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*Research Article*

# Local Boundedness of Weak Solutions for Nonlinear Parabolic Problem with $p(x)$-Growth

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We study the nonlinear parabolic problem with $p(x)$-growth conditions in the space $W^{1,x}L^{p(x)}(Q)$ and give a local boundedness theorem of weak solutions for the following equation $(\partial u/\partial t) + A(u) = 0$, where $A(u) = -\operatorname{div} a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u)$, $a(x,t,u,\nabla u)$ and $a_0(x,t,u,\nabla u)$ satisfy $p(x)$-growth conditions with respect to $u$ and $\nabla u$.

## 1. Introduction

The study of variational problems with nonstandard growth conditions is an interesting topic in recent years. $p(x)$-growth problems can be regarded as a kind of nonstandard growth problems and they appear in nonlinear elastic, electrorheological fluids and other physics phenomena. Many results have been obtained on this kind of problems, for example [1–9].

Let $Q$ be $\Omega \times (0,T)$, where $T > 0$ is given. In [8], the authors studied the following equation:

$$u_t - \operatorname{div}\left(|Du|^{p(x,t)-2} Du\right) = 0, \tag{1.1}$$

where $p_1 = \inf_{(x,t)\in Q} p(x,t) > \max\{1; 2N/(N+2)\}$, $p(x,t)$ is dependent on the space variable $x$ and the time variable $t$, $u$ is the local weak solution in the space $W^{1,p(x,t)}_{\text{loc}}(Q) \cap C(0,T; L^2_{\text{loc}}(\Omega))$, and the authors proved the local boundedness of the local weak solution in $Q$. In this paper, we will study the following more general problem:

$$\frac{\partial u}{\partial t} + A(u) = 0, \quad \text{in } Q, \tag{1.2}$$

$$u(x,t) = 0, \quad \text{on } \partial\Omega \times (0,T), \tag{1.3}$$

$$u(x,0) = \psi(x), \quad \text{in } \Omega, \tag{1.4}$$

where $\psi(x)$ is a given function in $L^2(\Omega)$ and $A : W_0^{1,x}L^{p(x)}(Q) \to W^{-1,x}L^{q(x)}(Q)$ is an elliptic operator of the form $A(u) = -\operatorname{div} a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u)$ with the coefficients $a$ and $a_0$ satisfying the classical Leray-Lions conditions. In [10], we have proved the existence of the solutions of (1.2)–(1.4) and have gotten $u \in W^{1,x}L^{p(x)}(Q) \cap L^\infty(0,T;L^2(\Omega))$; in this paper we will give the local boundedness theorem of the weak solutions in the framework space $W^{1,x}L^{p(x)}(Q)$, which can be considered as a special case of the space $W^{1,p(x,t)}(Q)$.

Many authors have already studied the boundedness of weak solutions of parabolic equation with $p$-growth conditions, where $p$ is a constant, for example [8, 11–15]. The boundedness of the weak solutions plays a central role in many aspects. Based on the boundedness, we can further study the regularity of the solutions. For example, first in [15] the author studied the equation

$$u_t - \operatorname{div}\ a(x,t,u,\nabla u) = b(x,t,u,\nabla u) \tag{1.5}$$

and got $L^\infty_{\text{loc}}$-estimates of the degenerate parabolic equation with $p$-growth conditions for $p > 1$, where $p$ is a constant, then in [16] the authors established the Hölder continuity of the equation for the singular case $1 < p < 2$, and in [17] the authors discussed Harnack estimates for the bounded solutions of the above parabolic equation for $p \geq 2$.

The space $W^{1,x}L^{p(x)}(Q)$ provides a suitable framework to discuss some physical problems. In [18], the authors studied a functional with variable exponent, $1 \leq p(x) \leq 2$, which provided a model for image denoising, enhancement, and restoration. Because in [18] the direction and speed of diffusion at each location depended on the local behavior, $p(x)$ only depended on the location $x$ in the image. Consider that the space $W^{1,x}L^{p(x)}(Q)$ was introduced and discussed in [10] and [19], we think that the space $W^{1,x}L^{p(x)}(Q)$ is a reasonable framework to discuss the $p(x)$-growth problem (1.2)–(1.4), where $p(x)$ only depends on the space variable $x$ similar to [18].

In this paper, let $a : Q \times R \times R^N \to R^N$ and $a_0 : Q \times R \times R^N \to R$ be the operators such that for any $s \in R$ and $\xi \in R^N$, $a(x,t,s,\xi)$ and $a_0(x,t,s,\xi)$ are both continuous in $(t,s,\xi)$ for a.e. $x \in \Omega$ and measurable in $x$ for all $(t, s, \xi) \in (0, T) \times R \times R^n$. They also satisfy that for a.e. $(x, t) \in Q$, any $s \in R$ and $\xi \neq \xi^* \in R^N$:

$$|a(x,t,s,\xi)| \le \alpha\left(|s|^{p(x)-1} + |\xi|^{p(x)-1}\right), \tag{1.6}$$

$$|a_0(x,t,s,\xi)| \le \alpha\left(|s|^{p(x)-1} + |\xi|^{p(x)-1}\right), \tag{1.7}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi^*)](\xi - \xi^*) > 0, \tag{1.8}$$

$$a(x,t,s,\xi)\xi + a_0(x,t,s,\xi)s \ge \beta\left(|\xi|^{p(x)} + |s|^{p(x)}\right), \tag{1.9}$$

where $\alpha, \beta > 0$ are constants.

Throughout this paper, unless special statement, we always suppose that $p(x)$ is $*$-continuous on $\overline{\Omega}$, that is, $\lim_{y \to x, y \in \overline{\Omega}} p(y) = p(x)$ for every $x \in \overline{\Omega}$, and satisfy

$$1 < p^- = \inf_{\Omega} p(x) \le p(x) \le \sup_{\Omega} p(x) = p^+ < \infty; \tag{1.10}$$

$q(x)$ is the conjugate function of $p(x)$.

*Definition 1.1.* A function $u \in W^{1,x}L^{p(x)}(Q) \cap L^\infty(0,T;L^2(\Omega))$ is called a weak solution of (1.2)–(1.4) if

$$-\int_Q u \frac{\partial \varphi}{\partial t} dx\,dt + \int_\Omega u\varphi\,dx\Big|_0^T + \int_Q \left[a(x,t,u,\nabla u)\nabla\varphi + a_0(x,t,u,\nabla u)\varphi\right] dx\,dt = 0 \tag{1.11}$$

for all $\varphi \in C^1(0,T;C_0^\infty(\Omega))$.

We will prove the following local boundedness theorem.

**Theorem 1.2.** *Let $p^- > \max\{1, 2N/(N+2)\}$. If $u$ is a nonnegative local weak solution of (1.2)–(1.4), then $u$ is locally bounded in $Q$. Moreover, there exists a constant $C = C(N, p_\rho^+, p_\rho^-, \rho)$ such that for any $Q(\rho^{p_\rho^+}, \rho) \in Q$ and any $\sigma \in (0,1)$,*

$$\sup_{Q(\sigma\rho^{p_\rho^+}, \sigma\rho)} u \le \max\left\{1, C(1-\sigma)^{-p_\rho^+(N+p_\rho^-)/N(q-\delta)}\left(\frac{1}{\left|Q\left(\rho^{p_\rho^+},\rho\right)\right|}\int_{Q(\rho^{p_\rho^+},\rho)} u^\delta\,dx\,dt\right)^{p_\rho^-/N(q-\delta)}\right\}, \tag{1.12}$$

*where for all $(x_0, t_0) \in Q$, $K_\rho = \{x \in \Omega \mid \max_{1\le i\le N}|x_i - x_{0,i}| < \rho\}$, $p_\rho^+ = \sup_{K_\rho} p(x)$, $p_\rho^- = \inf_{K_\rho} p(x)$, $Q(\rho^{p_\rho^+}, \rho) = K_\rho \times (t_0 - \rho^{p_\rho^+}, t_0)$, and $\max\{p_\rho^+, 2\} \le \delta < q = ((N+2)/N)p_\rho^-$.*

## 2. Preliminaries

We first recall some facts on spaces $L^{p(x)}(\Omega)$, $W^{m,p(x)}(\Omega)$, and $W^{m,x}L^{p(x)}(Q)$. For the details, see [19–21].

Although we assume (1.10) holds in this paper, in this section we introduce the general spaces $L^{p(x)}(\Omega)$, $W^{m,p(x)}(\Omega)$, and $W^{m,x}L^{p(x)}(Q)$.

Denote

$$E = \{\omega : \omega \text{ is a measurable function on } \Omega\}, \tag{2.1}$$

where $\Omega \subset R^N$ is an open subset.

Let $p(x) : \Omega \to [1, \infty]$ be an element in $E$. Denote $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$. For $u \in E$, we define

$$\rho(u) = \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} dx + \operatorname*{ess\,sup}_{x \in \Omega_\infty} |u(x)|. \tag{2.2}$$

The space $L^{p(x)}(\Omega)$ is

$$L^{p(x)}(\Omega) = \{u \in E : \exists \lambda > 0,\ \rho(\lambda u) < \infty\} \tag{2.3}$$

endowed with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda > 0 : \rho\left(\frac{u}{\lambda}\right) \le 1\right\}. \tag{2.4}$$

We define the conjugate function $q(x)$ of $p(x)$ by

$$q(x) = \begin{cases} \infty, & \text{if } p(x) = 1; \\ 1, & \text{if } p(x) = \infty; \\ \dfrac{p(x)}{p(x) - 1}, & \text{if } 1 < p(x) < \infty. \end{cases} \tag{2.5}$$

**Lemma 2.1** (see [21]). (1) *The dual space of* $L^{p(x)}(\Omega)$ *is* $L^{q(x)}(\Omega)$ *if* $1 \le p(x) < \infty$.

(2) *The space* $L^{p(x)}(\Omega)$ *is reflexive if and only if* (1.10) *is satisfied.*

**Lemma 2.2** (see [21]). *If* $1 \le p(x) < \infty$, $C_0^\infty(\Omega)$ *is dense in the space* $L^{p(x)}(\Omega)$ *and* $L^{p(x)}(\Omega)$ *is separable.*

**Lemma 2.3** (see [21]). *Let* $1 \le p(x) \le \infty$, *for every* $u(x) \in L^{p(x)}(\Omega)$ *and* $v(x) \in L^{q(x)}(\Omega)$, *we have*

$$\int_\Omega |u(x)v(x)| dx \le C\|u(x)\|_{L^{p(x)}(\Omega)} \|v(x)\|_{L^{q(x)}(\Omega)}, \tag{2.6}$$

*where* $C$ *is only dependent on* $p(x)$ *and* $\Omega$, *not dependent on* $u(x), v(x)$.

Next let $m > 0$ be an integer. For each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_i$ are nonnegative integers and $|\alpha| = \Sigma_{i=1}^{n} \alpha_i$, and denote by $D^\alpha$ the distributional derivative of order $\alpha$ with respect to the variable $x$.

We now introduce the generalized Lebesgue-Sobolev space $W^{m,p(x)}(\Omega)$ which is defined as

$$W^{m,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \le m \right\}. \tag{2.7}$$

$W^{m,p(x)}(\Omega)$ is a Banach space endowed with the norm

$$\|u\| = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^{p(x)}(\Omega)}. \tag{2.8}$$

The space $W_0^{m,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{m,p(x)}(\Omega)$. The dual space $(W_0^{m,p(x)}(\Omega))^*$ is denoted by $W^{-m,q(x)}(\Omega)$ equipped with the norm

$$\|f\|_{W^{-m,q(x)}(\Omega)} = \inf \Sigma_{|\alpha| \le m} \|f_\alpha\|_{L^{q(x)}(\Omega)}, \tag{2.9}$$

where infimum is taken on all possible decompositions

$$f = \Sigma_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha f_\alpha, \quad f_\alpha \in L^{q(x)}(\Omega). \tag{2.10}$$

**Lemma 2.4** (see [21]). (1) *$W^{m,p(x)}(\Omega)$ and $W_0^{m,p(x)}(\Omega)$ are separable if $1 \le p(x) < \infty$.*

(2) *$W^{m,p(x)}(\Omega)$ and $W_0^{m,p(x)}(\Omega)$ are reflexive if* (1.10) *holds.*

We define the space $W^{m,x}L^{p(x)}(Q)$ as the following:

$$W^{m,x}L^{p(x)}(Q) = \left\{ u \in L^{p(x)}(Q) : D^\alpha u \in L^{p(x)}(Q),\ |\alpha| \le m \right\}. \tag{2.11}$$

$W^{m,x}L^{p(x)}(Q)$ is a Banach space with the norm $\|u\| = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^{p(x)}(Q)}$, where $p(x)$ is independent of $t$.

The space $W_0^{m,x}L^{p(x)}(Q)$ is defined as the closure of $C_0^\infty(Q)$ in $W^{m,x}L^{p(x)}(Q)$, and $W_0^{m,x}L^{p(x)}(Q) \hookrightarrow L^{p(x)}(Q)$ is continuous embedding. Let $\overline{M}$ be the number of multiindexes $\alpha$ which satisfies $0 \le |\alpha| \le m$, then the space $W_0^{m,x}L^{p(x)}(Q)$ can be considered as a close subspace of the product space $\Pi_{i=1}^{\overline{M}} L^{p(x)}(Q)$. So if $1 < p(x) < \infty$, $\Pi_{i=1}^{\overline{M}} L^{p(x)}(Q)$ is reflexive and further we can get that the space $W_0^{m,x}L^{p(x)}(Q)$ is reflexive. The dual space $(W_0^{m,x}L^{p(x)}(Q))^*$ is denoted by $W^{-m,x}L^{q(x)}(Q)$ equipped with the norm

$$\|f\|_{W^{-m,x}L^{q(x)}(Q)} = \sup_{\|u\|_{W_0^{m,x}L^{p(x)}(Q)} \le 1} |\langle f, u \rangle| = \inf \Sigma_{|\alpha| \le m} \|f_\alpha\|_{L^{q(x)}(Q)}, \tag{2.12}$$

where infimum is taken on all possible decompositions

$$f = \Sigma_{|\alpha| \le m} (-1)^{|\alpha|} D_x^\alpha f_\alpha, \quad f_\alpha \in L^{q(x)}(Q). \tag{2.13}$$

Next, we will introduce some results in [22].

**Lemma 2.5.** *Let* $\{Y_n\}, n = 0, 1, 2, \ldots$, *be a sequence of positive numbers, satisfying the inequalities* $Y_{n+1} \leq Cb^n Y_n^{1+\alpha}$, *where* $C, b > 1$ *and* $\alpha > 0$ *are given numbers. If* $Y_0 \leq C^{-1/\alpha} b^{-1/\alpha^2}$, *then* $\{Y_n\}$ *converges to 0 as* $n \to \infty$.

**Lemma 2.6.** *There exists a constant* $C$ *depending only on* $N, r, m$, *such that for every* $v \in L^\infty(0, T; L^m(\Omega)) \cap L^r(0, T; W_0^{1,r}(\Omega))$,

$$\int_Q |v(x,t)|^q dx\,dt \leq C^q \left( \int_Q |Dv(x,t)|^r dx\,dt \right) \left( \sup_{0<t<T} \int_\Omega |v(x,t)|^m dx \right)^{r/N}, \tag{2.14}$$

*where* $q = r((N+m)/N)$.

*Remark 2.7.* In [10], we have gotten that for the Galerkin solutions $u_n \in C^1(0, T; C_0^\infty(\Omega))$, $u_n \to u$ strongly in $L^1(Q)$, $u_n \rightharpoonup u$ weakly in $W^{1,x}L^{p(x)}(Q)$, $a(x, t, u_n, \nabla u_n) \rightharpoonup a(x, t, u, \nabla u)$ weakly in $L^{q(x)}(Q)$ and $a_0(x, t, u_n, \nabla u_n) \rightharpoonup a_0(x, t, u, \nabla u)$ weakly in $L^{q(x)}(Q)$.

## 3. Proof of the Theorem

Suppose that $u$ is a weak solution of (1.2)–(1.4), then there exists $\delta > max\{p^+, 2\}$ such that

$$\int_Q |u|^\delta dx\,dt < \infty. \tag{3.1}$$

Indeed, by Young's inequality, we have

$$\int_{Q\cap\{p^-<p(x)\}} |\nabla u|^{p^-} dx\,dt + \int_{Q\cap\{p^-=p(x)\}} |\nabla u|^{p^-} dx\,dt \leq |Q| + \int_Q |\nabla u|^{p(x)} dx\,dt < \infty, \tag{3.2}$$

where $|Q|$ is the Lebesgue measure of $Q$. Since $W_0^{1,x}L^{p(x)}(Q) \hookrightarrow W_0^{1,x}L^{p^-}(Q) = L^{p^-}(0, T; W_0^{1,p^-}(\Omega))$ and $u \in W^{1,x}L^{p(x)}(Q) \cap L^\infty(0, T; L^2(\Omega))$, we can get $u \in L^\infty(0, T; L^2(\Omega)) \cap L^{p^-}(0, T; W_0^{1,p^-}(\Omega))$. Then by Lemma 2.6, we get

$$\int_Q |u|^\delta dx\,dt \leq C^\delta \left( \int_Q |Du|^{p^-} dx\,dt \right) \left( \sup_{0<t<T} \int_\Omega |u|^2 dx \right)^{2/N}, \tag{3.3}$$

where $\delta = ((N+2)/N)p^-$. Thus the desired result is obtained.

We define $u_+ = \max\{u, 0\}$. Fix a point $(x_0, t_0)$ in $Q$. Let $0 < \rho < 1$, $0 < \theta < 1$, and $Q(\theta, \rho) \equiv K_\rho \times (t_0 - \theta, t_0) \subset Q$. Fix $\sigma \in (0, 1)$ and consider the sequences

$$\rho_m = \sigma\rho + \frac{1-\sigma}{2^m}\rho, \quad \theta_m = \sigma\theta + \frac{1-\sigma}{2^m}\theta, \quad m = 0, 1, 2, \ldots, \tag{3.4}$$

and the corresponding cylinders $Q_m = Q(\theta_m, \rho_m)$. It follows from the definitions that

$$Q_0 = Q(\theta, \rho), \qquad Q_\infty = Q(\sigma\theta, \sigma\rho). \tag{3.5}$$

We consider also the boxes $\tilde{Q}_m = Q(\tilde{\theta}_m, \tilde{\rho}_m)$, where for $m = 0, 1, 2, \ldots,$

$$\tilde{\rho}_m = \frac{\rho_m + \rho_{m+1}}{2}, \qquad \tilde{\theta}_m = \frac{\theta_m + \theta_{m+1}}{2}. \tag{3.6}$$

For these boxes, we have the inclusion

$$Q_{m+1} \subset \tilde{Q}_m \subset Q_m, \quad m = 0, 1, 2, \ldots. \tag{3.7}$$

We introduce the sequence of increasing levels

$$k_m = k - \frac{k}{2^m}, \quad m = 0, 1, 2, \ldots, \ k > 0 \text{ to be chosen.} \tag{3.8}$$

Let $\{u_n\}$ be the Galerkin solutions in [10]. Similarly, we can get $u_n - u$ is bounded in $L^\delta(Q)$. Since $u_n - u$ converges to 0 in $L^1(Q)$, by interpolation inequality, we have

$$\|u_n - u\|_{L^{p^+}(Q)} \le \|u_n - u\|_{L^1(Q)}^{\lambda} \|u_n - u\|_{L^\delta(Q)}^{1-\lambda}, \tag{3.9}$$

where $0 < \lambda < 1$, $1/p^+ = \lambda + \delta/(1-\lambda)$. Furthermore, $u_n \to u$ strongly in $L^{p^+}(Q)$. Since $L^{p^+}(Q) \hookrightarrow L^{p(x)}(Q)$, $u_n \to u$ strongly in $L^{p(x)}(Q)$. In the same way, we obtain that $u_n \to u$ strongly in $L^2(Q)$; furthermore, we get $\|u_n(t) - u(t)\|_{L^2(\Omega)} \to 0$ for a.e. $t \in [0, T]$.

Let $Q_m^t = K_{\rho_m} \times (t_0 - \theta_m, t)$ and $\zeta$ be the smooth cutoff function satisfying

$$\begin{gathered} 0 \le \zeta \le 1, \quad \zeta \equiv 0 \quad \text{on } \partial K_{\rho_m} \times (t_0 - \theta_m, t_0) \cup K_{\rho_m} \times \{t\}, \quad \zeta \equiv 1 \quad \text{in } \tilde{Q}_m, \\ |\nabla\zeta| \le \frac{2^{m+2}}{(1-\sigma)\rho}, \qquad 0 \le \zeta_t \le \frac{2^{m+2}}{(1-\sigma)\theta}. \end{gathered} \tag{3.10}$$

Take $\varphi = (u_n - k_{m+1})_+ \zeta^{p_\rho^+}$ as the testing function in the following equation:

$$\int_{Q_m^t} \varphi \frac{\partial u_n}{\partial t} dx\, dt + \int_{Q_m^t} a(x, t, u_n, \nabla u_n) \nabla\varphi\, dx\, dt + \int_{Q_m^t} a_0(x, t, u_n, \nabla u_n)\varphi\, dx\, dt = 0. \tag{3.11}$$

First, by $\|u_n(t) - u(t)\|_{L^2(\Omega)} \to 0$ for a.e. $t \in [0,T]$ and $u_n \to u$ strongly in $L^2(Q)$, we get

$$\begin{aligned}
&\lim_{n\to\infty}\int_{Q_m^t} \varphi \frac{\partial u_n}{\partial t}\,dx\,dt \\
&\quad= \lim_{n\to\infty}\frac{1}{2}\int_{Q_m^t}\frac{\partial}{\partial t}(u_n - k_{m+1})_+^2 \zeta^{p_\rho^+}\,dx\,dt \\
&\quad= \lim_{n\to\infty}\Bigg(\frac{1}{2}\int_{K_{\rho m}}(u_n - k_{m+1})_+^2\zeta^{p_\rho^+}(x,t)\,dx - \frac{1}{2}\int_{K_{\rho m}}(u_n - k_{m+1})_+^2\zeta^{p_\rho^+}(x,t_0-\theta_m)\,dx \\
&\qquad\qquad - \frac{p_\rho^+}{2}\int_{Q_m^t}(u_n - k_{m+1})_+^2\zeta^{p_\rho^+-1}|\zeta_t|\,dx\,dt\Bigg) \\
&\quad= \frac{1}{2}\int_{K_{\rho m}}(u - k_{m+1})_+^2\zeta^{p_\rho^+}(x,t)\,dx - \frac{1}{2}\int_{K_{\rho m}}(u - k_{m+1})_+^2\zeta^{p_\rho^+}(x,t_0-\theta_m)\,dx \\
&\qquad - \frac{p_\rho^+}{2}\int_{Q_m^t}(u - k_{m+1})_+^2\zeta^{p_\rho^+-1}|\zeta_t|\,dx\,dt.
\end{aligned} \tag{3.12}$$

By Fatou's lemma, we get

$$\begin{aligned}
&\lim_{n\to\infty}\left(\int_{Q_m^t} a(x,t,u_n,\nabla u_n)\nabla(u_n - k_{m+1})_+\zeta^{p_\rho^+}\,dx\,dt + \int_{Q_m^t\cap\{u_n > k_{m+1}\}} \mathrm{a}_0(x,t,u_n,\nabla u_n)u_n\zeta^{p_\rho^+}\,dx\,dt\right) \\
&\quad\geq \int_{Q_m^t} a(x,t,u,\nabla u)\nabla(u - k_{m+1})_+\zeta^{p_\rho^+}\,dx\,dt + \int_{Q_m^t\cap\{u > k_{m+1}\}} a_0(x,t,u,\nabla u)u\zeta^{p_\rho^+}\,dx\,dt.
\end{aligned} \tag{3.13}$$

Because $(u_n)_+ \to u_+$ strongly in $L^{p(x)}(Q)$ and $a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u)$ weakly in $L^{q(x)}(Q)$, we get

$$\lim_{n\to\infty}\int_{Q_m^t} a(x,t,u_n,\nabla u_n)(u_n - k_{m+1})_+\zeta^{p_\rho^+-1}\nabla\zeta\,dx\,dt = \int_{Q_m^t} a(x,t,u,\nabla u)(u - k_{m+1})_+\zeta^{p_\rho^+-1}\nabla\zeta\,dx\,dt. \tag{3.14}$$

Since $(u_n)_+ \to u_+$ strongly in $L^{p(x)}(Q)$ and $a_0(x,t,u_n,\nabla u_n) \rightharpoonup a_0(x,t,u,\nabla u)$ weakly in $L^{q(x)}(Q)$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left(\int_{Q_m^t} a_0(x,t,u_n,\nabla u_n)(u_n - k_{m+1})_+\zeta^{p_\rho^+}\,dx\,dt - \int_{Q_m^t\cap\{u_n > k_{m+1}\}} a_0(x,t,u_n,\nabla u_n)u_n\zeta^{p_\rho^+}\,dx\,dt\right) \\
&\quad= \int_{Q_m^t} a_0(x,t,u,\nabla u)(u - k_{m+1})_+\zeta^{p_\rho^+}\,dx\,dt - \int_{Q_m^t\cap\{u > k_{m+1}\}} a_0(x,t,u,\nabla u)u\zeta^{p_\rho^+}\,dx\,dt.
\end{aligned} \tag{3.15}$$

Then for the remaining parts of (3.11), we get

$$\begin{aligned}
I &= \lim_{n\to\infty}\int_{Q_m^t} a(x,t,u_n,\nabla u_n)\nabla\varphi + a_0(x,t,u_n,\nabla u_n)\varphi\,dx\,dt \\
&= \lim_{n\to\infty}\Bigg(\int_{Q_m^t} a(x,t,u_n,\nabla u_n)\nabla(u_n-k_{m+1})_+\zeta^{p_\rho^+}\,dx\,dt \\
&\qquad + p_\rho^+\int_{Q_m^t} a(x,t,u_n,\nabla u_n)(u_n-k_{m+1})_+\zeta^{p_\rho^+-1}\nabla\zeta\,dx\,dt \\
&\qquad + \int_{Q_m^t} a_0(x,t,u_n,\nabla u_n)(u_n-k_{m+1})_+\zeta^{p_\rho^+}\,dx\,dt \\
&\qquad + \int_{Q_m^t\cap\{u_n>k_{m+1}\}} a_0(x,t,u_n,\nabla u_n)u_n\zeta^{p_\rho^+}\,dx\,dt \\
&\qquad - \int_{Q_m^t\cap\{u_n>k_{m+1}\}} a_0(x,t,u_n,\nabla u_n)u_n\zeta^{p_\rho^+}\,dx\,dt\Bigg) \qquad (3.16)\\
&\geq \int_{Q_m^t} a(x,t,u,\nabla u)\nabla(u-k_{m+1})_+\zeta^{p_\rho^+}\,dx\,dt \\
&\quad + \int_{Q_m^t\cap\{u>k_{m+1}\}} a_0(x,t,u,\nabla u)u\zeta^{p_\rho^+}\,dx\,dt \\
&\quad + p_\rho^+\int_{Q_m^t} a(x,t,u,\nabla u)(u-k_{m+1})_+\zeta^{p_\rho^+-1}\nabla\zeta\,dx\,dt \\
&\quad + \int_{Q_m^t} a_0(x,t,u,\nabla u)(u-k_{m+1})_+\zeta^{p_\rho^+}\,dx\,dt \\
&\quad - \int_{Q_m^t\cap\{u>k_{m+1}\}} a_0(x,t,u,\nabla u)u\zeta^{p_\rho^+}\,dx\,dt.
\end{aligned}$$

By (1.6), (1.7), and (1.9),

$$\begin{aligned}
I \geq{}& \beta\left(\int_{Q_m^t}|\nabla(u-k_{m+1})_+|^{p(x)}\zeta^{p_\rho^+}\,dx\,dt + \int_{Q_m^t\cap\{u>k_{m+1}\}}|u|^{p(x)}\zeta^{p_\rho^+}\,dx\,dt\right) \\
&- p_\rho^+\alpha\int_{Q_m^t\cap\{u>k_{m+1}\}}|u|^{p(x)}\zeta^{p_\rho^+-1}|\nabla\zeta|\,dx\,dt \\
&- p_\rho^+\alpha\int_{Q_m^t}|\nabla(u-k_{m+1})_+|^{p(x)-1}(u-k_{m+1})_+\zeta^{p_\rho^+-1}|\nabla\zeta|\,dx\,dt \\
&- \alpha\int_{Q_m^t\cap\{u>k_{m+1}\}}|u|^{p(x)}\zeta^{p_\rho^+}\,dx\,dt - \alpha\int_{Q_m^t}|\nabla(u-k_{m+1})_+|^{p(x)-1}|u|\zeta^{p_\rho^+}\,dx\,dt. \qquad (3.17)
\end{aligned}$$

As $(p_\rho^+ - 1)(p(x))/(p(x)-1) > p_\rho^+$, by Young's inequality and Hölder's inequality, we have

$$\begin{aligned}
&\int_{Q_m^t} |\nabla(u-k_{m+1})_+|^{p(x)-1}(u-k_{m+1})_+\zeta^{p_\rho^+-1}|\nabla\zeta|dx\,dt \\
&\quad\le \varepsilon\int_{Q_m^t} |\nabla(u-k_{m+1})_+|^{p(x)}\zeta^{p_\rho^+}dx\,dt + C(\varepsilon)\int_{Q_m^t}(u-k_{m+1})_+^{p(x)}|\nabla\zeta|^{p(x)}dx\,dt \\
&\quad\le \varepsilon\int_{Q_m^t} |\nabla(u-k_{m+1})_+|^{p(x)}\zeta^{p_\rho^+}dx\,dt + C(\varepsilon)\int_{Q_m^t}(u-k_{m+1})_+^{p_\rho^+}|\nabla\zeta|^{p_\rho^+}dx\,dt \\
&\qquad + C(\varepsilon)\int_{Q_m^t}\chi[(u-k_{m+1})_+>0]dx\,dt.
\end{aligned} \tag{3.18}$$

In the same way, by $p_\rho^+(p(x)/(p(x)-1)) > p_\rho^+$ and Young's inequality, we have

$$\begin{aligned}
\int_{Q_m^t} |\nabla(u-k_{m+1})_+|^{p(x)-1}|u|\zeta^{p_\rho^+}dx\,dt &\le \varepsilon\int_{Q_m^t}|\nabla(u-k_{m+1})_+|^{p(x)}\zeta^{p_\rho^+}dx\,dt \\
&\quad + C(\varepsilon)\int_{Q_m^t\cap\{u>k_{m+1}\}}|u|^{p(x)}dx\,dt.
\end{aligned} \tag{3.19}$$

For a set $A$, meas $A$ is the Lebesgue measure of $A$. Let $|A_{m+1}| \equiv \text{meas}\{(x,t)\in Q_m \mid u(x,t) > k_{m+1}\}$ and $\varepsilon\alpha = \beta/4$. By (3.11)–(3.19), we get

$$\begin{aligned}
&\sup_{t_0-\theta_m<t<t_0}\int_{K_{\rho m}}(u-k_{m+1})_+^2\zeta^{p_\rho^+}dx + \int_{Q_m}|\nabla(u-k_{m+1})_+|^{p(x)}\zeta^{p_\rho^+}dx\,dt \\
&\quad\le \int_{Q_m}(u-k_{m+1})_+^2\zeta^{p_\rho^+-1}|\zeta_t|dx\,dt + C\int_{Q_m}(u-k_{m+1})_+^{p_\rho^+}|\nabla\zeta|^{p_\rho^+}dx\,dt + C|A_{m+1}| \\
&\qquad + C\int_{Q_m\cap\{u>k_{m+1}\}}|u|^{p(x)}\zeta^{p_\rho^+-1}|\nabla\zeta|dx\,dt + C\int_{Q_m\cap\{u>k_{m+1}\}}|u|^{p(x)}dx\,dt.
\end{aligned} \tag{3.20}$$

Moreover, we observe that for $s>0$ to be determined later,

$$\begin{aligned}
\int_{Q_m}(u-k_m)_+^s dx\,dt &\ge \int_{Q_m}(u-k_m)_+^s\chi[u>k_{m+1}]dx\,dt \\
&\ge (k_{m+1}-k_m)^s|A_{m+1}| \\
&= \frac{k^s}{2^{(m+1)s}}|A_{m+1}|,
\end{aligned} \tag{3.21}$$

thus we get

$$|A_{m+1}| \le \frac{2^{(m+1)s}}{k^s}\int_{Q_m}(u-k_m)_+^s dx\,dt. \tag{3.22}$$

Then for $s = 2$ and $s = p_\rho^+$ in (3.22), by Hölder inequality, we obtain respectively

$$\begin{aligned}\int_{Q_m}(u-k_{m+1})_+^2 dx\,dt &\le \left(\int_{Q_m}(u-k_{m+1})_+^\delta dx\,dt\right)^{2/\delta}|A_{m+1}|^{1-2/\delta}\\ &\le C\frac{2^{(\delta-2)m}}{k^{\delta-2}}\int_{Q_m}(u-k_m)_+^\delta dx\,dt,\end{aligned} \tag{3.23}$$

$$\begin{aligned}\int_{Q_m}(u-k_{m+1})_+^{p_\rho^+} dx\,dt &\le \left(\int_{Q_m}(u-k_{m+1})_+^\delta dx\,dt\right)^{p_\rho^+/\delta}|A_{m+1}|^{1-p_\rho^+/\delta}\\ &\le C\frac{2^{(\delta-p_\rho^+)m}}{k^{\delta-p_\rho^+}}\int_{Q_m}(u-k_m)_+^\delta dx\,dt.\end{aligned} \tag{3.24}$$

For the integral involving $|u|^{p(x)}$, first we write $k_m = k_{m+1}((2^{m+1}-2)/(2^{m+1}-1))$, then we obtain

$$\begin{aligned}\int_{Q_m}(u-k_m)_+^\delta dx\,dt &\ge \int_{Q_m}(u-k_m)_+^\delta \chi[u>k_{m+1}]dx\,dt\\ &\ge \int_{Q_m}|u|^\delta\left(1-\frac{2^{m+1}-2}{2^{m+1}-1}\right)^\delta \chi[u>k_{m+1}]dx\,dt\\ &\ge \frac{C}{2^{m\delta}}\int_{Q_m}|u|^\delta\chi[u>k_{m+1}]dx\,dt.\end{aligned} \tag{3.25}$$

By Young's inequality and (3.25), we get

$$\begin{aligned}&\int_{Q_m\cap\{u>k_{m+1}\}}|u|^{p(x)}\zeta^{p_\rho^+-1}|\nabla\zeta|+|u|^{p(x)}dx\,dt\\ &\quad\le C\frac{2^m}{(1-\sigma)\rho}\int_{Q_m\cap\{u>k_{m+1}\}}|u|^{p(x)}dx\,dt\\ &\quad\le C\frac{2^m}{(1-\sigma)\rho}\left(\int_{Q_m\cap\{u>k_{m+1}\}}|u|^\delta dx\,dt+|A_{m+1}|\right)\\ &\quad\le C\frac{2^m}{(1-\sigma)\rho}\left(2^{m\delta}\int_{Q_m}(u-k_m)_+^\delta dx\,dt+|A_{m+1}|\right).\end{aligned} \tag{3.26}$$

Let $1 < k \le (1/\rho^{p_\rho^+-1})^{(1/\delta-p_\rho^+)}$, then $1/\rho \le 1/\rho^{p_\rho^+} k^{\delta-p_\rho^+}$. By (3.20)–(3.24) and (3.26), we obtain

$$\begin{aligned}
&\sup_{t_0-\theta_m<t<t_0} \int_{K_{\rho m}} (u-k_{m+1})_+^2 \zeta^{p_\rho^+} dx + \int_{Q_m} |\nabla(u-k_{m+1})_+|^{p(x)} \zeta^{p_\rho^+} dx\, dt \\
&\quad \le C\left( \frac{2^{(\delta-2)m}}{k^{\delta-2}} \frac{2^{m+2}}{(1-\sigma)\theta} + \frac{2^{(\delta-p_\rho^+)m}}{k^{\delta-p_\rho^+}} \frac{2^{m+2}}{(1-\sigma)\rho} + \frac{2^{(m+1)\delta}}{k^\delta} + \frac{2^m}{(1-\sigma)\rho} 2^{m\delta} + \frac{2^m}{(1-\sigma)\rho} \frac{2^{(m+1)\delta}}{k^\delta} \right) \\
&\qquad \times \int_{Q_m} (u-k_m)_+^\delta dx\, dt \\
&\quad \le C \frac{2^{m(1+\delta)}}{(1-\sigma)^{p_\rho^+}} \left( \frac{1}{\theta k^{\delta-2}} + \frac{1}{\rho^{p_\rho^+} k^{\delta-p_\rho^+}} \right) \int_{Q_m} (u-k_m)_+^\delta dx\, dt.
\end{aligned} \tag{3.27}$$

By Young's inequality,

$$\begin{aligned}
\int_{\widetilde{Q}_m} |\nabla(u-k_{m+1})_+|^{p_\rho^-} dx\, dt &\le \int_{\widetilde{Q}_m} |\nabla(u-k_{m+1})_+|^{p(x)} dx\, dt + \left| A_{m+1} \cap \widetilde{Q}_m \right| \\
&\le \int_{\widetilde{Q}_m} |\nabla(u-k_{m+1})_+|^{p(x)} dx\, dt + |A_{m+1}|.
\end{aligned} \tag{3.28}$$

Moreover, by (3.27), we can get

$$\begin{aligned}
&\sup_{t_0-\theta_m<t<t_0} \int_{K_{\tilde\rho m}} (u-k_{m+1})_+^2 dx + \int_{\widetilde{Q}_m} |\nabla(u-k_{m+1})_+|^{p_\rho^-} dx\, dt \\
&\quad \le C \frac{2^{m(1+\delta)}}{(1-\sigma)^{p_\rho^+}} \left( \frac{1}{\theta k^{\delta-2}} + \frac{1}{\rho^{p_\rho^+} k^{\delta-p_\rho^+}} \right) \int_{Q_m} (u-k_m)_+^\delta dx\, dt.
\end{aligned} \tag{3.29}$$

Next we define the smooth cutoff function $\widetilde{\zeta}_m$ in $\widetilde{Q}_m$

$$\begin{gathered}
0 \le \widetilde{\zeta}_m \le 1, \quad \widetilde{\zeta}_m \equiv 0 \quad \text{on } \partial K_{\tilde\rho_m} \times \left(t_0 - \widetilde{\theta}_m, t_0\right), \\
\widetilde{\zeta}_m \equiv 1 \quad \text{in } Q_{m+1}, \quad \left|\nabla \widetilde{\zeta}_m\right| \le \frac{2^{m+2}}{(1-\sigma)\rho}.
\end{gathered} \tag{3.30}$$

For the function $(u-k_{m+1})_+\widetilde{\zeta}_m$, by Lemma 2.6 and (3.29), we get

$$\begin{aligned}
&\int_{\widetilde{Q}_m}(u-k_{m+1})_+^q\widetilde{\zeta}_m^q dx\,dt \\
&\quad\le C\left(\int_{\widetilde{Q}_m}|\nabla(u-k_{m+1})_+|^{p_\rho^-}dx\,dt+\int_{\widetilde{Q}_m}|(u-k_{m+1})_+|^{p_\rho^-}\left|\nabla\widetilde{\zeta}_m\right|^{p_\rho^-}dx\,dt\right) \\
&\quad\quad\times\left(\sup_{t_0-\theta_m<t<t_0}\int_{K_{\widetilde{\rho}_m}}(u-k_{m+1})_+^2dx\right)^{p_\rho^-/N} \\
&\quad\le C\left(\frac{2^{m(1+\delta)}}{(1-\sigma)^{p_\rho^+}}\left(\frac{1}{\theta k^{\delta-2}}+\frac{1}{\rho^{p_\rho^+}k^{\delta-p_\rho^+}}\right)\right)^{1+p_\rho^-/N}\left(\int_{Q_m}(u-k_m)_+^\delta dx\,dt\right)^{1+p_\rho^-/N}.
\end{aligned}\tag{3.31}$$

Finally, we define $Y_m=(1/|Q_m|)\int_{Q_m}(u-k_m)_+^\delta dx\,dt$, $m=0,1,2,\ldots$. Let $\theta=\rho^{p_\rho^+}$; by Hölder inequality, we obtain

$$\begin{aligned}
Y_{m+1}&=\frac{1}{|Q_{m+1}|}\int_{Q_{m+1}}(u-k_{m+1})_+^\delta dx\,dt \\
&\le C\left(\frac{1}{\left|\widetilde{Q}_m\right|}\int_{\widetilde{Q}_m}(u-k_{m+1})_+^\delta\widetilde{\zeta}_m^\delta dx\,dt\right) \\
&\le C\left(\frac{1}{\left|\widetilde{Q}_m\right|}\int_{\widetilde{Q}_m}(u-k_{m+1})_+^q\widetilde{\zeta}_m^q dx\,dt\right)^{\delta/q}\left(\frac{|A_{m+1}|}{|Q_m|}\right)^{1-\delta/q} \\
&\le C\left(\frac{1}{\left|\widetilde{Q}_m\right|}\int_{\widetilde{Q}_m}(u-k_{m+1})_+^q\widetilde{\zeta}_m^q dx\,dt\right)^{\delta/q}\left(\frac{2^{m\delta}}{k^\delta}Y_m\right)^{1-\delta/q} \\
&\le\frac{Cb^m}{(\rho(1-\sigma))^{p_\rho^+((N+p_\rho^-)/N)\delta/q}k^{\delta/q(q-\delta)}}Y_m^{1+\delta p_\rho^-/Nq},
\end{aligned}\tag{3.32}$$

where $b=2^{\delta(1+\delta p_\rho^-/qN+(1/q)(1+p_\rho^-/N))}$. Then by Lemma 2.5, we have $Y_m\to 0$ as $m\to\infty$, provided $k=\max\{\overline{k},1\}$ is chosen to satisfy

$$Y_0=\frac{1}{\left|Q\left(\rho^{p_\rho^+},\rho\right)\right|}\int_{Q(\rho^{p_\rho^+},\rho)}u^\delta dx\,dt=C\overline{k}^{(q-\delta)N/p_\rho^-}(1-\sigma)^{((N+p_\rho^-)/p_\rho^-)p_\rho^+}.\tag{3.33}$$

By $Y_m\to 0$, we can get $\int_{Q_0}(u-k_m)_+^\delta\chi_{Q_m}dx\,dt\to 0$ as $m\to\infty$. Since $(u-k_m)_+^\delta\chi_{Q_m}\le(|u|+k)^\delta$ and $(u-k_m)_+^\delta\chi_{Q_m}\to(u-k)_+^\delta\chi_{Q(\sigma\theta,\sigma\rho)}$ a.e. in $Q_0$, by Lebesuge's theorem we get $\int_{Q_0}(u-k_m)_+^\delta\chi_{Q_m}dx\,dt\to\int_{Q_0}(u-k)_+^\delta\chi_{Q(\sigma\theta,\sigma\rho)}dx\,dt=0$. So we obtain $u\le k$ a.e. in $Q(\sigma\theta,\sigma\rho)$.

Thus we get

$$\sup_{Q(\sigma\rho^{p_\rho^+},\sigma\rho)} u \leq \max\left\{1, C(1-\sigma)^{-p_\rho^+(N+p_\rho^-)/N(q-\delta)}\left(\frac{1}{\left|Q\left(\rho^{p_\rho^+},\rho\right)\right|}\int_{Q(\rho^{p_\rho^+},\rho)} u^\delta dx\,dt\right)^{p_\rho^-/N(q-\delta)}\right\}. \tag{3.34}$$

*Remark 3.1.* In this paper, we study the boundedness of weak solution in the case $p^- > max\{1, 2N/(N+2)\}$. For the singular case $1 < p^- \leq \max\{1, 2N/(N+2)\}$, the conditions in the paper are not enough. In [22], there is a counterexample in §13 of Chapter XII. The author studied the solutions of the homogeneous equation

$$\begin{gathered} u_t - \operatorname{div}|Du|^{p-2}Du = 0, \quad \text{in } Q, \\ u \in C_{\text{loc}}\left(0,T; L^2_{\text{loc}}(\Omega)\right) \cap L^p_{\text{loc}}\left(0,T; W^{1,p}_{\text{loc}}(\Omega)\right), \quad p > 1, \end{gathered} \tag{3.35}$$

where

$$u \in L^1_{\text{loc}}(Q), \quad u \overline{\in} L^{1+\varepsilon}_{\text{loc}}(Q) \quad \forall \varepsilon \in (0,1), \quad p = \frac{2N}{N+1}, \tag{3.36}$$

and proved that the solution $u$ is unbounded in $Q$.

*Remark 3.2.* In general, we consider the equation

$$\frac{\partial u}{\partial t} + A(u) = f(x,t) \geq 0, \quad \text{in } Q, \tag{3.37}$$

where

$$f(x,t)^{\delta/(\delta-1)} \in L^{(N+p^-)/(1-h_0)p^-}(Q), \tag{3.38}$$

$h_0 \in (0,1]$ and $A : W_0^{1,x}L^{p(x)}(Q) \to W^{-1,x}L^{q(x)}(Q)$ is an elliptic operator of the form $A(u) = -\operatorname{div} a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u)$. $a(x,t,s,\xi)$ and $a_0(x,t,s,\xi)$ satisfy that for a.e. $(x,t) \in Q$, any $s \in R$ and $\xi \neq \xi^* \in R^N$:

$$\begin{gathered} |a(x,t,s,\xi)| \leq \alpha\left(C(x,t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}\right), \\ |a_0(x,t,s,\xi)| \leq \alpha\left(C(x,t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}\right), \\ [a(x,t,s,\xi) - a(x,t,s,\xi^*)](\xi - \xi^*) > 0, \\ a(x,t,s,\xi)\xi + a_0(x,t,s,\xi)s \geq \beta\left(|\xi|^{p(x)} + |s|^{p(x)}\right), \end{gathered} \tag{3.39}$$

where $C(x,t) \geq 0$, $C(x,t)^{p(x)/(p(x)-1)} \in L^{(N+p^-)/(1-h_0)p^-}(Q)$, and $\alpha, \beta > 0$ are constants.

Similarly, we can get the following theorem.

**Theorem 3.3.** *Let* $p^- > max\{1, 2N/(N+2)\}$. *If* $u$ *is a nonnegative local weak solution of* (3.37), (1.3), *and* (1.4), *then* $u$ *is locally bounded in* $Q$. *Moreover, there exists a constant* $C = C(N, p_\rho^+, p_\rho^-, \rho)$ *such that for any* $Q(\rho^{p_\rho^+}, \rho) \in Q$ *and any* $\sigma \in (0,1)$,

$$\sup_{Q(\sigma\rho^{p_\rho^+}, \sigma\rho)} u \le \max\left\{1, C(1-\sigma)^{-p_\rho^+(N+p_\rho^-)/N(q-\delta)} \left(\frac{1}{\left|Q\left(\rho^{p_\rho^+}, \rho\right)\right|} \int_{Q(\rho^{p_\rho^+}, \rho)} u^\delta \, dx \, dt\right)^{\tilde{h}/(q-\delta)}\right\}, \tag{3.40}$$

*where for all* $(x_0, t_0) \in Q$, $K_\rho = \{x \in \Omega \mid \max_{1\le i\le N}|x_i - x_{0,i}| < \rho\}$, $p_\rho^+ = \sup_{K_\rho} p(x)$, $p_\rho^- = \inf_{K_\rho} p(x)$, $Q(\rho^{p_\rho^+}, \rho) = K_\rho \times (t_0 - \rho^{p_\rho^+}, t_0)$, *and* $\max\{p_\rho^+, 2\} \le \delta < q = ((N+2)/N)p_\rho^-$, $\tilde{h} = h_0(p_\rho^-/N) \in (0, p_\rho^-/N]$.

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