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Research Article

On Complete Convergence for Weighted Sums of φ-Mixing Random Variables

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Some results on complete convergence for weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ are presented, where $\{X_n, n \geq 1\}$ is a sequence of φ-mixing random variables and $\{a_{ni}, n \geq 1, i \geq 1\}$ is an array of constants. They generalize the corresponding results for *i.i.d* sequence to the case of φ-mixing sequence.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Let n and m be positive integers. Write $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$. Given σ-algebras $\mathcal{B}, \mathcal{R}$ in $\mathcal{F}$, let

$$\varphi(\mathcal{B}, \mathcal{R}) = \sup_{A \in \mathcal{B}, B \in \mathcal{R}, P(A) > 0} |P(B \mid A) - P(B)|. \quad (1.1)$$

Define the φ-mixing coefficients by

$$\varphi(n) = \sup_{k \geq 1} \varphi\left(\mathcal{F}_1^k, \mathcal{F}_{k+n}^{\infty}\right), \quad n \geq 0. \quad (1.2)$$

Definition 1.1. A random variable sequence $\{X_n, n \geq 1\}$ is said to be a φ-mixing random variable sequence if $\varphi(n) \downarrow 0$ as $n \to \infty$.

φ-mixing random variables were introduced by Dobrushin [1] and many applications have been found. See, for example, Dobrushin [1], Utev [2], and Chen [3] for central limit theorem, Herrndorf [4] and Peligrad [5] for weak invariance principle, Sen [6, 7] for weak convergence of empirical processes, Shao [8] for almost sure invariance principles, Hu and Wang [9] for large deviations, and so forth. When these are compared with the corresponding results of independent random variable sequences, there still remains much to be desired.

Throughout the paper, let $I(A)$ be the indicator function of the set A. We assume that $\phi(x)$ is a positive increasing function on $(0,\infty)$ satisfying $\phi(x)\uparrow\infty$ as $x\to\infty$ and $\psi(x)$ is the inverse function of $\phi(x)$. Since $\phi(x)\uparrow\infty$, it follows that $\psi(x)\uparrow\infty$. For easy notation, we let $\phi(0)=0$ and $\psi(0)=0$. $a_n=O(b_n)$ denotes that there exists a positive constant C such that $|a_n/b_n|\le C$. C denotes a positive constant which may be different in various places.

Let $\{X,X_n,\ n\ge 1\}$ be a sequence of *i.i.d.* random variables and let $\{a_{ni},\ n\ge 1, i\ge 1\}$ be an array of constants. The almost sure limiting behavior of weighted sums $\sum_{i=1}^n a_{ni}X_i$ was studied by many authors; see, for example, Choi and Sung [10], Cuzick [11], Wu [12], and Sung [13, 14], and so forth.

The main purpose of this paper is to extend the complete convergence for weighted sums $\sum_{i=1}^n a_{ni}X_i$ of *i.i.d.* random variables to the case of φ-mixing random variables.

Definition 1.2. A sequence $\{X_n,\ n\ge 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C, such that

$$P(|X_n|>x)\le CP(|X|>x) \tag{1.3}$$

for all $x\ge 0$ and $n\ge 1$.

Definition 1.3. A double array $\{a_{ni},\ n\ge 1,\ i\ge 1\}$ of real numbers is said to be a Toeplitz array if $\lim_{n\to\infty}a_{ni}=0$ for each $i\ge 1$ and

$$\sum_{i=1}^{\infty}|a_{ni}|\le C \tag{1.4}$$

for all $n\ge 1$, where C is a positive constant.

Lemma 1.4. *Let $\{X_n,\ n\ge 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha>0$ and $b>0$, the following statement holds:*

$$E|X_n|^{\alpha}I(|X_n|\le b)\le C\left\{E|X|^{\alpha}I(|X|\le b)+b^{\alpha}P(|X|>b)\right\}, \tag{1.5}$$

where C is a positive constant.

Lemma 1.5 (cf. [15, Lemma 1.2.8]). *Let $\{X_n,\ n\ge 1\}$ be a sequence of φ-mixing random variables. Let $X\in L_p(\mathcal{F}_1^k)$, $Y\in L_q(\mathcal{F}_{k+n}^{\infty})$, $p\ge 1$, $q\ge 1$, and $1/p+1/q=1$. Then*

$$|EXY-EXEY|\le 2(\varphi(n))^{1/p}(E|X|^p)^{1/p}(E|Y|^q)^{1/q}. \tag{1.6}$$

Lemma 1.6 (cf. [8, Lemma 2.2]). *Let $\{X_n,\ n\ge 1\}$ be a φ-mixing sequence. Put $T_a(n)=\sum_{i=a+1}^{a+n}X_i$. Suppose that there exists an array $\{C_{a,n}\}$ of positive numbers such that*

$$ET_a^2(n)\le C_{a,n}\quad \textit{for every } a\ge 0,\ n\ge 1. \tag{1.7}$$

Then for every $q \geq 2$, there exists a constant C depending only on q and $\varphi(\cdot)$ such that

$$E\left(\max_{1\leq j\leq n}\left|T_a(j)\right|^q\right) \leq C\left[C_{a,n}^{q/2} + E\left(\max_{a+1\leq i\leq a+n}|X_i|^q\right)\right] \tag{1.8}$$

for every $a \geq 0$ and $n \geq 1$.

Lemma 1.7. *Let $\{X_n, n \geq 1\}$ be a sequence of φ-mixing random variables satisfying $\sum_{n=1}^{\infty}\varphi^{1/2}(n) < \infty$. $q \geq 2$. Assume that $EX_n = 0$ and $E|X_n|^q < \infty$ for each $n \geq 1$. Then there exists a constant C depending only on q and $\varphi(\cdot)$ such that*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=a+1}^{a+j}X_i\right|^q\right) \leq C\left[\sum_{i=a+1}^{a+n}E|X_i|^q + \left(\sum_{i=a+1}^{a+n}EX_i^2\right)^{q/2}\right] \tag{1.9}$$

for every $a \geq 0$ and $n \geq 1$. In particular, one has

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}X_i\right|^q\right) \leq C\left[\sum_{i=1}^{n}E|X_i|^q + \left(\sum_{i=1}^{n}EX_i^2\right)^{q/2}\right] \tag{1.10}$$

for every $n \geq 1$.

Proof. By Lemma 1.5, we can see that

$$\begin{aligned}
E\left(\sum_{i=a+1}^{a+n}X_i\right)^2 &\leq \sum_{i=a+1}^{a+n}EX_i^2 + 4\sum_{a+1\leq i<j\leq a+n}\varphi^{1/2}(j-i)\left(EX_i^2\right)^{1/2}\left(EX_j^2\right)^{1/2} \\
&\leq \sum_{i=a+1}^{a+n}EX_i^2 + 2\sum_{k=1}^{n-1}\sum_{i=a+1}^{a+n-k}\varphi^{1/2}(k)\left(EX_i^2 + EX_{k+i}^2\right) \\
&\leq \left(1 + 4\sum_{k=1}^{\infty}\varphi^{1/2}(k)\right)\sum_{i=a+1}^{a+n}EX_i^2 \doteq C_{a,n},
\end{aligned} \tag{1.11}$$

which implies (1.7). By Lemma 1.6, we can get the desired result (1.9) immediately. The proof is complete. □

Lemma 1.8. *Assume that the inverse function $\psi(x)$ of $\phi(x)$ satisfies*

$$\psi(n)\sum_{i=1}^{n}\frac{1}{\psi(i)} = O(n). \tag{1.12}$$

If $E[\phi(|X|)] < \infty$, then

$$\sum_{n=1}^{\infty}\frac{1}{\psi(n)}E|X|I(|X| > \psi(n)) < \infty. \tag{1.13}$$

Proof. The proof is similar to that of Lemma 1 by Sung [14]. So we omit it. □

2. Main Results and Their Proofs

Theorem 2.1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed φ-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$, $EX = 0$, $EX^2 < \infty$, and $E[\phi(|X|)] < \infty$. Assume that the inverse function $\psi(x)$ of $\phi(x)$ satisfies (1.12). Let $\{a_{ni}, n \geq 1, i \geq 1\}$ be an array of constants such that*

(i) $\max_{1\leq i\leq n}|a_{ni}| = O(1/\psi(n))$;

(ii) $\sum_{i=1}^{n} a_{ni}^2 = O(\log^{-1-\alpha} n)$ *for some* $\alpha > 0$.

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon\right) < \infty. \tag{2.1}$$

Proof. For each $n \geq 1$, denote

$$\begin{gathered} X_j^{(n)} = X_j I\left(\left|X_j\right| \leq \psi(n)\right), \quad T_j^{(n)} = \sum_{i=1}^{j}\left(a_{ni}X_i^{(n)} - Ea_{ni}X_i^{(n)}\right), \quad 1 \leq j \leq n, \\ A = \bigcap_{i=1}^{n}\left(X_i = X_i^{(n)}\right) = \bigcap_{i=1}^{n}(|X_i| \leq \psi(n)), \qquad B = \overline{A} = \bigcup_{i=1}^{n}\left(X_i \neq X_i^{(n)}\right) = \bigcup_{i=1}^{n}(|X_i| > \psi(n)), \\ E_n = \left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon\right). \end{gathered} \tag{2.2}$$

It is easy to check that

$$\begin{aligned} \sum_{i=1}^{j} a_{ni}X_i &= \sum_{i=1}^{j} a_{ni}X_i I\left(|X_i| \leq \psi(n)\right) + \sum_{i=1}^{j} a_{ni}X_i I\left(|X_i| > \psi(n)\right) \\ &= T_j^{(n)} + \sum_{i=1}^{j} Ea_{ni}X_i^{(n)} + \sum_{i=1}^{j} a_{ni}X_i I\left(|X_i| > \psi(n)\right), \\ E_n = E_nA + E_nB &= \left(\max_{1\leq j\leq n}\left|T_j^{(n)} + \sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\right| > \varepsilon\right) + E_nB \\ &\subset \left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \varepsilon - \max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\right|\right) + B. \end{aligned} \tag{2.3}$$

Therefore

$$\begin{aligned} P(E_n) &\le P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right| > \varepsilon - \max_{1\le j\le n}\left|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\right|\right) + P(B) \\ &\le P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right| > \varepsilon - \max_{1\le j\le n}\left|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\right|\right) + \sum_{i=1}^{n} P\left(|X_i| > \psi(n)\right). \end{aligned} \tag{2.4}$$

Firstly, we will show that

$$\max_{1\le j\le n}\left|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.5}$$

It follows from Lemma 1.8 and Kronecker's lemma that

$$\frac{1}{\psi(n)}\sum_{i=1}^{n} E|X|I\left(|X| > \psi(i)\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.6}$$

By $EX = 0$, condition (i), (2.6), and $\psi(n) \uparrow \infty$, we can see that

$$\begin{aligned} \max_{1\le j\le n}\left|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\right| &= \max_{1\le j\le n}\left|\sum_{i=1}^{j} Ea_{ni}X_i I\left(|X_i| > \psi(n)\right)\right| \\ &\le \sum_{i=1}^{n} E|a_{ni}X_i| I\left(|X_i| > \psi(n)\right) \\ &\le \sum_{i=1}^{n} |a_{ni}| E|X| I\left(|X| > \psi(n)\right) \\ &\le \frac{1}{\psi(n)}\sum_{i=1}^{n} E|X|I\left(|X| > \psi(i)\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{2.7}$$

which implies (2.5). By (2.4) and (2.5), we can see that, for sufficiently large n,

$$P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon\right) \le P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right) + \sum_{i=1}^{n} P\left(|X_i| > \psi(n)\right). \tag{2.8}$$

To prove (2.1), it suffices to show that

$$\begin{gathered} \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right) < \infty, \\ \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P\left(|X_i| > \psi(n)\right) < \infty. \end{gathered} \tag{2.9}$$

By Markov's inequality, Lemma 1.7, $EX^2 < \infty$, and condition (ii), we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right) &\le C\sum_{n=1}^{\infty} n^{-1} E\left(\max_{1\le j\le n}\left|T_j^{(n)}\right|^2\right) \\ &\le C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} E\left|a_{ni}X_i^{(n)}\right|^2 \\ &= C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} a_{ni}^2 EX^2 I(|X| \le \psi(n)) \\ &\le C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} a_{ni}^2 \\ &\le C\sum_{n=1}^{\infty} n^{-1}\log^{-1-\alpha} n < \infty.\end{aligned} \tag{2.10}$$

It follows from $E[\phi(|X|)] < \infty$ that

$$\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P(|X_i| > \psi(n)) = \sum_{n=1}^{\infty} P(|X| > \psi(n)) = \sum_{n=1}^{\infty} P(\phi(|X|) > n) \le CE\left[\phi(|X|)\right] < \infty. \tag{2.11}$$

We complete the proof of the theorem. □

Theorem 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of φ-mixing random variables satisfying $\sum_{n=1}^{\infty}\varphi^{1/2}(n) < \infty$ and let $\{a_{ni}, n \ge 1, i \ge 1\}$ be an array of real numbers. Let $\{b_n, n \ge 1\}$ be an increasing sequence of positive integers and let $\{c_n, n \ge 1\}$ be a sequence of positive real numbers. If for some $q \ge 2$, $0 < t < 2$, and for any $\varepsilon > 0$, the following conditions are satisfied:*

$$\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\left(|a_{ni}X_i| \ge \varepsilon b_n^{1/t}\right) < \infty, \tag{2.12}$$

$$\sum_{n=1}^{\infty} c_n b_n^{-q/t} \sum_{i=1}^{b_n} |a_{ni}|^q E|X_i|^q I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right) < \infty, \tag{2.13}$$

$$\sum_{n=1}^{\infty} c_n b_n^{-q/t}\left[\sum_{i=1}^{b_n} a_{ni}^2 EX_i^2 I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right)\right]^{q/2} < \infty, \tag{2.14}$$

then

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\le i\le b_n}\left|\sum_{j=1}^{i}\left[a_{nj}X_j - a_{nj}EX_j I\left(\left|a_{nj}X_j\right| < \varepsilon b_n^{1/t}\right)\right]\right| \ge \varepsilon b_n^{1/t}\right\} < \infty. \tag{2.15}$$

Proof. Note that if the series $\sum_{n=1}^{\infty} c_n$ is convergent, then (2.15) holds. Therefore, we will consider only such sequences $\{c_n, n \ge 1\}$ for which the series $\sum_{n=1}^{\infty} c_n$ is divergent.

Let

$$Y_i^{(n)} = a_{ni}X_i I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right), \quad S'_{ni} = \sum_{j=1}^{i} Y_j^{(n)}, \quad n \ge 1,\ i \ge 1,$$

$$A = \bigcap_{i=1}^{b_n}\left\{Y_i^{(n)} = a_{ni}X_i\right\}, \qquad B = \overline{A} = \bigcup_{i=1}^{b_n}\left\{Y_i^{(n)} \ne a_{ni}X_i\right\} = \bigcup_{i=1}^{b_n}\left(|a_{ni}X_i| \ge \varepsilon b_n^{1/t}\right), \tag{2.16}$$

$$E_n = \left\{\max_{1\le i\le b_n}\left|\sum_{j=1}^{i}\left[a_{nj}X_j - a_{nj}EX_j I\left(\left|a_{nj}X_j\right| < \varepsilon b_n^{1/t}\right)\right]\right| \ge \varepsilon b_n^{1/t}\right\}.$$

Therefore

$$\begin{aligned} P&\left\{\max_{1\le i\le b_n}\left|\sum_{j=1}^{i}\left[a_{nj}X_{nj} - a_{nj}EX_j I\left(\left|a_{nj}X_j\right| < \varepsilon b_n^{1/t}\right)\right]\right| \ge \varepsilon b_n^{1/t}\right\} \\ &= P(E_n) = P(E_nA) + P(E_nB) \le P(E_nA) + P(B) \\ &\le \sum_{i=1}^{b_n} P\left(|a_{ni}X_i| \ge \varepsilon b_n^{1/t}\right) + \varepsilon^{-q} b_n^{-q/t} E\left(\max_{1\le i\le b_n}\left|S'_{ni} - ES'_{ni}\right|\right)^q. \end{aligned} \tag{2.17}$$

Using the C_r inequality and Jensen's inequality, we can estimate $E|Y_i^{(n)} - EY_i^{(n)}|^q$ in the following way:

$$E\left|Y_i^{(n)} - EY_i^{(n)}\right|^q \le C|a_{ni}|^q E|X_i|^q I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right). \tag{2.18}$$

By (2.17), (2.18), and Lemma 1.7, we can get

$$\begin{aligned} P&\left\{\max_{1\le i\le b_n}\left|\sum_{j=1}^{i}\left[a_{nj}X_j - a_{nj}EX_j I\left(\left|a_{nj}X_j\right| < \varepsilon b_n^{1/t}\right)\right]\right| \ge \varepsilon b_n^{1/t}\right\} \\ &\le C\sum_{i=1}^{b_n} P\left(|a_{ni}X_i| \ge \varepsilon b_n^{1/t}\right) + Cb_n^{-q/t}\sum_{i=1}^{b_n}|a_{ni}|^q E|X_i|^q I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right) \\ &\quad + Cb_n^{-q/t}\left[\sum_{i=1}^{b_n} a_{ni}^2 EX_i^2 I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right)\right]^{q/2}. \end{aligned} \tag{2.19}$$

Therefore, we can conclude that (2.15) holds by (2.12), (2.13), (2.14), and (2.19). □

Theorem 2.3. *Let $1 \leq p \leq 2$ and let $\{X_n, n \geq 1\}$ be a sequence of φ-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$, $EX_n = 0$, and $E|X_n|^p < \infty$ for $n \geq 1$. Let $\{a_{ni}, n \geq 1, i \geq 1\}$ be an array of real numbers satisfying the following condition:*

$$\sum_{i=1}^{n} |a_{ni}|^p E|X_i|^p = O\left(n^{\delta}\right) \quad \text{as } n \longrightarrow \infty \tag{2.20}$$

for some $0 < \delta \leq 2/q$ and $q > 2$. Then for any $\varepsilon > 0$ and $\alpha p \geq 1$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq i\leq n}\left|\sum_{j=1}^{i} a_{nj}X_j\right| \geq \varepsilon n^{\alpha}\right) < \infty. \tag{2.21}$$

Proof. Take $c_n = n^{\alpha p-2}$, $b_n = n$, and $1/t = \alpha$ in Theorem 2.2. By (2.20) we have

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\left(|a_{ni}X_i| \geq \varepsilon b_n^{1/t}\right) &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} \frac{|a_{ni}|^p E|X_i|^p}{n^{\alpha p}} \leq C\sum_{n=1}^{\infty} n^{-2+\delta} < \infty, \\
\sum_{n=1}^{\infty} c_n b_n^{-q/t} \sum_{i=1}^{b_n} |a_{ni}|^q E|X_i|^q I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right) &\leq \sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^{n} |a_{ni}|^p E|X_i|^p \leq C\sum_{n=1}^{\infty} n^{-2+\delta} < \infty, \\
\sum_{n=1}^{\infty} c_n b_n^{-q/t} \left[\sum_{i=1}^{b_n} a_{ni}^2 EX_i^2 I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right)\right]^{q/2} &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha pq/2} \left(\sum_{i=1}^{n} |a_{ni}|^p E|X_i|^p\right)^{q/2} \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha pq/2+\delta q/2} \leq C\sum_{n=1}^{\infty} n^{\alpha p(1-q/2)-1} < \infty
\end{aligned} \tag{2.22}$$

following from $\delta q/2 \leq 1$. By the assumption $EX_n = 0$ for $n \geq 1$ and (2.20) we get

$$\begin{aligned}
\frac{1}{n^{\alpha}} \max_{1\leq i\leq n}\left|\sum_{j=1}^{i} a_{nj}EX_j I\left(\left|a_{nj}X_j\right| < \varepsilon n^{\alpha}\right)\right| &\leq \frac{1}{n^{\alpha}} \sum_{j=1}^{n} \left|a_{nj}EX_j I\left(\left|a_{nj}X_j\right| < \varepsilon n^{\alpha}\right)\right| \\
&= \frac{1}{n^{\alpha}} \sum_{j=1}^{n} \left|a_{nj}EX_j I\left(\left|a_{nj}X_j\right| \geq \varepsilon n^{\alpha}\right)\right| \\
&\leq \frac{1}{n^{\alpha p}} \sum_{j=1}^{n} \left|a_{nj}\right|^p E\left|X_j\right|^p \leq Cn^{\delta-\alpha p} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty
\end{aligned} \tag{2.23}$$

following from $\delta < 1$ and $\alpha p \geq 1$. We get the desired result by Theorem 2.2 immediately. The proof iscompleted. □

Theorem 2.4. *Let $1 \le p \le 2$ and let $\{X_n, n \ge 1\}$ be a sequence of φ-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$, $EX_n = 0$, and $E|X_n|^p < \infty$ for $n \ge 1$. Assume that the random variables are stochastically dominated by a random variable X such that $E|X|^p < \infty$ and let $\{a_{ni}, n \ge 1, i \ge 1\}$ be an array of real numbers satisfying the following condition:*

$$\sum_{i=1}^{n} |a_{ni}|^p = O\left(n^{\delta}\right) \quad \text{as } n \longrightarrow \infty \tag{2.24}$$

for some $0 < \delta \le 2/q$ and $q > 2$. Then for any $\varepsilon > 0$ and $\alpha p \ge 1$, (2.21) holds.

Proof. The proof is similar to that of Theorem 2.3. We only need to note that

$$\begin{aligned} E|X_n|^p &= \int_0^{\infty} t^p \, dP(|X_n| \le t) \\ &= -\int_0^{\infty} t^p \, dP(|X_n| > t) \\ &= -\lim_{t \to \infty} t^p P(|X_n| > t) + \int_0^{\infty} P(|X_n| > t) \, dt^p \\ &= 0 + p \int_0^{\infty} t^{p-1} P(|X_n| > t) \, dt \\ &\le Cp \int_0^{\infty} t^{p-1} P(|X| > t) \, dt \\ &= CE|X|^p < \infty \end{aligned} \tag{2.25}$$

for each $n \ge 1$. □

Theorem 2.5. *Let $\{X_n, n \ge 1\}$ be a sequence of φ-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ and let $\{a_{ni}, n \ge 1, i \ge 1\}$ be a Toeplitz array. Assume that the random variables are stochastically dominated by a random variable X. If for some $0 < t < 2$ and $\delta > 1/t$,*

$$\sup_{i \ge 1} |a_{ni}| = O\left(n^{1/t - \delta}\right), \qquad E|X|^{\beta} < \infty, \tag{2.26}$$

where $\beta = \max(2/\delta, 1 + 1/\delta)$, then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\max_{1 \le i \le n} \left| \sum_{j=1}^{i} a_{nj} X_j \right| \ge \varepsilon n^{1/t}\right) < \infty. \tag{2.27}$$

Proof. Take $c_n = 1$, $b_n = n$ for $n \ge 1$ and $q \ge \max(2, 1 + 1/\delta)$ in Theorem 2.2. Then we can see that (2.12) and (2.13) are satisfied. In fact, by (1.4) and (2.26) we have

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\left(|a_{ni}X_i| \ge \varepsilon b_n^{1/t}\right) &= \sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left(|a_{ni}X_i| \ge \varepsilon n^{1/t}\right) \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left(|a_{ni}X| \ge C n^{1/t}\right) \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left(|X| \ge C n^{\delta}\right) \\
&= C\sum_{n=1}^{\infty} n \sum_{k=n}^{\infty} P\left(Ck^{\delta} \le |X| < C(k+1)^{\delta}\right) \\
&\le C\sum_{k=1}^{\infty} k^2 P\left(Ck^{\delta} \le |X| < C(k+1)^{\delta}\right) \\
&\le CE|X|^{2/\delta} < \infty,
\end{aligned} \tag{2.28}$$

and by Lemma 1.4, (1.5), and (2.26) we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n b_n^{-q/t} \sum_{i=1}^{b_n} |a_{ni}|^q E|X_i|^q I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right) \\
&\quad = \sum_{n=1}^{\infty} n^{-q/t} \sum_{i=1}^{n} |a_{ni}|^q E|X_i|^q I\left(|a_{ni}X_i| < \varepsilon n^{1/t}\right) \\
&\quad \le C\sum_{n=1}^{\infty} n^{-q/t} \sum_{i=1}^{n} |a_{ni}|^q \left[E|X|^q I\left(|a_{ni}X| < \varepsilon n^{1/t}\right) + \frac{n^{q/t}}{|a_{ni}|^q} P\left(|a_{ni}X| \ge \varepsilon n^{1/t}\right)\right] \\
&\quad \le C\sum_{n=1}^{\infty} n^{-(1+1/\delta)/t} \sum_{i=1}^{n} |a_{ni}|^{1+1/\delta} E|X|^{1+1/\delta} + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left(|a_{ni}X| \ge \varepsilon n^{1/t}\right) \\
&\quad \le C\sum_{n=1}^{\infty} n^{-1/t-1} E|X|^{1+1/\delta} \sum_{i=1}^{n} |a_{ni}| + CE|X|^{2/\delta} \\
&\quad \le C\sum_{n=1}^{\infty} n^{-1/t-1} + CE|X|^{2/\delta} < \infty.
\end{aligned} \tag{2.29}$$

In order to prove that (2.14) holds, we should consider the following two cases.

In the case $\delta > 1$, by Lemma 1.4, (1.5), (2.26), and C_r inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n b_n^{-q/t} \left[\sum_{i=1}^{b_n} a_{ni}^2 E X_i^2 I\left(|a_{ni} X_i| < \varepsilon b_n^{1/t} \right) \right]^{q/2} \\
&\quad = \sum_{n=1}^{\infty} n^{-q/t} \left[\sum_{i=1}^{n} a_{ni}^2 E X_i^2 I\left(|a_{ni} X_i| < \varepsilon n^{1/t} \right) \right]^{q/2} \\
&\quad \le C \sum_{n=1}^{\infty} n^{-q/2t - q/2\delta t} \left(\sum_{i=1}^{n} |a_{ni}|^{1+1/\delta} E|X|^{1+1/\delta} \right)^{q/2} + C \sum_{n=1}^{\infty} \sum_{i=1}^{n} P\left(|a_{ni} X| \ge \varepsilon n^{1/t} \right) \\
&\quad \le C \sum_{n=1}^{\infty} n^{-q/2t - q/2\delta t} n^{(1/\delta)(1/t - \delta)(q/2)} \left(E|X|^{1+1/\delta} \right)^{q/2} \left(\sum_{i=1}^{n} |a_{ni}| \right)^{q/2} + C E|X|^{2/\delta} \\
&\quad \le C \sum_{n=1}^{\infty} n^{-q/2t - q/2} + C E|X|^{2/\delta} \\
&\quad = C \sum_{n=1}^{\infty} n^{-(q/2)(1+1/t)} + C E|X|^{2/\delta} < \infty.
\end{aligned} \tag{2.30}$$

In the case $0 < \delta \le 1$, we can get

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n b_n^{-q/t} \left[\sum_{i=1}^{b_n} a_{ni}^2 E X_i^2 I\left(|a_{ni} X_i| < \varepsilon b_n^{1/t} \right) \right]^{q/2} \\
&\quad = \sum_{n=1}^{\infty} n^{-q/t} \left[\sum_{i=1}^{n} a_{ni}^2 E X_i^2 I\left(|a_{ni} X_i| < \varepsilon n^{1/t} \right) \right]^{q/2} \\
&\quad \le C \sum_{n=1}^{\infty} n^{-q/t} n^{(1/t - \delta)(q/2)} \left(\sum_{i=1}^{n} |a_{ni}| E X^2 \right)^{q/2} + C \sum_{n=1}^{\infty} \sum_{i=1}^{n} P\left(|a_{ni} X| \ge \varepsilon n^{1/t} \right) \\
&\quad \le C \sum_{n=1}^{\infty} n^{-q/2t - q\delta/2} \left(E X^2 \right)^{q/2} \left(\sum_{i=1}^{n} |a_{ni}| \right)^{q/2} + C E|X|^{2/\delta} \\
&\quad \le C \sum_{n=1}^{\infty} n^{-(q/2)(\delta + 1/t)} + C E|X|^{2/\delta} < \infty.
\end{aligned} \tag{2.31}$$

To complete the proof of the theorem, we only need to prove

$$n^{-1/t} \max_{1 \le i \le n} \left| \sum_{j=1}^{i} a_{nj} E X_j I\left(|a_{nj} X_j| < \varepsilon n^{1/t} \right) \right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.32}$$

Indeed, by Lemma 1.4, it follows that

$$\begin{aligned} n^{-1/t}\max_{1\le i\le n}\left|\sum_{j=1}^{i}a_{nj}EX_jI\left(\left|a_{nj}X_j\right|<\varepsilon n^{1/t}\right)\right| &\le Cn^{-1/t}\sum_{j=1}^{n}\left|a_{nj}\right|E|X|+C\sum_{j=1}^{n}P\left(\left|a_{nj}X\right|\ge\varepsilon n^{1/t}\right) \\ &\le Cn^{-1/t}+C\sum_{j=1}^{n}P\left(\left|a_{nj}X\right|\ge\varepsilon n^{1/t}\right)\longrightarrow 0,\quad \text{as } n\longrightarrow\infty. \end{aligned} \tag{2.33}$$

Thus we get the desired result. □

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