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Research Article

On a Multiple Hilbert's Inequality with Parameters

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By introducing multiparameters and conjugate exponents and using Hadamard's inequality and the way of real analysis, we estimate the weight coefficients and give a multiple more accurate Hilbert's inequality, which is an extension of some published results. We also prove that the constant factor in the new inequality is the best possible and consider its equivalent form.

1. Introduction

In 1908, Weyl published the following famous Hilbert's inequality (cf. [1]). If $a_n, b_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{m=1}^{\infty}a_m^2\sum_{n=1}^{\infty}b_n^2\right)^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. In 1934, Hardy proved the following more accurate Hilbert's inequality (cf. [2]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_{\mathrm{m}} b_n}{m+n-1} < \pi\left(\sum_{m=1}^{\infty}a_m^2\sum_{n=1}^{\infty}b_n^2\right)^{1/2}, \tag{1.2}$$

where the constant factor π is the best possible. For $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$, the equivalent forms of (1.1) and (1.2) are given as follows (cf. [2]):

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^2 < \pi^2\sum_{m=1}^{\infty}a_m^2, \tag{1.3}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n-1}\right)^2 < \pi^2\sum_{m=1}^{\infty}a_m^2, \tag{1.4}$$

where the constant factor π^2 is the best possible. Inequalities (1.1)–(1.4) are important in analysis and their applications (cf. [3]). In near one century, there are many improvements, generalizations and, applications of (1.1)–(1.4) in numerous literatures and monographs of mathematics (cf. [2–18]). Yang and Huang also considered the multiple Hilbert-type integral inequality (cf. [19, 20]). Recently, Yang summarized the methods of introducing parameters and estimating the weight coefficients to extend Hilbert-type inequalities for the past 100 years. Some representative results are as follows (cf. [21, 22]):

(i) if $p, r > 1$, $1/p + 1/q = 1/r + 1/s = 1$, $0 < \alpha \le 1$, $0 < \lambda \le \min\{r, s\}$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n-1)^{\lambda}} < B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\times\left\{\sum_{m=1}^{\infty}\left(m-\frac{1}{2}\right)^{p(1-\lambda/r)-1}a_m^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right)^{q(1-\lambda/s)-1}b_n^q\right\}^{1/q}, \tag{1.5}$$

$$\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right)^{p\lambda/s-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n-1)^{\lambda}}\right]^p < \left[B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\right]^p\sum_{m=1}^{\infty}\left(m-\frac{1}{2}\right)^{p(1-\lambda/r)-1}a_m^p, \tag{1.6}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m-1/2)^{\alpha}+(n-1/2)^{\alpha}} &< \frac{\pi}{\alpha\sin(\pi/r)}\times\left\{\sum_{m=1}^{\infty}\left(m-\frac{1}{2}\right)^{p(1-\alpha/r)-1}a_m^p\right\}^{1/p}\\ &\quad\times\left\{\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right)^{q(1-\alpha/s)-1}b_n^q\right\}^{1/q},\end{aligned} \tag{1.7}$$

$$\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right)^{p\alpha/s-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m-1/2)^{\alpha}+(n-1/2)^{\alpha}}\right]^p < \left[\frac{\pi}{\alpha\sin(\pi/r)}\right]^p\sum_{m=1}^{\infty}\left(m-\frac{1}{2}\right)^{p(1-\alpha/r)-1}a_m^p, \tag{1.8}$$

(ii) if $p_i, r_i > 1$, $\sum_{i=1}^{n}(1/p_i) = \sum_{i=1}^{n}(1/r_i) = 1$, $0 < \alpha \le 1$, $0 < \lambda\alpha \le \min_{1\le i\le n}\{r_i\}$, then

$$\sum_{m_n=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{1}{\left(\sum_{i=1}^{n}m_i{}^{\alpha}\right)^{\lambda}}\prod_{i=1}^{n}a_{m_i}^{(i)} < \frac{\alpha^{1-n}}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(\frac{\lambda}{r_i}\right)\left\{\sum_{m_i=1}^{\infty}m_i{}^{p_i(1-\lambda\alpha/r_i)-1}\left(a_{m_i}^{(i)}\right)^{p_i}\right\}^{1/p_i}. \tag{1.9}$$

The constant factors in the above five inequalities are all the best possible. Inequalities (1.5) and (1.7) are generalizations of inequality (1.2), and inequality (1.9) is a multiple extension of (1.1). Inequalities (1.6) and (1.8) are the equivalent forms of (1.5) and (1.7), which are extensions of (1.4).

In this paper, by introducing multi-parameters and conjugate exponents and using Hadamard's inequality, we estimate the weight coefficients and give a multiple more accurate Hilbert 's inequality, which is an extension of inequalities (1.5), (1.7), and (1.9). We also prove that the constant factor in the new inequality is the best possible and consider its equivalent form.

2. Some Lemmas

Lemma 2.1. *If* $n \in \mathbf{N} \setminus \{1\}$, $p_i, r_i > 1$ $(i = 1, \ldots, n)$, $\sum_{i=1}^{n}(1/p_i) = \sum_{i=1}^{n}(1/r_i) = 1$, $\lambda > 0 < \alpha < 2$, $\beta \geq -1/2$, $\lambda\alpha \max\{1/(2-\alpha), 1\} \leq \min_{1 \leq i \leq n}\{r_i\}$, *then*

$$A := \prod_{i=1}^{n}\left[(m_i+\beta)^{(\lambda\alpha/r_i-1)(1-p_i)}\prod_{j=1(j\neq i)}^{n}(m_j+\beta)^{\lambda\alpha/r_j-1}\right]^{1/p_i} = 1. \tag{2.1}$$

Proof. We find the following:

$$\begin{aligned} A &= \prod_{i=1}^{n}\left[(m_i+\beta)^{(\lambda\alpha/r_i-1)(1-p_i)+1-\lambda\alpha/r_i}\prod_{j=1}^{n}(m_j+\beta)^{\lambda\alpha/r_j-1}\right]^{1/p_i} \\ &= \prod_{i=1}^{n}\left[(m_i+\beta)^{p_i(1-\lambda\alpha/r_i)}\prod_{j=1}^{n}(m_j+\beta)^{\lambda\alpha/r_j-1}\right]^{1/p_i} \\ &= \prod_{i=1}^{n}(m_i+\beta)^{1-\lambda\alpha/r_i}\left[\prod_{j=1}^{n}(m_j+\beta)^{\lambda\alpha/r_j-1}\right]^{\sum_{i=1}^{n}(1/p_i)} = 1, \end{aligned} \tag{2.2}$$

and then (2.1) is valid. □

Lemma 2.2. *If* $\lambda, y > 0$, $r > 1$, $1/r + 1/s = 1$, $0 < \alpha < 2$, $\beta \geq -1/2$, $\lambda\alpha \max\{1/(2-\alpha), 1\} \leq r$, *then*

$$\frac{\Gamma(\lambda/r)\Gamma(\lambda/s)}{\alpha\Gamma(\lambda)}\left[1 - O\left(\frac{1}{y^{\lambda/r}}\right)\right] < \sum_{m=1}^{\infty}\frac{y^{\lambda/s}(m+\beta)^{\lambda\alpha/r-1}}{\left[y+(m+\beta)^{\alpha}\right]^{\lambda}} < \frac{\Gamma(\lambda/r)\Gamma(\lambda/s)}{\alpha\Gamma(\lambda)}. \tag{2.3}$$

Proof. For fixed $y > 0$, we set

$$f(x) := \frac{y^{\lambda/s}(x+\beta)^{(\lambda\alpha/r)-1}}{\left[y+(x+\beta)^{\alpha}\right]^{\lambda}}, \quad x \in (-\beta, \infty). \tag{2.4}$$

In virtue of $\alpha + \lambda\alpha/r - 2 \leq 0$ and $\lambda\alpha/r - 1 \leq 0$, we find $(-1)^i f^{(i)}(x) > 0$, $(i = 1,2)$. Putting $u = (x+\beta)^\alpha/y$, we have the following:

$$\int_{-\beta}^{\infty} f(x)dx = \frac{1}{\alpha}\int_0^{\infty} \frac{u^{\lambda/r-1}}{(1+u)^\lambda}du = \frac{\Gamma(\lambda/r)\Gamma(\lambda/s)}{\alpha\Gamma(\lambda)}. \tag{2.5}$$

Since $-\beta \leq 1/2$, by the following Hadamard's inequality (cf. [5]):

$$f(m) < \int_{m-1/2}^{m+1/2} f(x)dx (m \in \mathbf{N}), \tag{2.6}$$

it follows that

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{y^{\lambda/s}(m+\beta)^{\lambda\alpha/r-1}}{\left[y+(m+\beta)^\alpha\right]^\lambda} &= \sum_{m=1}^{\infty} f(m) < \sum_{m=1}^{\infty}\int_{m-1/2}^{m+1/2} f(x)dx \\ &= \int_{1/2}^{\infty} f(x)dx \leq \int_{-\beta}^{\infty} f(x)dx = \frac{\Gamma(\lambda/r)\Gamma(\lambda/s)}{\alpha\Gamma(\lambda)}, \end{aligned} \tag{2.7}$$

and then we have the right-hand side of (2.3). Since

$$\begin{aligned} \int_{-\beta}^{1} f(x)dx &= \int_0^{(x+\beta)^\alpha/y} \frac{u^{\lambda/r-1}}{\alpha(1+u)^\lambda}du \\ &< \frac{1}{\alpha}\int_0^{(x+\beta)^\alpha/y} u^{\lambda/r-1}du = \frac{r(1+\beta)^{\lambda\alpha/r}}{\lambda\alpha y^{\lambda/r}}, \end{aligned} \tag{2.8}$$

and $f(x)$ is strictly decreasing in $(-\beta, \infty)$, we get

$$\begin{aligned} \sum_{m=1}^{\infty} f(m) &> \int_1^{\infty} f(x)dx = \int_{-\beta}^{\infty} f(x)dx - \int_{-\beta}^{1} f(x)dx \\ &> \frac{\Gamma(\lambda/r)\Gamma(\lambda/s)}{\alpha\Gamma(\lambda)} - \frac{r(1+\beta)^{\lambda\alpha/r}}{\lambda\alpha y^{\lambda/r}}. \end{aligned} \tag{2.9}$$

Hence, we prove that the left-hand side of (2.3) is valid. □

Lemma 2.3. *As the assumption of Lemma 2.1, define the weight coefficients $\omega_i(m_i) = \omega(m_i; r_1, \ldots, r_n)$ as*

$$\omega_i(m_i) := (m_i+\beta)^{\lambda\alpha/r_i}\sum_{m_n=1}^{\infty}\cdots\sum_{m_{i+1}=1}^{\infty}\sum_{m_{i-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{j=1(j\neq i)}^{n}(m_j+\beta)^{\lambda\alpha/r_j-1}}{\left[\sum_{i=1}^{n}(m_i+\beta)^\alpha\right]^\lambda} \tag{2.10}$$

$(i = 1, \ldots, n)$, *then there exists* $\delta_n > 0$, *such that*

$$\frac{\alpha^{1-n}}{\Gamma(\lambda)} \prod_{j=1}^{n} \Gamma\left(\frac{\lambda}{r_j}\right) \left[1 - O\left(\frac{1}{(m_n + \beta)^{\delta_n}}\right)\right] < \omega_n(m_n)$$
$$= (m_n + \beta)^{\lambda\alpha/r_n} \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{\prod_{j=1}^{n-1} (m_j + \beta)^{\lambda\alpha/r_j - 1}}{\left[\sum_{i=1}^{n} (m_i + \beta)^{\alpha}\right]^{\lambda}} < \frac{\alpha^{1-n}}{\Gamma(\lambda)} \prod_{j=1}^{n} \Gamma\left(\frac{\lambda}{r_j}\right). \tag{2.11}$$

Moreover, for any $i \in \{1, \ldots, n\}$, *it follows that*

$$\omega_i(m_i) < \frac{\alpha^{1-n}}{\Gamma(\lambda)} \prod_{j=1}^{n} \Gamma\left(\frac{\lambda}{r_j}\right). \tag{2.12}$$

Proof. We prove (2.11) by mathematical induction. For $n = 2$, we set $r = r_1$ and $s = r_2$ satisfying $1/r + 1/s = 1$. Putting $m = m_1$, $y = (m_2 + \beta)^{\alpha}$, $\delta_2 = \lambda\alpha/r > 0$, we have the following:

$$\omega_2(m_2) = \sum_{m_1=1}^{\infty} \frac{(m_1 + \beta)^{\lambda\alpha/r_1 - 1} (m_2 + \beta)^{\lambda\alpha/r_2}}{\left[(m_1 + \beta)^{\alpha} + (m_2 + \beta)^{\alpha}\right]^{\lambda}} = \sum_{m=1}^{\infty} \frac{y^{\lambda/s} (m + \beta)^{\lambda\alpha/r - 1}}{\left[y + (m + \beta)^{\alpha}\right]^{\lambda}}, \tag{2.13}$$

and then (2.11) is valid by using inequality (2.3).

Assuming that for $n(\geq 2)$, (2.11) is valid, then for $n + 1$, setting $y = \sum_{i=2}^{n+1} (m_i + \beta)^{\alpha} (> (m_{n+1} + \beta)^{\alpha})$, $s_1 = (1 - 1/r_1)^{-1}$, by (2.3), we have the following:

$$\frac{\Gamma(\lambda/r_1)\Gamma(\lambda/s_1)}{\alpha\Gamma(\lambda)} \left[1 - O_1\left(\frac{1}{y^{\lambda/r_1}}\right)\right] < \sum_{m_1=1}^{\infty} \frac{y^{\lambda/s_1} (m_1 + \beta)^{\lambda\alpha/r_1 - 1}}{\left[y + (m_1 + \beta)^{\alpha}\right]^{\lambda}} < \frac{\Gamma(\lambda/r_1)\Gamma(\lambda/s_1)}{\alpha\Gamma(\lambda)}. \tag{2.14}$$

Setting $\widetilde{\lambda} = \lambda/s_1$, $\widetilde{r}_j = r_{j+1}/s_1$, $\widetilde{m}_j = m_{j+1}$ $(j = 1, \ldots, n)$, we find $\sum_{j=1}^{n} (1/\widetilde{r}_j) = 1$, $\alpha\widetilde{\lambda} \max\{1/(2 - \alpha), 1\} \leq \min_{1 \leq i \leq n}\{\widetilde{r}_i\}$. By the assumption of induction, it follows that

$$\omega_{n+1}(m_{n+1}) = (\widetilde{m}_n + \beta)^{\widetilde{\lambda}\alpha/\widetilde{r}_n} \times \sum_{\widetilde{m}_{n-1}=1}^{\infty} \cdots \sum_{\widetilde{m}_1=1}^{\infty} \frac{\prod_{j=1}^{n-1} (\widetilde{m}_j + \beta)^{\widetilde{\lambda}\alpha/\widetilde{r}_j - 1}}{\left[\sum_{i=1}^{n} (\widetilde{m}_i + \beta)^{\alpha}\right]^{\widetilde{\lambda}}}$$
$$\times \left\{ \sum_{m_1=1}^{\infty} \frac{y^{\lambda/s_1} (m_1 + \beta)^{\lambda\alpha/r_1 - 1}}{\left[y + (m_1 + \beta)^{\alpha}\right]^{\lambda}} \right\}$$
$$< (\widetilde{m}_n + \beta)^{\widetilde{\lambda}\alpha/\widetilde{r}_n} \sum_{\widetilde{m}_{n-1}=1}^{\infty} \cdots \sum_{\widetilde{m}_1=1}^{\infty} \frac{\prod_{j=1}^{n-1} (\widetilde{m}_j + \beta)^{\widetilde{\lambda}\alpha/\widetilde{r}_j - 1}}{\left[\sum_{i=1}^{n} (\widetilde{m}_i + \beta)^{\alpha}\right]^{\widetilde{\lambda}}} \cdot \frac{\Gamma(\lambda/r_1)\Gamma(\lambda/s_1)}{\alpha\Gamma(\lambda)} \tag{2.15}$$
$$< \frac{\alpha^{1-n}}{\Gamma(\widetilde{\lambda})} \prod_{i=1}^{n} \Gamma\left(\frac{\widetilde{\lambda}}{\widetilde{r}_i}\right) \cdot \frac{\Gamma(\lambda/r_1)\Gamma(\widetilde{\lambda})}{\alpha\Gamma(\lambda)} = \frac{\alpha^{1-(n+1)}}{\Gamma(\lambda)} \prod_{i=1}^{n+1} \Gamma\left(\frac{r_i}{\lambda}\right),$$

$$
\begin{aligned}
\omega_{n+1}(m_{n+1}) &> (\widetilde{m}_n+\beta)^{\widetilde{\lambda}\alpha/\widetilde{r}_n} \times \sum_{\widetilde{m}_{n-1}=1}^{\infty}\cdots\sum_{\widetilde{m}_1=1}^{\infty}\frac{\prod_{j=1}^{n-1}(\widetilde{m}_j+\beta)^{\widetilde{\lambda}\alpha/\widetilde{r}_j-1}}{\left[\sum_{i=1}^{n}(\widetilde{m}_i+\beta)^{\alpha}\right]^{\widetilde{\lambda}}}\cdot\frac{\Gamma(\lambda/r_1)\Gamma(\lambda/s_1)}{\alpha\Gamma(\lambda)} \\
&\quad\times\left[1-O_1\left(\frac{1}{y^{\lambda/r_1}}\right)\right] \\
&> \frac{\Gamma(\lambda/r_1)\Gamma(\lambda/s_1)}{\alpha\Gamma(\lambda)}\left[(\widetilde{m}_n+\beta)^{\widetilde{\lambda}\alpha/\widetilde{r}_n}\sum_{\widetilde{m}_{n-1}=1}^{\infty}\cdots\sum_{\widetilde{m}_1=1}^{\infty}\frac{\prod_{j=1}^{n-1}(\widetilde{m}_j+\beta)^{\widetilde{\lambda}\alpha/\widetilde{r}_j-1}}{\left[\sum_{i=1}^{n}(\widetilde{m}_i+\beta)^{\alpha}\right]^{\widetilde{\lambda}}}-\gamma\right] \\
&> \frac{\alpha^{1-(n+1)}}{\Gamma(\lambda)}\prod_{i=1}^{n+1}\Gamma\left(\frac{\lambda}{r_i}\right)\times\left[1-\widetilde{O}_2\left(\frac{1}{(\widetilde{m}_n+\beta)^{\widetilde{\delta}_n}}\right)\right]-\frac{\Gamma(\lambda/r_1)\Gamma(\lambda/s_1)}{\alpha\Gamma(\lambda)}\gamma,
\end{aligned}
\tag{2.16}
$$

where $\widetilde{\delta}_n > 0$ and

$$
\begin{aligned}
0 < \gamma &:= (\widetilde{m}_n+\beta)^{\widetilde{\lambda}\alpha/\widetilde{r}_n}\sum_{\widetilde{m}_{n-1}=1}^{\infty}\cdots\sum_{\widetilde{m}_1=1}^{\infty}\frac{\prod_{j=1}^{n-1}(\widetilde{m}_j+\beta)^{\widetilde{\lambda}\alpha/\widetilde{r}_j-1}}{\left[\sum_{i=1}^{n}(\widetilde{m}_i+\beta)^{\alpha}\right]^{\widetilde{\lambda}}}\widetilde{O}_1\left(\frac{1}{(m_{n+1}+\beta)^{\alpha\lambda/r_1}}\right) \\
&< \frac{\alpha^{1-n}}{\Gamma(\lambda/s_1)}\prod_{i=2}^{n+1}\Gamma\left(\frac{r_i}{\lambda}\right)\times\widetilde{O}_1\left(\frac{1}{(m_{n+1}+\beta)^{\alpha\lambda/r_1}}\right).
\end{aligned}
\tag{2.17}
$$

Setting $\delta_{n+1} = \min\{\widetilde{\delta}_n, \alpha\lambda/r_1\} > 0$, by (2.16), we have the following:

$$
\omega_{n+1}(m_{n+1}) > \frac{\alpha^{1-(n+1)}}{\Gamma(\lambda)}\prod_{i=1}^{n+1}\Gamma\left(\frac{\lambda}{r_i}\right)\times\left[1-O\left(\frac{1}{(m_{n+1}+\beta)^{\delta_{n+1}}}\right)\right],
\tag{2.18}
$$

and then by (2.15), (2.18), and mathematical induction, (2.11) is valid. Setting $\widetilde{m}_j = m_j$, $\widetilde{r}_j = r_j$ $(j = 1,\ldots,i-1)$, $\widetilde{m}_j = m_{j+1}$, $\widetilde{r}_j = r_{j+1}$ $(j = i,\ldots,n-1)$, $\widetilde{m}_n = m_i$, $\widetilde{r}_n = r_i$, then we have the following:

$$
\omega_i(m_i) = \omega(\widetilde{m}_n;\widetilde{r}_1,\ldots,\widetilde{r}_n) < \frac{\alpha^{1-n}}{\Gamma(\lambda)}\prod_{j=1}^{n}\Gamma\left(\frac{\lambda}{\widetilde{r}_j}\right) = \frac{\alpha^{1-n}}{\Gamma(\lambda)}\prod_{j=1}^{n}\Gamma\left(\frac{\lambda}{r_j}\right).
\tag{2.19}
$$

Hence, (2.12) is valid. □

3. Main Results

Theorem 3.1. *Suppose that* $n \in \mathbf{N} \setminus \{1\}$, $p_i, r_i > 1$ $(i = 1, \ldots, n)$, $\sum_{i=1}^{n}(1/p_i) = \sum_{i=1}^{n}(1/r_i) = 1$, $1/q_n = 1 - 1/p_n$, $\lambda > 0$, $0 < \alpha < 2$, $\beta \geq -1/2$, $\lambda\alpha \max\{1/(2-\alpha), 1\} \leq \min_{1\leq i\leq n}\{r_i\}$, $a_{m_i}^{(i)} \geq 0$ $(m_i \in \mathbf{N})$, *such that*

$$0 < \sum_{m_i=1}^{\infty} (m_i + \beta)^{p_i(1-\lambda\alpha/r_i)-1} \left(a_{m_i}^{(i)}\right)^{p_i} < \infty \quad (i = 1, \ldots, n), \tag{3.1}$$

then one has the following equivalent inequalities:

$$\begin{aligned} I &:= \sum_{m_n=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{1}{\left[\sum_{i=1}^{n} (m_i + \beta)^{\alpha}\right]^{\lambda}} \prod_{i=1}^{n} a_{m_i}^{(i)} \\ &< \frac{\alpha^{1-n}}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(\frac{\lambda}{r_i}\right) \left\{ \sum_{m_i=1}^{\infty} (m_i + \beta)^{p_i(1-\lambda\alpha/r_i)-1} \left(a_{m_i}^{(i)}\right)^{p_i} \right\}^{1/p_i}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} J &:= \left\{ \sum_{m_n=1}^{\infty} (m_n + \beta)^{\lambda\alpha q_n/r_n - 1} \left[\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{\prod_{i=1}^{n-1} a_{m_i}^{(i)}}{\left[\sum_{i=1}^{n} (m_i + \beta)^{\alpha}\right]^{\lambda}} \right]^{q_n} \right\}^{1/q_n} \\ &< \frac{\Gamma(\lambda/r_n)}{\alpha^{n-1}\Gamma(\lambda)} \prod_{i=1}^{n-1} \Gamma\left(\frac{\lambda}{r_i}\right) \left\{ \sum_{m_i=1}^{\infty} (m_i + \beta)^{p_i(1-\lambda\alpha/r_i)-1} \left(a_{m_i}^{(i)}\right)^{p_i} \right\}^{1/p_i}. \end{aligned} \tag{3.3}$$

Proof. Since $1/p_n + 1/q_n = 1$, by (2.1) and Hölder's inequality (cf. [5]), we find that

$$\begin{aligned} &\left[\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{\prod_{i=1}^{n-1} a_{m_i}^{(i)}}{\left[\sum_{i=1}^{n} (m_i + \beta)^{\alpha}\right]^{\lambda}} \right]^{q_n} \\ &= \left\{ \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{1}{\left[\sum_{i=1}^{n} (m_i + \beta)^{\alpha}\right]^{\lambda}} \left[(m_n + \beta)^{(\lambda\alpha/r_n - 1)(1-p_n)} \prod_{j=1}^{n-1} (m_j + \beta)^{\lambda\alpha/r_j - 1} \right]^{1/p_n} \right. \\ &\quad \left. \times \prod_{i=1}^{n-1} \left[(m_i + \beta)^{(\lambda\alpha/r_i - 1)(1-p_i)} \prod_{j=1 (j\neq i)}^{n} (m_j + \beta)^{\lambda\alpha/r_j - 1} \right]^{1/p_i} a_{m_i}^{(i)} \right\}^{q_n} \\ &\leq \left\{ \omega_n(m_n)(m_n + \beta)^{p_n(1-\lambda\alpha/r_n)-1} \right\}^{q_n/p_n} \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{1}{\left[\sum_{i=1}^{n} (m_i + \beta)^{\alpha}\right]^{\lambda}} \\ &\quad \times \prod_{i=1}^{n-1} \left[(m_i + \beta)^{(\lambda\alpha/r_i - 1)(1-p_i)} \prod_{j=1 (j\neq i)}^{n} (m_j + \beta)^{\lambda\alpha/r_j - 1} \right]^{q_n/p_i} \left(a_{m_i}^{(i)}\right)^{q_n} \end{aligned}$$

$$\leq \left(\frac{\prod_{i=1}^{n}\Gamma(\lambda/r_i)}{\alpha^{n-1}\Gamma(\lambda)}\right)^{q_n/p_n}(m_n+\beta)^{1-\lambda\alpha q_n/r_n}\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{1}{\left[\sum_{i=1}^{n}(m_i+\beta)^{\alpha}\right]^{\lambda}}$$
$$\times\prod_{i=1}^{n-1}\left[(m_i+\beta)^{(\lambda\alpha/r_i-1)(1-p_i)}\prod_{j=1(j\neq i)}^{n}(m_j+\beta)^{\lambda\alpha/r_j-1}\right]^{q_n/p_i}\left(a_{m_i}^{(i)}\right)^{q_n}, \tag{3.4}$$

$$J\leq \left(\frac{\prod_{i=1}^{n}\Gamma(\lambda/r_i)}{\alpha^{n-1}\Gamma(\lambda)}\right)^{1/p_n}$$
$$\times\left\{\sum_{m_n=1}^{\infty}\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{1}{\left[\sum_{i=1}^{n}(m_i+\beta)^{\alpha}\right]^{\lambda}}\times\prod_{i=1}^{n-1}\left[(m_i+\beta)^{(\lambda\alpha/r_i-1)(1-p_i)}\prod_{j=1(j\neq i)}^{n}(m_j+\beta)^{\lambda\alpha/r_j-1}\right]^{q_n/p_i}\right.$$
$$\left.\times\left(a_{m_i}^{(i)}\right)^{q_n}\right\}^{1/q_n}$$
$$=\left(\frac{\prod_{i=1}^{n}\Gamma(\lambda/r_i)}{\alpha^{n-1}\Gamma(\lambda)}\right)^{1/p_n}\left\{\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\left[\sum_{m_n=1}^{\infty}\frac{(m_n+\beta)^{\lambda\alpha/r_n-1}}{\left[\sum_{i=1}^{n}(m_i+\beta)^{\alpha}\right]^{\lambda}}\right]\right.$$
$$\left.\times\prod_{i=1}^{n-1}\left[(m_i+\beta)^{p_i(1-\lambda\alpha/r_i)-1}(m_i+\beta)^{\lambda\alpha/r_i}\prod_{\substack{j=1\\(j\neq i)}}^{n-1}(m_j+\beta)^{\lambda\alpha/r_j-1}\right]^{q_n/p_i}\left(a_{m_i}^{(i)}\right)^{q_n}\right\}^{1/q_n}. \tag{3.5}$$

For $n\geq 3$, since $\sum_{i=1}^{n-1}(q_n/p_i)=1$, by Hölder's inequality again in (3.5), we have the following:

$$J\leq \left(\frac{\prod_{i=1}^{n}\Gamma(\lambda/r_i)}{\alpha^{n-1}\Gamma(\lambda)}\right)^{1/p_n}\prod_{i=1}^{n-1}\left\{\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\sum_{m_n=1}^{\infty}\frac{(m_n+\beta)^{\lambda\alpha/r_n-1}}{\left[\sum_{i=1}^{n}(m_i+\beta)^{\alpha}\right]^{\lambda}}\right.$$
$$\left.\times\left[(m_i+\beta)^{p_i(1-\lambda\alpha/r_i)-1}(m_i+\beta)^{\lambda\alpha/r_i}\prod_{\substack{j=1\\(j\neq i)}}^{n-1}(m_j+\beta)^{\lambda\alpha/r_j-1}\right]\left(a_{m_i}^{(i)}\right)^{p_i}\right\}^{1/p_i} \tag{3.6}$$
$$=\left(\frac{\prod_{i=1}^{n}\Gamma(\lambda/r_i)}{\alpha^{n-1}\Gamma(\lambda)}\right)^{1/p_n}\prod_{i=1}^{n-1}\left\{\sum_{m_i=1}^{\infty}\omega_i(m_i)(m_i+\beta)^{p_i(1-\lambda\alpha/r_i)-1}\left(a_{m_i}^{(i)}\right)^{p_i}\right\}^{1/p_i}.$$

Note that for $n=2$, by (3.5), we directly get (3.6). Hence, (3.3) is valid by (3.6) and (2.12).

Since $1/q_n + 1/p_n = 1$, by Hölder's inequality once again, it follows that

$$I = \sum_{m_n=1}^{\infty}\left[(m_n+\beta)^{\lambda\alpha/r_n-1/q_n}\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{i=1}^{n-1}a_{m_i}^{(i)}}{\left[\sum_{i=1}^{n}(m_i+\beta)^{\alpha}\right]^{\lambda}}\right]\times\left[(m_n+\beta)^{1/q_n-\lambda\alpha/r_n}a_{m_n}^{(n)}\right]$$
$$\leq J\left\{\sum_{m_n=1}^{\infty}(m_n+\beta)^{p_n(1-\lambda\alpha/r_n)-1}\left(a_{m_n}^{(n)}\right)^{p_n}\right\}^{1/p_n}. \tag{3.7}$$

By (3.3), we have (3.2). On the other hand, assuming that (3.2) is valid, setting

$$a_{m_n}^{(n)} := (m_n+\beta)^{\lambda\alpha q_n/r_n-1}\left[\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{i=1}^{n-1}a_{m_i}^{(i)}}{\left[\sum_{i=1}^{n}(m_i+\beta)^{\alpha}\right]^{\lambda}}\right]^{q_n-1}, \tag{3.8}$$

then we find that

$$J = \left\{\sum_{m_n=1}^{\infty}(m_n+\beta)^{p_n(1-\lambda\alpha/r_n)-1}\left(a_{m_n}^{(n)}\right)^{p_n}\right\}^{1/q_n} = I^{1/q_n}. \tag{3.9}$$

By (3.2), it follows that $J < \infty$. If $J = 0$, then (3.3) is naturally valid. Suppose that $J > 0$, by (3.2), we find that

$$0 < \sum_{m_n=1}^{\infty}(m_n+\beta)^{p_n(1-\lambda\alpha/r_n)-1}\left(a_{m_n}^{(n)}\right)^{p_n} = J^{q_n} = I$$
$$< \frac{\prod_{i=1}^{n}\Gamma(\lambda/r_i)}{\alpha^{n-1}\Gamma(\lambda)}\prod_{i=1}^{n}\left\{\sum_{m_i=1}^{\infty}(m_i+\beta)^{p_i(1-\lambda\alpha/r_i)-1}\left(a_{m_i}^{(i)}\right)^{p_i}\right\}^{1/p_i} < \infty. \tag{3.10}$$

Dividing out J^{q_n/p_n} into two sides of (3.10), we have the following:

$$\left\{\sum_{m_n=1}^{\infty}(m_n+\beta)^{p_n(1-\lambda\alpha/r_n)-1}\left(a_{m_n}^{(n)}\right)^{p_n}\right\}^{1/q_n} = J$$
$$< \frac{\prod_{i=1}^{n}\Gamma(\lambda/r_i)}{\alpha^{n-1}\Gamma(\lambda)}\prod_{i=1}^{n-1}\left\{\sum_{m_i=1}^{\infty}(m_i+\beta)^{p_i(1-\lambda\alpha/r_i)-1}\left(a_{m_i}^{(i)}\right)^{p_i}\right\}^{1/p_i}. \tag{3.11}$$

Then (3.3) is valid, which is equivalent to (3.2). □

Theorem 3.2. *Let the assumptions of Theorem 3.1 be fulfilled, then the same constant factor* $(\alpha^{1-n}/\Gamma(\lambda))\prod_{i=1}^{n}\Gamma(\lambda/r_i)$ *in* (3.2) *and* (3.3) *is the best possible.*

Proof. By (2.11) and

$$\lim_{N\to\infty}(m_n+\beta)^{\lambda\alpha/r_n}\sum_{m_{n-1}=1}^{N}\cdots\sum_{m_1=1}^{N}\frac{\prod_{j=1}^{n-1}(m_j+\beta)^{\lambda\alpha/r_j-1}}{\left[\sum_{i=1}^{n}(m_i+\beta)^{\alpha}\right]^{\lambda}}=\omega_n(m_n), \tag{3.12}$$

there exists $N_0\in\mathbf{N}$, such that for $N>N_0$,

$$(m_n+\beta)^{\lambda\alpha/r_n}\sum_{m_{n-1}=1}^{N}\cdots\sum_{m_1=1}^{N}\frac{\prod_{j=1}^{n-1}(m_j+\beta)^{\lambda\alpha/r_j-1}}{\left[\sum_{i=1}^{n}(m_i+\beta)^{\alpha}\right]^{\lambda}}>\frac{\alpha^{1-n}}{\Gamma(\lambda)}\prod_{j=1}^{n}\Gamma\left(\frac{\lambda}{r_j}\right)\left[1-O\left(\frac{1}{(m_n+\beta)^{\delta_n}}\right)\right], \tag{3.13}$$

where $\delta_n>0$. Setting

$$\widetilde{a}_{m_i}^{(i)}:=\begin{cases}(m_i+\beta)^{\lambda\alpha/r_i-1}, & m_i\le N,\\ 0, & m_i>N\ ,\end{cases}\qquad (i=1,\dots,n) \tag{3.14}$$

we find that

$$\begin{aligned}\widetilde{I}&:=\sum_{m_n=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{1}{\left[\sum_{i=1}^{n}(m_i+\beta)^{\alpha}\right]^{\lambda}}\prod_{i=1}^{n}\widetilde{a}_{m_i}^{(i)}\\&=\sum_{m_n=1}^{N}\frac{(m_n+\beta)^{\lambda\alpha/r_n}}{m_n+\beta}\sum_{m_{n-1}=1}^{N}\cdots\sum_{m_1=1}^{N}\frac{\prod_{i=1}^{n-1}(m_i+\beta)^{\lambda\alpha/r_i-1}}{\left[\sum_{i=1}^{n}(m_i+\beta)^{\alpha}\right]^{\lambda}}\\&>\sum_{m_n=1}^{N}\frac{1}{m_n+\beta}\cdot\frac{\alpha^{1-n}}{\Gamma(\lambda)}\prod_{j=1}^{n}\Gamma\left(\frac{\lambda}{r_j}\right)\left[1-O\left(\frac{1}{(m_n+\beta)^{\delta_n}}\right)\right]\\&=\frac{\alpha^{1-n}}{\Gamma(\lambda)}\prod_{j=1}^{n}\Gamma\left(\frac{\lambda}{r_j}\right)\left(\sum_{m_n=1}^{N}\frac{1}{m_n+\beta}\right)\\&\quad\times\left\{1-\left(\sum_{m_n=1}^{N}\frac{1}{m_n+\beta}\right)^{-1}\sum_{m_n=1}^{N}O\left(\frac{1}{(m_n+\beta)^{\delta_n+1}}\right)\right\}.\end{aligned} \tag{3.15}$$

If there exists a constant $k\le(\alpha^{1-n}/\Gamma(\lambda))\prod_{i=1}^{n}\Gamma(\lambda/r_i)$, such that (3.2) is still valid as we replace $(\alpha^{1-n}/\Gamma(\lambda))\prod_{i=1}^{n}\Gamma(\lambda/r_i)$ by k, then in particular, we have the following:

$$\widetilde{I}<k\prod_{i=1}^{n}\left\{\sum_{m_i=1}^{\infty}(m_i+\beta)^{p_i(1-\lambda\alpha/r_i)-1}\left(\widetilde{a}_{m_i}^{(i)}\right)^{p_i}\right\}^{1/p_i}=k\sum_{m_n=1}^{N}\frac{1}{m_n+\beta}. \tag{3.16}$$

In virtue of (3.15) and (3.16), it follows that

$$\frac{\alpha^{1-n}}{\Gamma(\lambda)}\prod_{j=1}^{n}\Gamma\left(\frac{\lambda}{r_j}\right)\left\{1-\left(\sum_{m_n=1}^{N}\frac{1}{m_n+\beta}\right)^{-1}\sum_{m_n=1}^{N}O\left(\frac{1}{(m_n+\beta)^{\delta_n+1}}\right)\right\}<k. \tag{3.17}$$

For $N \to \infty$, we have $(\alpha^{1-n}/\Gamma(\lambda))\prod_{i=1}^{n}\Gamma(\lambda/r_i) \le k$. Hence, $k = (\alpha^{1-n}/\Gamma(\lambda))\prod_{i=1}^{n}\Gamma(\lambda/r_i)$ is the best value of (3.2).

We conform that the constant factor $(\alpha^{1-n}/\Gamma(\lambda))\prod_{i=1}^{n}\Gamma(\lambda/r_i)$ in (3.3) is the best possible, otherwise we can get a contradiction by (3.7) that the constant factor in (3.2) is not the best possible. □

Remarks 3.3. (i) When $0 < \alpha \le 1$, the assumption $\lambda\alpha\max\{1/(2-\alpha),1\} \le \min_{1\le i\le n}\{r_i\}$ of two theorems becomes $\lambda\alpha \le \min_{1\le i\le n}\{r_i\}$. (ii) When $0 < \alpha \le 1$, $\beta = 0$, (3.2) reduces to (1.9). (iii) For $n = 2$, $r_1 = r, r_2 = s, p_1 = p, p_2 = q$, setting $\alpha = 1$, $\beta = -1/2$ in (3.2), then $\Gamma(\lambda/r_1)\Gamma(\lambda/r_2)/\Gamma(\lambda) = B(\lambda/r,\ \lambda/s)$, we obtain (1.5). Setting $\beta = -1/2$, $\lambda = 1$ in (3.2), we get (1.7).

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