2)}\sum_{k=0}^{p-1}T\left(\left[\frac{a+p^n\overline{f}k}{p^n\overline{f}}\right]_{q^{p^n\overline{f}}}\right)(-1)^{a+k}\\
&\quad=\rho^{-(n+1)}(-1)^a\sum_{k=0}^{p-1}T\left(\left[\frac{k+a/p^n\overline{f}}{p}\right]_{(q^{p^n\overline{f}})^p}\right)(-1)^k+B\rho^{-(n+2)}(-1)^a\sum_{k=0}^{p-1}T\left(\left[\frac{a}{p^n\overline{f}}+k\right]_{q^{p^n\overline{f}}}\right)(-1)^k\\
&\quad=\rho^{-(n+1)}(-1)^aAT\left(\left[\frac{a}{p^n\overline{f}}\right]_{q^{p^n\overline{f}}}\right)+\rho^{-(n+1)}B(-1)^aT\left(\left[\frac{a}{p^{n-1}\overline{f}}\right]_{q^{p^{n-1}\overline{f}}}\right)\\
&\qquad+B\rho^{-(n+2)}(-1)^apT\left(\left[\frac{a}{p^n\overline{f}}\right]_{q^{p^n\overline{f}}}\right)\\
&\quad=\rho^{-(n+2)}(-1)^a(\rho A+Bp)T\left(\left[\frac{a}{p^n\overline{f}}\right]_{q^{p^n\overline{f}}}\right)+\rho^{-(n+1)}B(-1)^aT\left(\left[\frac{a}{p^{n-1}\overline{f}}\right]_{q^{p^{n-1}\overline{f}}}\right)\\
&\quad=\mu(I_{a,n}).
\end{aligned}\tag{2.3}$$

Thus, we have

$$\mu(I_{a,n})=\sum_{\substack{b\,(\mathrm{mod}\ p^{n+1}\overline{f})\\ b\equiv a\,(\mathrm{mod}\ p^n\overline{f})}}\mu(I_{b,n+1}).\tag{2.4}$$

Therefore we obtain the following theorem.

**Theorem 2.1.** *For $f\in\mathbb{N}$ with $f\equiv 1\,(\mathrm{mod}\,2)$ and $\overline{f}=[p,f]$, let $T$ be a $K$-valued function defined on $Q^{(f)}$ with the following properties.*

*There exist two constants $A,B\in K$ such that*

$$\begin{gathered}
\sum_{k=0}^{p-1}T\left(\left[\frac{x+k}{p}\right]_{q^p}\right)(-1)^k=AT\left([x]_q\right)+BT\left([p\ x]_{q^{1/p}}\right),\\
T\left([x+1]_q\right)=T\left([x]_q\right),
\end{gathered}\tag{2.5}$$

*for any $x \in Q^{(f)}$. Suppose that $\rho$ is a root of the equation $y^2 = Ay + Bp$. Then there exists a $K(\rho)$-measure $\mu$ on $\mathbb{Z}^*_{\overline{f}}$ such that*

$$\mu(I_{a,n}) = \rho^{-n}(-1)^a T\left(\left[\frac{a}{p^n\overline{f}}\right]_{q^{p^n\overline{f}}}\right) + B\rho^{-(n+1)}(-1)^a T\left(\left[\frac{a}{p^{n-1}\overline{f}}\right]_{q^{p^{n-1}\overline{f}}}\right), \tag{2.6}$$

*for any interval $I_{a,n}$.*

From (1.9), we note that

$$E_{n,q}(x) = [p]_q^n \frac{[2]_q}{[2]_{q^p}} \sum_{a=0}^{p-1} (-1)^a E_{n,q^p}\left(\frac{x+a}{p}\right). \tag{2.7}$$

Let $E_{m,q}(x)$ be the $m$th $q$-Euler polynomials and let $P_m([x]_q)$ be the $m$th $q$-Euler functions, that is, for $0 \le x < 1$,

$$P_m\left([x]_q\right) = E_{m,q}(x). \tag{2.8}$$

Note that $\lim_{q\to 1} P_m([x]_q) = P_m(x)$ is the Euler function. By (2.7), we see that

$$\frac{[2]_q}{[2]_{q^p}} [p]_q^m \sum_{a=0}^{p-1} (-1)^a P_m\left(\left[\frac{x+i}{p}\right]_{q^p}\right) = P_m\left([x]_q\right). \tag{2.9}$$

Thus, the $q$-Euler function $P_m([x]_q)$ satisfies the properties of Theorem 2.1 with constants

$$A = [p]_q^{-m} \frac{[2]_{q^p}}{[2]_q}, \qquad B = 0. \tag{2.10}$$

Then $\rho \neq 0$ is equal to $[p]_q^{-m}([2]_{q^p}/[2]_q)$, as $\rho^2 = A\rho + Bp$ reduces simply to $\rho^2 = [p]_q^{-m}([2]_{q^p}/[2]_q)\rho$. Therefore, we obtain the following theorem.

**Theorem 2.2.** *For $m \in \mathbb{Z}_+$, let the function $\mu_m = \mu_{m,q}$ be defined on $I_{a,n}$ as follows:*

$$\mu_m(I_{a,n}) = \left[\overline{f}p^n\right]_q^m \frac{[2]_q}{[2]_{q^{p^n\overline{f}}}} (-1)^a P_m\left(\left[\frac{a}{p^n\overline{f}}\right]_{q^{p^n\overline{f}}}\right). \tag{2.11}$$

*Then $\mu_m$ is a $\mathbb{Q}_p(q)$-measure on $\mathbb{Z}^*_{\overline{f}}$.*

For $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$ and $\overline{f} = [f, p\ ]$, let $\chi$ be a primitive Dirichlet character modulo $\overline{f}$. Then the generalized $q$-Euler numbers are defined as follows:

$$E_{n,\chi,q} = \left[\overline{f}\right]_q^n \frac{[2]_q}{[2]_{q^{\overline{f}}}} \sum_{a=0}^{\overline{f}-1} \chi(a)(-1)^a E_{n,q^{\overline{f}}}\left(\frac{a}{\overline{f}}\right). \tag{2.12}$$

From (2.12) and (2.7), we can easily derive the following Witt's formula:

$$\begin{aligned}
E_{n,\chi,q} &= \int_{\mathbb{Z}_{\overline{f}}} [x]_q^n q^{-x} \chi(x) d\mu_q(x) \\
&= [d]_q^n \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{\overline{f}-1} \chi(a)(-1)^a \int_{\mathbb{Z}_p} \left[\frac{a}{d} + x\right]_{q^{\overline{f}}} q^{-dx} d\mu_{q^d}(x) \\
&= \left[\overline{f}\right]_q^n \frac{[2]_q}{[2]_{q^{\overline{f}}}} \sum_{a=0}^{\overline{f}-1} \chi(a)(-1)^a \int_{\mathbb{Z}_p} \left[\frac{a}{\overline{f}} + x\right]_{q^{\overline{f}}} q^{-\overline{f}x} d\mu_{q^{\overline{f}}}(x) \\
&= \left[\overline{f}\right]_q^n \frac{[2]_q}{[2]_{q^{\overline{f}}}} \sum_{a=0}^{\overline{f}-1} \chi(a)(-1)^a E_{n,q^{\overline{f}}}\left(\frac{a}{\overline{f}}\right).
\end{aligned} \tag{2.13}$$

We can compute a $q$-analogue of the $p$-adic $q$-$l$-function by the following $p$-adic $q$-Mellin Mazur transform with respect to $\mu_m$.

Let

$$\begin{aligned}
L(\mu_m, \chi) &= \int_{\mathbb{Z}_{\overline{f}}^*} \chi(a) d\mu_m(a) \\
&= \lim_{\rho \to \infty} \sum_{\substack{a \pmod{p^\rho \overline{f}} \\ a \in \mathbb{Z},\ (a,p)=1}} \chi(a) \mu_m(I_{a,\rho}).
\end{aligned} \tag{2.14}$$

Since the character $\chi$ is constant on the interval $I_{a,0}$,

$$\begin{aligned}
L(\mu_m, \chi) &= \sum_{\substack{a \pmod{\overline{f}} \\ (a,p)=1}} \chi(a) \mu_m(I_{a,0}) \\
&= \sum_{\substack{a \pmod{\overline{f}} \\ (a,p)=1}} \chi(a) \left[\overline{f}\right]_q^m \frac{[2]_q}{[2]_{q^{\overline{f}}}} (-1)^a P_m\left(\left[\frac{a}{\overline{f}}\right]_{q^{\overline{f}}}\right) \\
&= E_{m,\chi,q} - \chi(p) \frac{[2]_q}{[2]_{q^p}} [p]_q^m E_{m,\chi,q^p},
\end{aligned} \tag{2.15}$$

where $E_{m,\chi,q}$ are the $m$th generalized $q$-Euler numbers attached to $\chi$. For $m \in \mathbb{Z}_+$, we have

$$\begin{aligned} L(\mu_m, \chi w^{-m}) &= E_{m,\chi w^{-m},q} - \chi w^{-m}(p) \frac{[2]_q}{[2]_{q^p}} [p]_q^m E_{m,\chi w^{-m},q^p} \\ &= l_{p,q}(-m, \chi). \end{aligned} \tag{2.16}$$

Assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$. Let $w$ be the Teichmüller character mod $p$. For $x \in \mathbb{Z}_f^*$, we set $\langle x \rangle_q = [x]_q / w(x)$. Note that $|\langle x \rangle_q - 1|_p < p^{-1/(p-1)}$ and $\langle x \rangle_q^s$ are defined by $\exp(s \log_p \langle x \rangle_q)$ for $|s|_p \leq 1$. For $s \in \mathbb{Z}_p$, we define

$$l_{p,q}(s, x) = \int_{\mathbb{Z}_f^*} \langle x \rangle_q^{-s} \chi(x) d\mu_q(x). \tag{2.17}$$

For (2.14), (2.16) and (2.17), we note that

$$l_{p,q}\left(-k, \chi w^k\right) = \int_{\mathbb{Z}_f^*} [x]_q^k \chi(x) d\mu_q(x) = \int_{\mathbb{Z}_f^*} \chi(x) d\mu_k(x). \tag{2.18}$$

Since $|\langle x \rangle_q - 1|_p < p^{-1/(p-1)}$ for $x \in \mathbb{Z}_f^*$, we have $\langle x \rangle_q^{p^n} \equiv 1 \pmod{p^n}$. Let $k \equiv k' (\operatorname{mod} p^n(p-1))$. Then we have

$$l_{p,q}\left(-k, \chi w^k\right) \equiv l_{p,q}\left(-k', \chi w^{k'}\right) \left(\operatorname{mod} p^n\right). \tag{2.19}$$

Therefore, we obtain the following theorem.

**Theorem 2.3.** *For $k \equiv k'(\operatorname{mod} p^n(p-1))$, we have*

$$L(\mu_k, \chi) \equiv L(\mu_{k'}, \chi) \left(\operatorname{mod} p^n\right). \tag{2.20}$$

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