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Research Article

Viscosity Approximation of Common Fixed Points for L-Lipschitzian Semigroup of Pseudocontractive Mappings in Banach Spaces

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We study the strong convergence of two kinds of viscosity iteration processes for approximating common fixed points of the pseudocontractive semigroup in uniformly convex Banach spaces with uniformly Gâteaux differential norms. As special cases, we get the strong convergence of the implicit viscosity iteration process for approximating common fixed points of the nonexpansive semigroup in Banach spaces satisfying some conditions. The results presented in this paper extend and generalize some results concerned with the nonexpansive semigroup in (Chen and He, 2007) and the pseudocontractive mapping in (Zegeye et al., 2007) to the pseudocontractive semigroup in Banach spaces under different conditions.

1. Introduction

Let E be a real Banach space with the dual space E^* and $J : E \to 2^{E^*}$ be a normalized duality mapping defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that (see, e.g., [1, pages 107–113])

(i) J is single-valued if E^* is strictly convex;

(ii) E is uniformly smooth if and only if J is single-valued and uniformly continuous on any bounded subset of E.

Let K be a nonempty closed convex subset of E. A mapping $T : K \to E$ is said to be

(i) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K, \tag{1.2}$$

(ii) L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in K, \tag{1.3}$$

(iii) k-strongly pseudocontractive if there exist a constant $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, \, j(x-y) \rangle \leq k\|x - y\|^2, \quad \forall x, y \in K, \tag{1.4}$$

(iv) pseudocontractive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x - y\|^2, \quad \forall x, y \in K. \tag{1.5}$$

It is easy to see that the pseudocontractive mapping is more general than the nonexpansive mapping.

A pseudocontractive semigroup is a family,

$$\Gamma := \{T(t) : t \geq 0\}, \tag{1.6}$$

of self-mappings on K such that

(1) $T(0)x = x$ for all $x \in K$;

(2) $T(s+t)x = T(s)T(t)x$ for all $x \in K$ and $s, t \geq 0$;

(3) $T(t)$ is pseudocontractive for each $t \geq 0$;

(4) for each $x \in K$, the mapping $T(\cdot)x$ from R^+ into K is continuous.

If the mapping $T(t)$ in condition (3) is replaced by

$(3)'$ $T(t)$ is nonexpansive for each $t \geq 0$;

then $\Gamma := \{T(t) : t \geq 0\}$ is said to be a nonexpansive semigroup on K.

We denote by $F(\Gamma)$ the common fixed points set of pseudocontractive semigroup Γ, that is,

$$F(\Gamma) = \bigcap_{t \in R^+} F(T(t)) = \{x \in K : T(t)x = x \text{ for each } t \geq 0\}. \tag{1.7}$$

In the sequel, we always assume that $F(\Gamma) \neq \emptyset$.

In recent decades, many authors studied the convergence of iterative algorithms for nonexpansive mappings, nonexpansive semigroup, and pseudocontractive mapping in Banach spaces (see, e.g., [2–14]). Let $\Gamma := \{T(t) : t \geq 0\}$ be a nonexpansive semigroup from K into itself and $f : K \to K$ be a contractive mapping. It follows from Banach's fixed theorem that the following implicit viscosity iteration process is well defined:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 1, \tag{1.8}$$

where $\alpha_n \in (0,1)$ and $T(t_n) \in \Gamma$. Some authors studied the convergence of iteration process (1.8) for nonexpansive mappings in certain Banach spaces (see [5, 10]). Recently, Xu [11] studied the following implicit iteration process: for any $u \in K$,

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 1, \tag{1.9}$$

where $\alpha_n \in (0,1)$, $T(t_n) \in \Gamma$, and obtained the convergence theorem as follows.

Theorem X (see [11]). *Let E be a uniformly convex Banach space having a weakly continuous duality map J_φ with gauge φ, K a nonempty closed convex subset of E and*

$$\Gamma := \{T(t) : t \geq 0\} \tag{1.10}$$

a nonexpansive semigroup on K such that $F = Fix(\Gamma) \neq \emptyset$. If

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0, \tag{1.11}$$

then $\{x_n\}$ generated by (1.9) *converges strongly to a member of F.*

Xu [11] also proposed the following problem.

Problem X (see [11]). We do not know if Theorem X holds in a uniformly convex and uniformly smooth Banach (e.g., L^p for $1 < p < \infty$).

This problem has been solved by Li and Huang [15] and Suzuki [8], respectively.

Moudafi's viscosity approximation method has been recently studied by many authors (see, e.g., [2, 3, 5, 10, 13, 15–17] and the references therein). Chen and He [3] studied the convergence of (1.8) constructed from a nonexpansive semigroup and a contractive mapping in a reflective Banach space with a weakly sequentially continuous duality mapping. Zegeye et al. [13] studied the convergence of (1.8) constructed from a pseudocontractive mapping and a contractive mapping.

On the other hand, many authors (see [2, 3, 5, 13]) studied the following explicit viscosity iteration process: for any given $y_0 \in K$,

$$y_{n+1} = [1 - \lambda_n(1 + \theta_n)]y_n + \lambda_n T(t_n)y_n + \lambda_n\theta_n f(y_n), \quad \forall n \geq 0, \tag{1.12}$$

where $T(t_n) \in \Gamma$, $\lambda_n, \theta_n \in (0,1)$ and $\lambda_n(1 + \theta_n) \in (0,1]$. Chen and He [3] studied the convergence of (1.12) constructed from a nonexpansive semigroup and obtained some convergence results.

An interesting work is to extend some results involving nonexpansive mapping, nonexpansive semigroup, and pseudocontractive mapping to the semigroup of pseudocontractive mappings. Li and Huang [15] generalized some corresponding results to pseudocontractive semigroup in Banach spaces. Some further study concerned with approximating common fixed points of the semigroup of pseudocontractive mappings in Banach spaces, we refer to Li and Huang [16].

Motivated by the works mentioned above, in this paper, we study the convergence of implicit viscosity iteration process (1.8) constructed from the pseudocontractive semigroup $\Gamma := \{T(t) : t \in R^+\}$ and k-strongly pseudocontractive mapping in uniformly convex Banach spaces with uniformly Gâteaux differential norms. As special cases, we obtain the convergence of the implicit iteration process for approximating the common fixed point of the nonexpansive semigroup in certain Banach spaces. We also study the convergence of the explicit viscosity iteration process (1.12) constructed from the pseudocontractive semigroup Γ and k-strongly pseudocontractive mapping in uniformly convex Banach spaces with uniformly Gâteaux differential norms. The results presented in this paper extend and generalize some results concerned with the nonexpansive semigroup in [3] and the pseudocontractive mapping in [13] to the pseudocontractive semigroup in Banach spaces under different conditions.

2. Preliminaries

A real Banach space E is said to have a weakly continuous duality mapping if J is single-valued and weak-to-weak* sequentially continuous (i.e., if each $\{x_n\}$ is a sequence in E weakly convergent to x, then $\{J(x_n)\}$ converges weakly* to $J(x)$). Obviously, if E has a weakly continuous duality mapping, then J is norm-to-weak* sequentially continuous. It is well known that l^p $(1 < p < \infty)$ posses duality mapping which is weakly continuous (see, e.g., [11]).

Let l^∞ be the Banach space of all bounded real-valued sequences. A Banach limit LIM (see [1]) is a linear continuous functional on l^∞ such that

$$\|\mathrm{LIM}\| = \mathrm{LIM}(1) = 1, \qquad \mathrm{LIM}(t_1, t_2, \dots) = \mathrm{LIM}(t_2, t_3, \dots) \tag{2.1}$$

for each $t = (t_1, t_2, \dots) \in l^\infty$. If LIM is a Banach limit, then it follows from [1, Theorem 1.4.4] that

$$\liminf_{n\to\infty} t_n \le \mathrm{LIM}(t) \le \limsup_{n\to\infty} t_n \tag{2.2}$$

for each $t = (t_1, t_2, \dots) \in l^\infty$.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at a point $p \in E$ if whenever $\{x_n\}$ is a sequence in $D(T)$ which converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx = p$.

For the sake of convenience, we restate the following lemmas that will be used.

Lemma 2.1 (see [18]). *Let E be a Banach space, K be a nonempty closed convex subset of E, and $T : K \to K$ be a strongly pseudocontractive and continuous mapping. Then T has a unique fixed point in K.*

Lemma 2.2 (see [19]). *Let E be a Banach space and J be the normalized duality mapping. Then for any $x, y \in E$ and $j(x+y) \in J(x+y)$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle. \tag{2.3}$$

Lemma 2.3 (see [12]). *Let $r > 0$. Then a real Banach space E is uniformly convex if and only if there exists a continuous and strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\left\|\lambda x + (1-\lambda)y\right\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|) \tag{2.4}$$

for all $x, y \in B_r$, $\lambda \in [0,1]$, where $B_r = \{x \in E : \|x\| \leq r\}$.

Lemma 2.4 (see [9]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\tau_n)a_n + \eta_n, \tag{2.5}$$

where $\tau_n \in (0,1)$, $\forall n \geq n_0$, $n_0 \in N$ is fixed, $\sum_{n=1}^{\infty}\tau_n = \infty$, and $\eta_n = o(\tau_n)$. Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

We first discuss the convergence of implicit viscosity iteration process (1.8) constructed from a pseudocontractive semigroup $\Gamma := \{T(t) : t \geq 0\}$.

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Banach space E. Let $\Gamma := \{T(t) : t \in R^+\}$ be an L-Lipschitzian semigroup of pseudocontractive mappings and $f : K \to K$ be an L_f-Lipschitzian k-strongly pseudocontractive mapping. Suppose that for any bounded subset $C \subset K$,*

$$\lim_{s\to 0}\sup_{x\in C}\left\|T(s)x - x\right\| = 0. \tag{3.1}$$

Then the sequence $\{x_n\}$ generated by (1.8) is well defined. Moreover, if

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty}\frac{\alpha_n}{t_n} = 0, \tag{3.2}$$

then $\lim_{n\to\infty}\|x_n - T(t)x_n\| = 0$ for any $t \in R^+$.

Proof. Let

$$T_n x := \alpha_n f(x) + (1-\alpha_n)T(t_n)x, \quad \forall n \geq 1. \tag{3.3}$$

Since

$$\langle T_n x - T_n y, j(x-y)\rangle = (1-\alpha_n)\langle T(t_n)x - T(t_n)y, j(x-y)\rangle + \alpha_n\langle f(x) - f(y), j(x-y)\rangle \\ \leq [1-\alpha_n(1-k)]\|x-y\|^2, \tag{3.4}$$

we know that T_n is strongly pseudocontractive and strongly continuous. It follows from Lemma 2.1 that T_n has a unique fixed point (say) $x_n \in K$, that is, $\{x_n\}$ generated by (1.8) is well defined.

Taking $p \in F(\Gamma)$, we have

$$\|x_n - p\|^2 = \alpha_n\langle f(x_n) - p, j(x_n - p)\rangle + (1-\alpha_n)\langle T(t_n)x_n - T(t_n)p, j(x_n - p)\rangle \\ \leq \alpha_n k\|x_n - p\|^2 + \alpha_n\langle f(p) - p, j(x_n - p)\rangle + (1-\alpha_n)\|x_n - p\|^2 \\ \leq [1-\alpha_n(1-k)]\|x_n - p\|^2 + \alpha_n\|f(p) - p\|\|x_n - p\|, \tag{3.5}$$

and so $\|x_n - p\| \leq (1/(1-k))\|f(p) - p\|$. This means $\{x_n\}$ is bounded. By the Lipschitzian conditions of Γ and f, it follows that $\{T(t_n)x_n\}$ and $\{f(x_n)\}$ are bounded. Therefore,

$$\|x_n - T(t_n)x_n\| = \alpha_n\|f(x_n) - T(t_n)x_n\| \longrightarrow 0. \tag{3.6}$$

For any given $t > 0$,

$$\|x_n - T(t)x_n\| = \sum_{k=0}^{[t/t_n]-1}\|T((k+1)t_n)x_n - T(kt_n)x_n\| + \|T(t)x_n - T([t/t_n]t_n)x_n\| \\ \leq [t/t_n]L\|x_n - T(t_n)x_n\| + L\|T(t - [t/t_n]t_n)x_n - x_n\| \\ \leq tL\frac{\alpha_n}{t_n}\|f(x_n) - T(t_n)x_n\| + L\max\{\|T(s)x_n - x_n\| : 0 \leq s \leq t_n\}, \tag{3.7}$$

where $[t/t_n]$ is the integral part of t/t_n. Since $\lim_{n\to\infty}(\alpha_n/t_n) = 0$ and $T(\cdot)x : R^+ \to K$ is continuous for any $x \in K$, it follows from (3.1) that

$$\lim_{n\to\infty}\|x_n - T(t)x_n\| = 0. \tag{3.8}$$

This completes the proof. □

Theorem 3.2. *Let E be a uniformly convex Banach space with the uniformly Gâteaux differential norm and K be a nonempty closed convex subset of E. Let $\Gamma := \{T(t) : t \in R^+\}$ be an L-Lipschitzian semigroup of pseudocontractive mappings satisfying* (3.1) *and let $f : K \to K$ be an L_f-Lipschitzian k-strongly pseudocontractive mapping. Suppose that $\{x_n\}$ is a sequence generated by* (1.8) *and*

(1) $\lim_{n\to\infty}(\alpha_n/t_n) = \lim_{n\to\infty}t_n = 0$;

(2) $LIM\|T(t)x_n - T(t)x^*\| \leq LIM\|x_n - x^*\|$, $\forall x^* \in C$, $t \in R^+$, *where* $C := \{x^* \in K : \Phi(x^*) = \min_{x\in K}\Phi(x)\}$ *with* $\Phi(x) = LIM\|x_n - x\|^2$ *for all* $x \in K$.

Then $\{x_n\}$ converges strongly to a common fixed point x^* of Γ that is the unique solution in $F(\Gamma)$ to the following variational inequality:

$$\langle f(x^*) - x^*, j(x^* - p)\rangle \geq 0, \quad \forall p \in F(\Gamma). \tag{3.9}$$

Proof. From Theorem 3.1, we know that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - T(t)x_n\| = 0$. It is easy to see that C is a nonempty bounded closed convex subset of K (see, e.g., [10]).

Now, we show that there exists a common fixed point of Γ in C. For any $t \in R^+$ and $x^* \in C$, it follows from $\lim_{n\to\infty}\|x_n - T(t)x_n\| = 0$ that

$$\begin{aligned}\Phi(T(t)x^*) &= \mathrm{LIM}\|x_n - T(t)x^*\|^2 \\ &= \mathrm{LIM}\|T(t)x_n - T(t)x^*\|^2 \\ &\leq \mathrm{LIM}\|x_n - x^*\|^2 \\ &= \Phi(x^*),\end{aligned} \tag{3.10}$$

and so

$$T(t)(C) \subset C. \tag{3.11}$$

Next, we prove that C is a singleton. In fact, since E is uniformly convex, by Lemma 2.3 that there exists a continuous and strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that, for any x_1^* and $x_2^* \in C$,

$$\left\|x_n - \frac{x_1^* + x_2^*}{2}\right\|^2 \leq \frac{1}{2}\|x_n - x_1^*\|^2 + \frac{1}{2}\|x_n - x_2^*\|^2 - \frac{1}{4}g(\|x_1^* - x_2^*\|). \tag{3.12}$$

Taking Banach limit LIM on the above inequality, it follows that

$$\begin{aligned}\frac{1}{4}g(\|x_1^* - x_2^*\|) &\leq \frac{1}{2}\mathrm{LIM}\|x_n - x_1^*\|^2 + \frac{1}{2}\mathrm{LIM}\|x_n - x_2^*\|^2 - \mathrm{LIM}\left\|x_n - \frac{x_1^* + x_2^*}{2}\right\|^2 \\ &\leq 0.\end{aligned} \tag{3.13}$$

This implies $x_1^* = x_2^*$ and so C is a singleton. Therefore, (3.11) implies that there exists $x^* \in C$ such that $x^* \in F(\Gamma)$.

For any $p \in F(\Gamma)$, from (1.8), we have

$$\begin{aligned}\langle x_n - f(x_n), j(x_n - p)\rangle &= \frac{1-\alpha_n}{\alpha_n}\langle T(t_n)x_n - x_n, j(x_n - p)\rangle \\ &= \frac{1-\alpha_n}{\alpha_n}\left[\langle T(t_n)x_n - T(t_n)p, j(x_n - p)\rangle - \langle x_n - p, j(x_n - p)\rangle\right] \\ &\leq 0.\end{aligned} \tag{3.14}$$

Since $x^* \in F(\Gamma)$, it follows from (3.14) that

$$\mathrm{LIM}\langle x_n - f(x_n), j(x_n - x^*)\rangle \le 0. \tag{3.15}$$

Furthermore, for any $t \in (0,1)$, by Lemma 2.2, we have

$$\begin{aligned}
\|x_n - x^* - t(f(x_n) - x^*)\|^2 &\le \|x_n - x^*\|^2 - 2t\langle f(x_n) - x^*, j(x_n - x^* - t(f(x_n) - x^*))\rangle \\
&\le \|x_n - x^*\|^2 - 2t\langle f(x_n) - x^*, j(x_n - x^*)\rangle \\
&\quad - 2t\langle f(x_n) - x^*, j(x_n - x^* - t(f(x_n) - x^*)) - j(x_n - x^*)\rangle, \\
\langle f(x_n) - x^*, j(x_n - x^*)\rangle &\le \frac{1}{2t}\left[\|x_n - x^*\|^2 - \|x_n - x^* - t(f(x_n) - x^*)\|^2\right] \\
&\quad - \langle f(x_n) - x^*, j(x_n - x^* - t(f(x_n) - x^*)) - j(x_n - x^*)\rangle.
\end{aligned} \tag{3.16}$$

For any $\epsilon > 0$, since E has a uniformly Gâteaux differential norm, we know that J is norm-to-weak* uniformly continuous on any bounded subset of E (see, e.g., [1, pages 107–113]) and so there exists sufficient small $\delta(\epsilon) > 0$ such that

$$\langle f(x_n) - x^*, j(x_n - x^*)\rangle \le \frac{1}{2t}\left[\|x_n - x^*\|^2 - \|x_n - x^* - t(f(x_n) - x^*)\|^2\right] + \epsilon, \quad \forall t \in (0,\delta). \tag{3.17}$$

This implies that

$$\mathrm{LIM}\langle f(x_n) - x^*, j(x_n - x^*)\rangle \le \frac{1}{2t}\left[\mathrm{LIM}\|x_n - x^*\|^2 - \mathrm{LIM}\|x_n - x^* - t(f(x_n) - x^*)\|^2\right] + \epsilon \le \epsilon. \tag{3.18}$$

By the arbitrariness of ϵ, it follows that

$$\mathrm{LIM}\langle f(x_n) - x^*, j(x_n - x^*)\rangle \le 0. \tag{3.19}$$

Adding inequalities (3.15) and (3.19), we have

$$\mathrm{LIM}\langle x_n - x^*, j(x_n - x^*)\rangle = \mathrm{LIM}\|x_n - x^*\|^2 \le 0. \tag{3.20}$$

This implies that there exists subsequence $\{x_{n_j}\} \subset \{x_n\}$ which converges strongly to x^*. From the proof of (3.20), we know that $\mathrm{LIM}\|x_{n_l} - x^*\|^2 \le 0$ for any subsequence $\{x_{n_l}\} \subset \{x_n\}$ and so there exists subsequence of $\{x_{n_l}\}$ which converges strongly to x^*. If there exists another subsequence $\{x_{n_k}\} \subset \{x_n\}$ which converges strongly to y^*, then it follows from Theorem 3.1 that $y^* \in F(\Gamma)$. From (3.14), we have

$$\begin{aligned}
&\langle x^* - f(x^*), j(x^* - y^*)\rangle \le 0, \\
&\langle y^* - f(y^*), j(y^* - x^*)\rangle \le 0.
\end{aligned} \tag{3.21}$$

Thus

$$\|x^* - y^*\|^2 \leq \langle f(x^*) - f(y^*), j(x^* - y^*) \rangle \leq k\|x^* - y^*\|^2. \tag{3.22}$$

This implies that $\|x^* - y^*\|^2 \leq 0$ and so $x^* = y^*$. Therefore, $\{x_n\}$ converges strongly to $x^* \in F(\Gamma)$. From (3.14) and the deduction above, we know that x^* is also the unique solution to the variational inequlity

$$\langle f(x^*) - x^*, j(x^* - p) \rangle \geq 0, \quad \forall p \in F(\Gamma). \tag{3.23}$$

This completes the proof. □

Remark 3.3. (1) Theorem 3.2 extends and generalizes Theorem 3.1 of [3] from nonexpansive semigroup to Lipschitzian pseudocontractive semigroup in Banach spaces with different conditions; (2) If Γ is a pseudocontractive mapping, then condition (3.1) is trivial.

If $\Gamma := \{T(t) : t \in R^+\}$ is a nonexpansive semigroup, then $\Gamma := \{T(t) : t \in R^+\}$ is an L-Lipschitzian semigroup of pseudocontractive mappings, condition (2) of Theorem 3.2 holds trivially. From Theorem 3.2, we have the following result.

Corollary 3.4. *Let E be a uniformly convex Banach space with the uniformly Gâteaux differential norm and K be a nonempty closed convex subset of E. Let $\Gamma := \{T(t) : t \in R^+\}$ be a nonexpansive semigroup satisfying* (3.1) *and let $f : K \to K$ be an L_f-Lipschitzian k-strongly pseudocontractive mapping. Suppose that $\{x_n\}$ is a sequence generated by* (1.8). *If*

$$\lim_{n\to\infty} \frac{\alpha_n}{t_n} = \lim_{n\to\infty} t_n = 0, \tag{3.24}$$

then $\{x_n\}$ converges strongly to a common fixed point x^* of Γ that is the unique solution in $F(\Gamma)$ to VI (3.9).

Theorem 3.5. *Let E be a uniformly smooth Banach space and K be a nonempty closed convex subset of E. Let $\Gamma := \{T(t) : t \in R^+\}$ be a nonexpansive semigroup satisfying* (3.1) *and let $f : K \to K$ be an L_f-Lipschitzian k-strongly pseudocontractive mapping. Suppose that $\{x_n\}$ is a sequence generated by* (1.8). *If*

$$\lim_{n\to\infty} \frac{\alpha_n}{t_n} = \lim_{n\to\infty} t_n = 0, \tag{3.25}$$

then $\{x_n\}$ converges strongly to a common fixed point x^* of Γ that is the unique solution in $F(\Gamma)$ to VI (3.9).

Proof. For the nonexpansive semigroup Γ, condition (2) of Theorem 3.2 is trivial and so formula (3.11) holds. Since uniformly smooth Banach space E has the fixed point property for nonexpansive mapping $T(t)$ (see, e.g., [10]), $T(t)$ has a fixed point $x^* \in C \cap F(\Gamma)$. The rest proof is similar to the proof of Theorem 3.2 and so we omit it. This completes the proof. □

Theorem 3.6. *Let E be a real Hilbert space and K be a nonempty closed convex subset of E. Let $\Gamma := \{T(t) : t \in R^+\}$ be an L-Lipschitzian semigroup of pseudocontractive mappings satisfying* (3.1) *and let $f : K \to K$ be an L_f-Lipschitzian k-strongly pseudocontractive mapping. Suppose that $\{x_n\}$ is a sequence generated by* (1.8). *If*

$$\lim_{n\to\infty} \frac{\alpha_n}{t_n} = \lim_{n\to\infty} t_n = 0, \tag{3.26}$$

then $\{x_n\}$ converges strongly to a common fixed point x^* of Γ that is the unique solution in $F(\Gamma)$ to the following variational inequality:

$$\langle f(x^*) - x^*, x^* - p\rangle \geq 0, \quad \forall p \in F(\Gamma). \tag{3.27}$$

Proof. From the proof of Theorem 3.1, we know that $\{x_n\}$ is bounded and so there exists subsequence $\{x_{n_j}\} \subset \{x_n\}$ which converges weakly to some point $x^* \in K$. By Theorem 3.1, we have

$$\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0. \tag{3.28}$$

It follows from [20, Theorem 3.18b] that $I - T(t)$ is demiclosed at zero for each $t \in R^+$, where I is an identity mapping. This implies that $x^* \in F(\Gamma)$.

In addition, from (1.8), we have

$$\begin{aligned} \|x_n - x^*\|^2 &= \alpha_n\langle f(x_n) - x^*, x_n - x^*\rangle + (1-\alpha_n)\langle T(t_n)x_n - x^*, x_n - x^*\rangle \\ &\leq \alpha_n\langle f(x_n) - f(x^*) + (f(x^*) - x^*), x_n - x^*\rangle \\ &\quad + (1-\alpha_n)\langle T(t_n)x_n - T(t_n)x^*, x_n - x^*\rangle \\ &\leq [1 - \alpha_n(1-k)]\|x_n - x^*\|^2 + \alpha_n\langle f(x^*) - x^*, x_n - x^*\rangle, \end{aligned} \tag{3.29}$$

and so

$$\|x_n - x^*\|^2 \leq \frac{1}{1-k}\langle f(x^*) - x^*, x_n - x^*\rangle. \tag{3.30}$$

This implies that $\{x_{n_j}\}$ converges strongly to $x^* \in F(\Gamma)$. Similar to the proof of Theorem 3.2, it is easy to show that $\{x_n\}$ converges strongly to $x^* \in F(\Gamma)$ that is also the unique solution to VI (3.27). This completes the proof. □

Now we turn to discuss the convergence of explicit viscosity iteration process (1.12) for approximating the common fixed point of the pseudocontractive semigroup $\Gamma := \{T(t) : t \geq 0\}$.

Theorem 3.7. *Let K be a nonempty closed convex subset of a real Banach space E. Let $\Gamma := \{T(t) : t \in R^+\}$ be an L-Lipschitzian semigroup of pseudocontractive mappings with $L \geq 1$ such that* (3.1) *holds. Let $f : K \to K$ be an L_f-Lipschitzian k-strongly pseudocontractive mapping. Suppose that* *the sequence* $\{y_n\}$ *is generated by* (1.12) *and the following conditions hold:*

(i) $\sum_{n=1}^{\infty}\lambda_n\theta_n = \infty$, $\lambda_n(1+\theta_n) \le 1$, *for all* $n \ge 0$;

(ii) $\theta_n/t_n \to 0$, $(\theta_{n-1}/\theta_n - 1)/(\lambda_n\theta_n) \to 0$, $t_n \to 0$ $(n \to \infty)$;

(iii) *there exists some constant* $\alpha > 0$ *such that*

$$\frac{\lambda_n}{\theta_n} \le \frac{1-k}{4L(2+L+L_f)(1+\alpha)}, \quad \forall n \ge 0; \tag{3.31}$$

(iv) *The following equation holds:*

$$\lim_{n\to\infty}\frac{\|T(t_n - t_{n-1})x - x\|}{\lambda_n\theta_n^2} = 0, \quad \forall x \in K. \tag{3.32}$$

Then $\lim_{n\to\infty}\|y_n - T(t)y_n\| = 0$ *for any* $t \in R^+$.

Proof. Let $\{x_n\}$ denote the sequence defined as in (1.8) with $\alpha_n = \theta_n/(1+\theta_n)$. By virtue of condition (ii) and Theorem 3.1, we know that $\{x_n\}$ is well defined and $\lim_{n\to\infty}\|x_n - T(t)x_n\| = 0$ for any $t \in R^+$. From (1.8), we have

$$(1+\theta_n)x_n = \theta_n f(x_n) + T(t_n)x_n, \quad \forall n \ge 0, \tag{3.33}$$

$$\lambda_n(1+\theta_n)x_n = \lambda_n\theta_n f(x_n) + \lambda_n T(t_n)x_n, \quad \forall n \ge 0. \tag{3.34}$$

To obtain the assertion of Theorem 3.7, we first give a serial of estimations: using (3.33), we get

$$\begin{aligned}
\|x_n - x_{n-1}\|^2 &= \langle x_n - x_{n-1}, j(x_n - x_{n-1})\rangle \\
&= \langle T(t_n)x_n - T(t_{n-1})x_{n-1} + \theta_n(f(x_n) - x_n) - \theta_{n-1}(f(x_{n-1}) - x_{n-1}), j(x_n - x_{n-1})\rangle \\
&\le \langle T(t_n)x_n - T(t_n)x_{n-1} + T(t_n)x_{n-1} - T(t_{n-1})x_{n-1} \\
&\quad + \theta_n(f(x_n) - f(x_{n-1}) - (x_n - x_{n-1}) + f(x_{n-1}) - x_{n-1}) \\
&\quad - \theta_{n-1}(f(x_{n-1}) - x_{n-1}), j(x_n - x_{n-1})\rangle \\
&\le [1 - \theta_n(1-k)]\|x_n - x_{n-1}\|^2 + (\theta_n - \theta_{n-1})\|f(x_{n-1}) - x_{n-1}\|\|x_n - x_{n-1}\| \\
&\quad + \|T(t_n - t_{n-1})T(t_{n-1})x_{n-1} - T(t_{n-1})x_{n-1}\|\|x_n - x_{n-1}\|,
\end{aligned} \tag{3.35}$$

which implies that

$$\|x_n - x_{n-1}\| \le \frac{\theta_n - \theta_{n-1}}{(1-k)\theta_n}\|f(x_{n-1}) - x_{n-1}\| + \frac{\|T(t_n - t_{n-1})z_n - z_n\|}{(1-k)\theta_n}, \tag{3.36}$$

where $z_n = T(t_{n-1})x_{n-1}$. From the proof of Theorem 3.1, we know that $\{x_n\}$ is bounded. Therefore, there exists a constant $M > 0$ such that

$$\|x_n - x_{n-1}\| \leq \frac{M}{1-k}\left|\frac{\theta_n - \theta_{n-1}}{\theta_n}\right| + \frac{\|T(t_n - t_{n-1})z_n - z_n\|}{(1-k)\theta_n}. \tag{3.37}$$

By using (1.12) and (3.33), we have

$$\begin{aligned}\|y_{n+1} - y_n\| &= \lambda_n \|T(t_n)y_n - (1+\theta_n)y_n + \theta_n f(y_n)\| \\ &\leq \lambda_n \|T(t_n)y_n - T(t_n)x_n - (1+\theta_n)(y_n - x_n) + \theta_n(f(y_n) - f(x_n))\| \\ &\leq \lambda_n [L\|y_n - x_n\| + (1+\theta_n)\|y_n - x_n\| + \theta_n L_f \|y_n - x_n\|] \\ &\leq (2 + L + L_f)\lambda_n \|y_n - x_n\|.\end{aligned} \tag{3.38}$$

It follows from (1.12) and (3.34) that

$$\begin{aligned}\|y_{n+1} - x_n\| &= \|[1 - \lambda_n(1+\theta_n)](y_n - x_n) + \lambda_n(T(t_n)y_n - T(t_n)x_n) + \lambda_n\theta_n(f(y_n) - f(x_n))\| \\ &\leq [1 - \lambda_n(1+\theta_n) + \lambda_n(L + L_f)]\|y_n - x_n\| \\ &\leq [1 + \lambda_n(L + L_f)]\|y_n - x_n\|.\end{aligned} \tag{3.39}$$

By virtue of (1.12), (3.34), and Lemma 2.2, we have

$$\begin{aligned}\|y_{n+1} - x_n\|^2 &= \|[1 - \lambda_n(1+\theta_n)](y_n - x_n) + \lambda_n(T(t_n)y_n - T(t_n)x_n) + \lambda_n\theta_n(f(y_n) - f(x_n))\|^2 \\ &\leq [1 - \lambda_n(1+\theta_n)]^2\|y_n - x_n\|^2 + 2\lambda_n\langle T(t_n)y_n - T(t_n)x_n, j(y_{n+1} - x_n)\rangle \\ &\quad + 2\lambda_n\theta_n\langle f(y_n) - f(x_n), j(y_{n+1} - x_n)\rangle \\ &\leq [1 - \lambda_n(1+\theta_n)]^2\|y_n - x_n\|^2 + 2\lambda_n\|y_{n+1} - x_n\|^2 + 2\lambda_n L\|y_{n+1} - y_n\|\|y_{n+1} - x_n\| \\ &\quad + 2\lambda_n\theta_n k\|y_{n+1} - x_n\|^2 + 2\lambda_n\theta_n L_f\|y_{n+1} - y_n\|\|y_{n+1} - x_n\|.\end{aligned} \tag{3.40}$$

Since $\theta_n \to 0$, then $\lambda_n \to 0$ by condition (iii). Thus for sufficient large n, we know

$$\begin{aligned}\|y_{n+1} - x_n\|^2 \leq{}& [1 - \lambda_n(1+\theta_n)]^2\|y_n - x_n\|^2 + 2\lambda_n^2 L(2 + L + L_f)(1+\alpha)\|y_n - x_n\|^2 \\ &+ 2\lambda_n(1 + k\theta_n)\|y_{n+1} - x_n\|^2.\end{aligned} \tag{3.41}$$

Consequently, by condition (iv) we can have

$$\begin{aligned}
\|y_{n+1}-x_n\|^2 &\le \frac{1-2\lambda_n(1+\theta_n)+\lambda_n^2(1+\theta_n)^2}{1-2\lambda_n(1+k\theta_n)}\|y_n-x_n\|^2 \\
&\quad + \frac{2\lambda_n^2 L(2+L+L_f)(1+\alpha)}{1-2\lambda_n(1+k\theta_n)}\|y_n-x_n\|^2 \\
&\le \left[1-2\lambda_n\theta_n\frac{1-k-(\lambda_n/2\theta_n)(1+\theta_n)^2}{1-2\lambda_n(1+k\theta_n)}\right]\|y_n-x_n\|^2 \\
&\quad + 2\lambda_n\theta_n\frac{L(2+L+L_f)(1+\alpha)(\lambda_n/\theta_n)}{1-2\lambda_n(1+k\theta_n)}\|y_n-x_n\|^2 \\
&\le \left[1-2\lambda_n\theta_n\frac{1-k-(1/4)(1-k)}{1-2\lambda_n(1+k\theta_n)}\right]\|y_n-x_n\|^2 \\
&\quad + 2\lambda_n\theta_n\frac{(1/4)(1-k)}{1-2\lambda_n(1+k\theta_n)}\|y_n-x_n\|^2 \\
&\le [1-2\lambda_n\theta_n(1-k)]\|y_n-x_n\|^2.
\end{aligned} \tag{3.42}$$

Squaring on both sides of (3.42) and using (3.37), we get

$$\begin{aligned}
\|y_{n+1}-x_n\| &\le [1-\lambda_n\theta_n(1-k)]\|y_n-x_n\| \\
&\le [1-\lambda_n\theta_n(1-k)](\|y_n-x_{n-1}\|+\|x_n-x_{n-1}\|) \\
&\le [1-\lambda_n\theta_n(1-k)]\|y_n-x_{n-1}\| + \frac{M}{1-k}\left|\frac{\theta_n-\theta_{n-1}}{\theta_n}\right| + \frac{\|T(t_n-t_{n-1})z_n-z_n\|}{(1-k)\theta_n}.
\end{aligned} \tag{3.43}$$

Setting $a_n := \|y_n - x_{n-1}\|$ and $\tau_n := \lambda_n\theta_n(1-k)$, then it follows from conditions (i)–(iv) that

$$a_{n+1} = (1-\tau_n)a_n + o(\tau_n). \tag{3.44}$$

By Lemma 2.4, we know that $a_n \to 0$, which implies that

$$\lim_{n\to\infty}\|y_{n+1}-x_n\| = 0. \tag{3.45}$$

Consequently, since $\|x_n - x_{n-1}\| \to 0$ by (3.37), we have

$$\lim_{n\to\infty}\|y_n-x_n\| = 0. \tag{3.46}$$

Now we prove that $\lim_{n\to\infty}\|y_n - T(t)y_n\| = 0$ for any $t \in R^+$. Since

$$\begin{aligned}\|y_n - T(t)y_n\| &\le \|y_n - x_n\| + \|x_n - T(t)x_n\| + \|T(t)x_n - T(t)y_n\| \\ &\le (1+L)\|y_n - x_n\| + \|x_n - T(t)x_n\|,\end{aligned} \tag{3.47}$$

by Theorem 3.1 and (3.46) we know that for any $t \in R^+$,

$$\lim_{n\to\infty}\|y_n - T(t)y_n\| = 0. \tag{3.48}$$

This completes the proof. □

Remark 3.8. An example for the conditions (i)–(iii) of Theorem 3.7 is given by

$$t_n = \frac{1}{\sqrt[4]{n+1}}, \qquad \theta_n = \frac{1}{\sqrt[3]{n+1}}, \qquad \lambda_n = \frac{1-k}{4L(2+L+L_f)(1+\alpha)}\theta_n \tag{3.49}$$

for all $n \ge 0$, where α is an any given positive real number. It is easy to see that the conditions with regard to λ_n and θ_n in Theorem 3.7 hold. If the mapping $T(\cdot)x : R^+ \to K$ is Lipschitz continuous for any $x \in K$, then condition (iv) in Theorem 3.7 also holds.

Theorem 3.9. *Let E be a uniformly convex Banach space with the uniformly Gâteaux differential norm and K be a nonempty closed convex subset of E. Let $\Gamma := \{T(t) : t \in R^+\}$ be an L-Lipschitzian semigroup of pseudocontractive mappings with $L \ge 1$ such that* (3.1) *holds. Let $f : K \to K$ be an L_f-Lipschitzian k-strongly pseudocontractive mapping. Suppose that the sequence $\{y_n\}$ is generated by* (1.12) *and conditions (i)–(iv) of Theorem 3.7 hold. Assume further that condition (2) of Theorem 3.2 holds, where $\{x_n\}$ is generated by* (1.8) *with $\alpha_n = \theta_n/(1+\theta_n)$. Then $\{y_n\}$ converges strongly to a common fixed point x^* of Γ that is the unique solution in $F(\Gamma)$ to VI* (3.9).

Proof. By Theorem3.2, we know that $\{x_n\}$ converges strongly to a fixed point x^* of Γ that is the unique solution in $F(\Gamma)$ to VI (3.9), where $\{x_n\}$ is generated by (1.8) with $\alpha_n = \theta_n/(1+\theta_n)$. It follows from (3.46) that $y_n \to x^* \in F(\Gamma)$. This completes the proof. □

Remark 3.10. (1) Theorem 3.9 extends Theorem 4.1 of [13] from Lipschitzian pseudocontractive mapping to Lipschitzian pseudocontractive semigroup in Banach spaces under different conditions; (2) Theorem 3.9 also extends Theorem 3.2 of [3] from nonexpansive semigroup to Lipschitzian pseudocontractive semigroup in Banach spaces under different conditions.

If $\Gamma := \{T(t) : t \in R^+\}$ is a nonexpansive semigroup, then $\Gamma := \{T(t) : t \in R^+\}$ is an L-Lipschitzian semigroup of pseudocontractive mappings, condition (2) of Theorem 3.2 holds trivially. Therefore, Theorem 3.9 gives the following result.

Corollary 3.11. *Let E be a uniformly convex Banach space with the uniformly Gâteaux differential norm and K be a nonempty closed convex subset of E. Let $\Gamma := \{T(t) : t \in R^+\}$ be a nonexpansive semigroup satisfying* (3.1) *and $f : K \to K$ be an L_f-Lipschitzian k-strongly pseudocontractive mapping. Suppose that the sequence $\{y_n\}$ is generated by* (1.12) *and conditions (i)–(iv) of Theorem 3.7 hold. Then $\{y_n\}$ converges strongly to a common fixed point x^* of Γ that is the unique solution in F(T) to VI* (3.9).

Theorem 3.12. *Let E be a uniformly smooth Banach space and K be a nonempty closed convex subset of E. Let $\Gamma := \{T(t) : t \in R^+\}$ be a nonexpansive semigroup satisfying* (3.1) *and let $f : K \to K$ be an L_f-Lipschitzian k-strongly pseudocontractive mapping. Suppose that the sequence $\{y_n\}$ is generated by* (1.12) *and conditions (i)–(iv) of Theorem 3.7 hold. Then $\{y_n\}$ converges strongly to a common fixed point x^* of Γ that is the unique solution in $F(\Gamma)$ to VI* (3.9).

Proof. Let $\{x_n\}$ denote the sequence defined as in (1.8) with $\alpha_n = \theta_n/(1+\theta_n)$. By Theorem 3.5, we know that $\{x_n\}$ converges strongly to a fixed point x^* of Γ that is the unique solution in $F(\Gamma)$ to VI (3.9). It follows from (3.46) that $y_n \to x^*$. This completes the proof. □

Theorem 3.13. *Let E be a real Hilbert space and K be a nonempty closed convex subset of E. Let $\Gamma := \{T(t) : t \in R^+\}$ be an L-Lipschitzian semigroup of pseudocontractive mappings with $L \geq 1$ such that* (3.1) *holds. Let $f : K \to K$ be an L_f-Lipschitzian k-strongly pseudocontractive mapping. Suppose that the sequence $\{y_n\}$ is generated by* (1.12) *and conditions (i)–(iv) of Theorem 3.7 hold. Then $\{y_n\}$ converges strongly to a common fixed point x^* of Γ that is the unique solution in $F(\Gamma)$ to VI* (3.27).

Proof. Let $\{x_n\}$ denote the sequence defined as in (1.8) with $\alpha_n = \theta_n/(1+\theta_n)$. By Theorem 3.6, we know that $\{x_n\}$ converges strongly to a fixed point x^* of Γ that is the unique solution in $F(\Gamma)$ to VI (3.27). It follows from (3.46) that $y_n \to x^*$. This completes the proof. □

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