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Research Article

Hybrid Projection Algorithms for Generalized Equilibrium Problems and Strictly Pseudocontractive Mappings

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The purpose of this paper is to consider the problem of finding a common element in the solution set of equilibrium problems and in the fixed point set of a strictly pseudocontractive mapping. Strong convergence of the purposed hybrid projection algorithm is obtained in Hilbert spaces.

1. Introduction and Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and $S : C \to C$ a nonlinear mapping. In this paper, we use $F(S)$ to denote the fixed point set of S. Recall that the mapping S is said to be nonexpansive if

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

S is said to be k-strictly pseudocontractive if there exists a constant $k \in [0,1)$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + k\|(x - Sx) - (y - Sy)\|^2, \quad \forall x, y \in C. \tag{1.2}$$

S is said to be pseudocontractive if

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \|(x - Sx) - (y - Sy)\|^2, \quad \forall x, y \in C. \tag{1.3}$$

The class of strictly pseudocontractive mappings was introduced by Browder and Petryshyn [1] in 1967. It is easy to see that the class of strictly pseudocontractive mappings falls into the class of nonexpansive mappings and the class of pseudocontractions.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C. \tag{1.4}$$

A is said to be inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.5}$$

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers and $A : C \to H$ an inverse-strongly monotone mapping. In this paper, we consider the following generalized equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

In this paper, the set of such an $x \in C$ is denoted by $\mathrm{EP}(F, A)$, that is,

$$\mathrm{EP}(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C\}. \tag{1.7}$$

To study the generalized equilibrium problems (1.6), we may assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y); \tag{1.8}$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semicontinuous.

Next, we give two special cases of the problem (1.6).

(I) If $A \equiv 0$, then the generalized equilibrium problem (1.6) is reduced to the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{1.9}$$

In this paper, the set of such an $x \in C$ is denoted by $\mathrm{EP}(F)$, that is,

$$\mathrm{EP}(F) = \{x \in C : F(x, y) \geq 0, \ \forall y \in C\}. \tag{1.10}$$

(II) If $F \equiv 0$, then the problem (1.6) is reduced to the following classical variational inequality. Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.11}$$

It is known that $x \in C$ is a solution to (1.11) if and only if x is a fixed point of the mapping $P_C(I - \rho A)$, where $\rho > 0$ is a constant and I is the identity mapping.

Recently, many authors studied the problems (1.6) and (1.9) based on iterative methods; see, for example, [2–18].

In 2007, Tada and Takahashi [17] considered the problem (1.9) and proved the following result.

Theorem TT. *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$–$(A4)$ and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$ and let*

$$\begin{aligned}
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\
w_n &= (1 - \alpha_n)x_n + \alpha_n S u_n, \\
C_n &= \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\
D_n &= \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap D_n} x,
\end{aligned} \tag{1.12}$$

for every $n \geq 1$, where $\{\alpha_n\} \subset [a, 1]$ for some $a \in (0, 1)$ and $\{r_n\} \subset [0, \infty)$ satisfies $\liminf_{n \to \infty} r_n > 0$. *Then, $\{x_n\}$ converges strongly to $P_{F(S) \cap EP(F)} x$.*

In this paper, we consider the generalized equilibrium problem (1.6) and a strictly pseudocontractive mapping based on the shrinking projection algorithm which was first introduced by Takahashi et al. [18]. A strong convergence of common elements of the fixed point sets of the strictly pseudocontractive mapping and of the solution sets of the generalized equilibrium problem is established in the framework of Hilbert spaces. The results presented in this paper improve and extend the corresponding results announced by Tada and Takahashi [17].

In order to prove our main results, we also need the following definitions and lemmas.

Lemma 1.1 (see [19]). *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a k-strict pseudocontraction. Then T is $(1 + k)/(1 - k)$-Lipschitz and $I - T$ is demiclosed, this is, if $\{x_n\}$ is a sequence in C with $x_n \rightharpoonup x$ and $x_n - T x_n \to 0$, then $x \in F(T)$.*

The following lemma can be found in [2, 3].

Lemma 1.2. *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{1.13}$$

Further, define

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\} \tag{1.14}$$

for all $r > 0$ and $x \in H$. Then, the following hold:

(a) *T_r is single-valued;*

(b) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| T_r x - T_r y \right\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{1.15}$$

(c) *$F(T_r) = EP(F)$;*

(d) *$EP(F)$ is closed and convex.*

Lemma 1.3 (see [1]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ a k-strict pseudocontraction with a fixed point. Define $S : C \to C$ by $S_a x = ax + (1 - a)Sx$ for each $x \in C$. If $a \in [k, 1)$, then S_a is nonexpansive with $F(S_a) = F(S)$.*

2. Main Results

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping, $B : C \to H$ a β-inverse-strongly monotone mapping, and $S : C \to C$*

a k-strict pseudocontraction. Let $\{r_n\}$ and $\{s_n\}$ be two positive real sequences. Assume that $\mathcal{F} := EP(F_1, A)\cap FP(F_2, B)\cap F(S)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner:

$$
\begin{gathered}
x_1 \in C,\\
C_1 = C,\\
F_1(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C,\\
F_2(v_n, v) + \langle Bx_n, v - v_n\rangle + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in C,\\
z_n = \gamma_n u_n + (1-\gamma_n)v_n,\\
y_n = \alpha_n x_n + (1-\alpha_n)\left(\beta_n z_n + (1-\beta_n)Sz_n\right),\\
C_{n+1} = \left\{w \in C_n : \left\|y_n - w\right\| \leq \|x_n - w\|\right\},\\
x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 1,
\end{gathered}
\tag{Υ}
$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{r_n\}$, and $\{s_n\}$ satisfy the following restrictions:

(a) $0 \leq \alpha_n \leq a < 1$;

(b) $0 \leq k \leq \beta_n < b < 1$;

(c) $0 \leq c \leq \gamma_n \leq d < 1$;

(d) $0 < e \leq r_n \leq f < 2\alpha$ and $0 < e' \leq s_n \leq f' < 2\beta$.

Then the sequence $\{x_n\}$ generated in (Υ) converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\mathcal{F}}x_1$.

Proof. Note that u_n can be rewritten as

$$u_n = T_{r_n}(x_n - r_n Ax_n), \quad \forall n \geq 1 \tag{2.1}$$

and v_n can be rewritten as

$$v_n = T_{s_n}(x_n - s_n Bx_n), \quad \forall n \geq 1. \tag{2.2}$$

Fix $p \in \mathcal{F}$. It follows that

$$p = Sp = T_{r_n}(p - r_n Ap) = T_{s_n}(p - s_n Bp), \quad \forall n \geq 1. \tag{2.3}$$

Note that $I - r_nA$ is nonexpansive for each $n \geq 1$. Indeed, for any $x, y \in C$, we see from the restriction (d) that

$$\begin{aligned}\|(I - r_nA)x - (I - r_nA)y\|^2 &= \|(x - y) - r_n(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2.\end{aligned} \tag{2.4}$$

This shows that $I - r_nA$ is nonexpansive for each $n \geq 1$. In a similar way, we can obtain that $I - s_nB$ is nonexpansive for each $n \geq 1$. It follows that

$$\|u_n - p\| \leq \|x_n - p\|, \qquad \|u_n - p\| \leq \|x_n - p\|. \tag{2.5}$$

This implies that

$$\|z_n - p\| \leq \gamma_n\|u_n - p\| + (1 - \gamma_n)\|v_n - p\| \leq \|x_n - p\|. \tag{2.6}$$

Now, we are in a position to show that C_n is closed and convex for each $n \geq 1$. From the assumption, we see that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for some $m \geq 1$. We show that C_{m+1} is closed and convex for the same m. Indeed, for any $w \in C_m$, we see that

$$\|y_m - w\| \leq \|x_m - w\| \tag{2.7}$$

is equivalent to

$$\|y_m\|^2 - \|x_m\|^2 - 2\langle w, y_m - x_m\rangle \geq 0. \tag{2.8}$$

Thus C_{m+1} is closed and convex. This shows that C_n is closed and convex for each $n \geq 1$.

Next, we show that $\mathcal{F} \subset C_n$ for each $n \geq 1$. From the assumption, we see that $\mathcal{F} \subset C = C_1$. Suppose that $\mathcal{F} \subset C_m$ for some $m \geq 1$. Putting

$$S_n = \beta_nI + (1 - \beta_n)S, \quad \forall n \geq 1, \tag{2.9}$$

we see from Lemma 1.3 that S_n is a nonexpansive mapping for each $n \geq 1$. For any $w \in \mathcal{F} \subset C_m$, we see from (2.6) that

$$\begin{aligned}\|y_m - w\| &= \|\alpha_mx_m + (1 - \alpha_m)S_mz_m - w\| \\ &\leq \alpha_m\|x_m - w\| + (1 - \alpha_m)\|z_m - w\| \\ &\leq \|x_m - w\|.\end{aligned} \tag{2.10}$$

This shows that $w \in C_{m+1}$. This proves that $\mathcal{F} \subset C_n$ for each $n \geq 1$. Note $x_n = P_{C_n} x_1$. For each $w \in \mathcal{F} \subset C_n$, we have

$$\|x_1 - x_n\| \leq \|x_1 - w\|. \tag{2.11}$$

In particular, we have

$$\|x_1 - x_n\| \leq \|x_1 - P_{\mathcal{F}} x_1\|. \tag{2.12}$$

This implies that $\{x_n\}$ is bounded. Since $x_n = P_{C_n} x_1$ and $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have

$$\begin{aligned} 0 &\leq \langle x_1 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1} \rangle \\ &\leq -\|x_1 - x_n\|^2 + \|x_1 - x_n\| \|x_1 - x_{n+1}\|. \end{aligned} \tag{2.13}$$

It follows that

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|. \tag{2.14}$$

This proves that $\lim_{n \to \infty} \|x_n - x_1\|$ exists. Notice that

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_1 + x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &\leq \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2. \end{aligned} \tag{2.15}$$

It follows that

$$\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0. \tag{2.16}$$

Since $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1}$, we see that

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|. \tag{2.17}$$

This implies that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \leq 2\|x_n - x_{n+1}\|. \tag{2.18}$$

From (2.16), we obtain that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{2.19}$$

On the other hand, we have

$$\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)S_n z_n\| = (1-\alpha_n)\|x_n - S_n z_n\|. \tag{2.20}$$

From the assumption $0 \le \alpha_n \le a < 1$ and (2.19), we have

$$\lim_{n\to\infty}\|x_n - S_n z_n\| = 0. \tag{2.21}$$

For any $p \in \mathcal{F}$, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(I - r_n A)x_n - T_{r_n}(I - r_n A)p\|^2 \\
&= \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + r_n^2\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2.
\end{aligned} \tag{2.22}$$

In a similar way, we also have

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - s_n(2\beta - s_n)\|Bx_n - Bp\|^2. \tag{2.23}$$

Note that

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)S_n z_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|S_n z_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\gamma_n\|u_n - p\|^2 + (1-\alpha_n)(1-\gamma_n)\|v_n - p\|^2.
\end{aligned} \tag{2.24}$$

Substituting (2.22) and (2.23) into (2.24), we arrive at

$$\begin{aligned}
\|y_n - p\|^2 \le{}& \|x_n - p\|^2 - (1-\alpha_n)\gamma_n r_n(2\alpha - r_n)\|Ax_n - Ap\|^2 \\
&- (1-\alpha_n)(1-\gamma_n)s_n(2\beta - s_n)\|Bx_n - Bp\|^2.
\end{aligned} \tag{2.25}$$

It follows that

$$\begin{aligned}(1-\alpha_n)\gamma_n r_n(2\alpha-r_n)\|Ax_n-Ap\|^2 &\le \|x_n-p\|^2-\|y_n-p\|^2\\ &\le (\|x_n-p\|+\|y_n-p\|)\|x_n-y_n\|.\end{aligned} \tag{2.26}$$

In view of the restrictions (a)–(d) and (2.19), we obtain that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{2.27}$$

It also follows from (2.25) that

$$\begin{aligned}(1-\alpha_n)(1-\gamma_n)s_n(2\beta-s_n)\|Bx_n-Bp\|^2 &\le \|x_n-p\|^2-\|y_n-p\|^2\\ &\le (\|x_n-p\|+\|y_n-p\|)\|x_n-y_n\|.\end{aligned} \tag{2.28}$$

By virtue of the restrictions (a)–(d) and (2.19), we get that

$$\lim_{n\to\infty}\|Bx_n-Bp\|=0. \tag{2.29}$$

On the other hand, we have from Lemma 1.1 that

$$\begin{aligned}\|u_n-p\|^2 &= \|T_{r_n}(I-r_nA)x_n-T_{r_n}(I-r_nA)p\|^2\\ &\le \langle (I-r_nA)x_n-(I-r_nA)p, u_n-p\rangle\\ &= \frac{1}{2}\left(\|(I-r_nA)x_n-(I-r_nA)p\|^2+\|u_n-p\|^2-\|(I-r_nA)x_n-(I-r_nA)p-(u_n-p)\|^2\right)\\ &\le \frac{1}{2}\left(\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n-r_n(Ax_n-Ap)\|^2\right)\\ &= \frac{1}{2}\left(\|x_n-p\|^2+\|u_n-p\|^2-\left(\|x_n-u_n\|^2-2r_n\langle x_n-u_n, Ax_n-Ap\rangle+r_n^2\|Ax_n-Ap\|^2\right)\right).\end{aligned} \tag{2.30}$$

This implies that

$$\|u_n-p\|^2 \le \|x_n-p\|^2-\|x_n-u_n\|^2+2r_n\|x_n-u_n\|\|Ax_n-Ap\|. \tag{2.31}$$

In a similar way, we can also obtain that

$$\|v_n-p\|^2 \le \|x_n-p\|^2-\|x_n-v_n\|^2+2s_n\|x_n-v_n\|\|Bx_n-Bp\|. \tag{2.32}$$

Substituting (2.31) and (2.32) into (2.24), we obtain that

$$\begin{aligned}\|y_n - p\|^2 &\le \|x_n - p\|^2 - (1-\alpha_n)\gamma_n\|x_n - u_n\|^2 + 2r_n(1-\alpha_n)\gamma_n\|x_n - u_n\|\|Ax_n - Ap\| \\ &\quad - (1-\alpha_n)(1-\gamma_n)\|x_n - v_n\|^2 + 2s_n(1-\alpha_n)(1-\gamma_n)\|x_n - v_n\|\|Bx_n - Bp\| \\ &\le \|x_n - p\|^2 - (1-\alpha_n)\gamma_n\|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\| \\ &\quad - (1-\alpha_n)(1-\gamma_n)\|x_n - v_n\|^2 + 2s_n\|x_n - v_n\|\|Bx_n - Bp\|.\end{aligned} \tag{2.33}$$

It follows that

$$\begin{aligned}(1-\alpha_n)\gamma_n\|x_n - u_n\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\| \\ &\quad + 2s_n\|x_n - v_n\|\|Bx_n - Bp\| \\ &\le (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| + 2r_n\|x_n - u_n\|\|Ax_n - Ap\| \\ &\quad + 2s_n\|x_n - v_n\|\|Bx_n - Bp\|.\end{aligned} \tag{2.34}$$

In view of the restrictions (a) and (c), we obtain from (2.27) and (2.29) that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{2.35}$$

It also follows from (2.33) that

$$\begin{aligned}(1-\alpha_n)(1-\gamma_n)\|x_n - v_n\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\| \\ &\quad + 2s_n\|x_n - v_n\|\|Bx_n - Bp\| \\ &\le (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| + 2r_n\|x_n - u_n\|\|Ax_n - Ap\| \\ &\quad + 2s_n\|x_n - v_n\|\|Bx_n - Bp\|.\end{aligned} \tag{2.36}$$

Thanks to the restrictions (a) and (c), we obtain from (2.27) and (2.29) that

$$\lim_{n\to\infty}\|x_n - v_n\| = 0. \tag{2.37}$$

Note that

$$\|z_n - x_n\| \le \gamma_n\|u_n - x_n\| + (1-\gamma_n)\|v_n - x_n\|. \tag{2.38}$$

From (2.35) and (2.37), we see that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{2.39}$$

On the other hand, we see from (2.21) that

$$\beta_n(z_n - x_n) + (1-\beta_n)(Sz_n - x_n) \longrightarrow 0 \tag{2.40}$$

as $n \to \infty$. In view of (2.39) and the restriction (b), we obtain that

$$\lim_{n\to\infty} \|x_n - Sz_n\| = 0. \tag{2.41}$$

Note that

$$\|Sx_n - x_n\| \le \|Sx_n - Sz_n\| + \|Sz_n - x_n\| \le \frac{1+k}{1-k}\|x_n - z_n\| + \|Sz_n - x_n\|. \tag{2.42}$$

It follows from (2.39) and (2.41) that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{2.43}$$

Since $\{x_n\}$ is bounded, we assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to ξ.

Next, we show that $\xi \in F(S) \cap \mathrm{EP}(F_1, A) \cap \mathrm{EP}(F_2, B)$. First, we prove that $\xi \in \mathrm{EP}(F_1, A)$. Since $u_n = T_{r_n}(x_n - r_n A x_n)$ for any $u \in C$, we have

$$F_1(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \ge 0. \tag{2.44}$$

From the condition (A2), we see that

$$\langle Ax_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \ge F_1(u, u_n). \tag{2.45}$$

Replacing n by n_i, we arrive at

$$\langle Ax_{n_i}, u - u_{n_i}\rangle + \left\langle u - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge F_1(u, u_{n_i}). \tag{2.46}$$

For any t with $0 < t \le 1$ and $u \in C$, let $u_t = tu + (1-t)\xi$. Since $u \in C$ and $\xi \in C$, we have $u_t \in C$. It follows from (2.46) that

$$\begin{aligned}\langle u_t - u_{n_i}, Au_t\rangle &\ge \langle u_t - u_{n_i}, Au_t\rangle - \langle Ax_{n_i}, u_t - u_{n_i}\rangle - \left\langle u_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F_1(u_t, u_{n_i}) \\ &= \langle u_t - u_{n_i}, Au_t - Au_{n_i}\rangle + \langle u_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle - \left\langle u_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F_1(u_t, u_{n_i}).\end{aligned} \tag{2.47}$$

Since A is Lipschitz continuous, we obtain from (2.35) that $Au_{n_i} - Ax_{n_i} \to 0$ as $i \to \infty$. On the other hand, we get from the monotonicity of A that

$$\langle u_t - u_{n_i}, Au_t - Au_{n_i} \rangle \geq 0. \tag{2.48}$$

It follows from (A4) and (2.47) that

$$\langle u_t - \xi, Au_t \rangle \geq F_1(u_t, \xi). \tag{2.49}$$

From (A1), (A4), and (2.49), we see that

$$\begin{aligned} 0 = F_1(u_t, u_t) &\leq tF_1(u_t, u) + (1-t)F_1(u_t, \xi) \\ &\leq tF_1(u_t, u) + (1-t)\langle u_t - \xi, Au_t \rangle \\ &= tF_1(u_t, u) + (1-t)t\langle u - \xi, Au_t \rangle, \end{aligned} \tag{2.50}$$

which yields that

$$F_1(u_t, u) + (1-t)\langle u - \xi, Au_t \rangle \geq 0. \tag{2.51}$$

Letting $t \to 0$ in the above inequality, we arrive at

$$F_1(\xi, u) + \langle u - \xi, A\xi \rangle \geq 0. \tag{2.52}$$

This shows that $\xi \in \mathrm{EP}(F_1, A)$. In a similar way, we can obtain that $\xi \in \mathrm{EP}(F_2, B)$.

Next, we show that $\xi \in F(S)$. We can conclude from Lemma 1.1 the desired conclusion easily. This proves that $\xi \in \mathcal{F}$. Put $\overline{x} = P_{\mathcal{F}} x_1$. Since $\overline{x} = P_{\mathcal{F}} x_1 \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}} x_1$, we have

$$\|x_1 - x_{n+1}\| \leq \|x_1 - \overline{x}\|. \tag{2.53}$$

On the other hand, we have

$$\begin{aligned} \|x_1 - \overline{x}\| &\leq \|x_1 - \xi\| \\ &\leq \liminf_{i \to \infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i \to \infty} \|x_1 - x_{n_i}\| \\ &\leq \|x_1 - \overline{x}\|. \end{aligned} \tag{2.54}$$

We, therefore, obtain that

$$\|x_1 - \xi\| = \lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - \overline{x}\|. \tag{2.55}$$

This implies $x_{n_i} \to \xi = \overline{x}$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we obtain that $x_n \to \overline{x}$ as $n \to \infty$. This completes the proof. □

If S is nonexpansive, then we have from Theorem 2.1 the following result immediately.

Corollary 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping, $B : C \to H$ a β-inverse-strongly monotone mapping, and $S : C \to C$ a nonexpansive mapping. Let $\{r_n\}$ and $\{s_n\}$ be two positive real sequences. Assume that $\mathcal{F} := EP(F_1, A) \cap FP(F_2, B) \cap F(S)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered} x_1 \in C, \\ C_1 = C, \\ F_1(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ F_2(v_n, v) + \langle Bx_n, v - v_n\rangle + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) S\big(\gamma_n u_n + (1-\gamma_n) v_n\big), \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1, \end{gathered} \tag{2.56}$$

where $\{\alpha_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$. Assume that $\{\alpha_n\}$, $\{\gamma_n\}$, $\{r_n\}$, and $\{s_n\}$ satisfy the following restrictions:

(a) $0 \leq \alpha_n \leq a < 1$;

(b) $0 \leq c \leq \gamma_n \leq d < 1$;

(c) $0 < e \leq r_n \leq f < 2\alpha$ *and* $0 < e' \leq s_n \leq f' < 2\beta$.

Then the sequence $\{x_n\}$ converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\mathcal{F}} x_1$.

As applications of Theorem 2.1, we consider the problems (1.9) and (1.11).

Theorem 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping, $B : C \to H$ a β-inverse-strongly monotone mapping, and $S : C \to C$ a k-strict pseudocontraction. Let $\{r_n\}$ and $\{s_n\}$ be two positive real sequences. Assume that $\mathcal{F} := VI(C,A) \cap VI(C,B) \cap F(S)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered}
x_1 \in C, \\
C_1 = C, \\
z_n = \gamma_n P_C(I - r_n A)x_n + (1-\gamma_n)P_C(I - s_n B)x_n, \\
y_n = \alpha_n x_n + (1-\alpha_n)\left(\beta_n z_n + (1-\beta_n)Sz_n\right), \\
C_{n+1} = \left\{w \in C_n : \left\|y_n - w\right\| \le \left\|x_n - w\right\|\right\}, \\
x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1,
\end{gathered} \tag{2.57}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{r_n\}$, and $\{s_n\}$ satisfy the following restrictions:

(a) $0 \le \alpha_n \le a < 1$;

(b) $0 \le k \le \beta_n < b < 1$;

(c) $0 \le c \le \gamma_n \le d < 1$;

(d) $0 < e \le r_n \le f < 2\alpha$ *and* $0 < e' \le s_n \le f' < 2\beta$.

Then the sequence $\{x_n\}$ converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\mathcal{F}} x_1$.

Proof. Putting $F_1 = F_2 \equiv 0$, we see that

$$\langle Ax_n, u - u_n \rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n \rangle \ge 0, \quad \forall u \in C, \tag{2.58}$$

is equivalent to

$$\langle x_n - r_n A x_n - u_n, u_n - u \rangle \ge 0, \quad \forall u \in C. \tag{2.59}$$

This implies that $u_n = P_C(x_n - r_n A x_n)$. We also have $v_n = P_C(x_n - s_n B x_n)$. We can obtain from Theorem 2.1 the desired results immediately. □

Corollary 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping, $B : C \to H$ a β-inverse-strongly monotone mapping, and $S : C \to C$ a nonexpansive mapping. Let $\{r_n\}$ and $\{s_n\}$ be two positive real sequences. Assume* *that* $\mathcal{F} := VI(C,A) \cap VI(C,B) \cap F(S)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$
\begin{gathered}
x_1 \in C, \\
C_1 = C, \\
z_n = \gamma_n P_C (I - r_n A) x_n + (1 - \gamma_n) P_C (I - s_n B) x_n, \\
y_n = \alpha_n x_n + (1 - \alpha_n) S z_n, \\
C_{n+1} = \{ w \in C_n : \|y_n - w\| \le \|x_n - w\| \}, \\
x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1,
\end{gathered}
\tag{2.60}
$$

where $\{\alpha_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$*. Assume that* $\{\alpha_n\}$*,* $\{\gamma_n\}$*,* $\{r_n\}$*, and* $\{s_n\}$ *satisfy the following restrictions:*

(a) $0 \le \alpha_n \le a < 1$;

(b) $0 \le c \le \gamma_n \le d < 1$;

(c) $0 < e \le r_n \le f < 2\alpha$ *and* $0 < e' \le s_n \le f' < 2\beta$.

Then the sequence $\{x_n\}$ *converges strongly to some point* $\overline{x}$*, where* $\overline{x} = P_{\mathcal{F}} x_1$*.*

Theorem 2.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* F_1 *and* F_2 *be two bifunctions from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* $(A1)$*–*$(A4)$*. Let* $S : C \to C$ *be a* k*-strict pseudocontraction. Let* $\{r_n\}$ *and* $\{s_n\}$ *be two positive real sequences. Assume that* $\mathcal{F} := EP(F_1) \cap FP(F_2) \cap F(S)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$
\begin{gathered}
x_1 \in C, \\
C_1 = C, \\
F_1(u_n, u) + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \ge 0, \quad \forall u \in C, \\
F_2(v_n, v) + \frac{1}{s_n} \langle v - v_n, v_n - x_n \rangle \ge 0, \quad \forall v \in C, \\
z_n = \gamma_n u_n + (1 - \gamma_n) v_n, \\
y_n = \alpha_n x_n + (1 - \alpha_n)(\beta_n z_n + (1 - \beta_n) S z_n), \\
C_{n+1} = \{ w \in C_n : \|y_n - w\| \le \|x_n - w\| \}, \\
x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1,
\end{gathered}
\tag{2.61}
$$

where $\{\alpha_n\}$*,* $\{\beta_n\}$*, and* $\{\gamma_n\}$ *are sequences in* $(0,1)$*. Assume that* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$*,* $\{r_n\}$*, and* $\{s_n\}$ *satisfy the following restrictions:*

(a) $0 \le \alpha_n \le a < 1$;

(b) $0 \le k \le \beta_n < b < 1$;

(c) $0 \le c \le \gamma_n \le d < 1$;

(d) $0 < e \le r_n \le f < \infty$ *and* $0 < e' \le s_n \le f' < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to some point* $\overline{x}$, *where* $\overline{x} = P_{\mathcal{F}} x_1$.

Proof. Putting $A = B = 0$, we can obtain from Theorem 2.1 the desired conclusion immediately. □

Remark 2.6. Theorem 2.5 is generalization of Theorem TT. To be more precise, we consider a pair of bifunctions and a strictly pseudocontractive mapping.

Let T : $C \to C$ be a k-strict pseudocontraction. It is known that $I - T$ is a $(1-k)/2$-inverse-strongly monotone mapping. The following results are not hard to derive.

Theorem 2.7. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F_1 *and* F_2 *be two bifunctions from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* $(A1)$–$(A4)$. *Let* $T_A : C \to C$ *be a* k_α*-strict pseudocontraction,* $B : C \to C$ *a* k_β*-strict pseudocontraction, and* $S : C \to C$ *a* k*-strict pseudocontraction. Let* $\{r_n\}$ *and* $\{s_n\}$ *be two positive real sequences. Assume that* $\mathcal{F} := EP(F_1, I-T_A) \cap FP(F_2, I-T_B) \cap F(S)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{gathered}
x_1 \in C, \\
C_1 = C, \\
F_1(u_n, u) + \langle (I - T_A)x_n, u - u_n \rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n \rangle \ge 0, \quad \forall u \in C, \\
F_2(v_n, v) + \langle (I - T_B)x_n, v - v_n \rangle + \frac{1}{s_n}\langle v - v_n, v_n - x_n \rangle \ge 0, \quad \forall v \in C, \\
z_n = \gamma_n u_n + (1 - \gamma_n) v_n, \\
y_n = \alpha_n x_n + (1 - \alpha_n)(\beta_n z_n + (1 - \beta_n) S z_n), \\
C_{n+1} = \{ w \in C_n : \| y_n - w \| \le \| x_n - w \| \}, \\
x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1,
\end{gathered} \tag{2.62}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$. *Assume that* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{r_n\}$, *and* $\{s_n\}$ *satisfy the following restrictions:*

(a) $0 \le \alpha_n \le a < 1$;

(b) $0 \le k \le \beta_n < b < 1$;

(c) $0 \le c \le \gamma_n \le d < 1$;

(d) $0 < e \le r_n \le f < 1 - k_\alpha$ *and* $0 < e' \le s_n \le f' < 1 - k_\beta$.

Then the sequence $\{x_n\}$ *converges strongly to some point* $\overline{x}$, *where* $\overline{x} = P_{\mathcal{F}} x_1$.

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