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Research Article

Auxiliary Principle for Generalized Strongly Nonlinear Mixed Variational-Like Inequalities

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We introduce and study a class of generalized strongly nonlinear mixed variational-like inequalities, which includes several classes of variational inequalities and variational-like inequalities as special cases. By applying the auxiliary principle technique and KKM theory, we suggest an iterative algorithm for solving the generalized strongly nonlinear mixed variational-like inequality. The existence of solutions and convergence of sequence generated by the algorithm for the generalized strongly nonlinear mixed variational-like inequalities are obtained. The results presented in this paper extend and unify some known results.

1. Introduction

It is well known that the auxiliary principle technique plays an efficient and important role in variational inequality theory. In 1988, Cohen [1] used the auxiliary principle technique to prove the existence of a unique solution for a variational inequality in reflexive Banach spaces, and suggested an innovative and novel iterative algorithm for computing the solution of the variational inequality. Afterwards, Ding [2], Huang and Deng [3], and Yao [4] obtained the existence of solutions for several kinds of variational-like inequalities. Fang and Huang [5] and Liu et al. [6] discussed some classes of variational inequalities involving various monotone mappings. Recently, Liu et al. [7, 8] extended the auxiliary principle technique to two new classes of variational-like inequalities and established the existence results for these variational-like inequalities.

Inspired and motivated by the results in [1–13], in this paper, we introduce and study a class of generalized strongly nonlinear mixed variational-like inequalities. Making use of the auxiliary principle technique, we construct an iterative algorithm for solving the generalized strongly nonlinear mixed variational-like inequality. Several existence results of solutions for the generalized strongly nonlinear mixed variational-like inequality involving strongly monotone, relaxed Lipschitz, cocoercive, relaxed cocoercive and generalized pseudocontractive mappings, and the convergence results of iterative sequence generated by the algorithm are given. The results presented in this paper extend and unify some known results in [9, 12, 13].

2. Preliminaries

In this paper, let $\mathbb{R} = (-\infty, +\infty)$, let H be a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, let K be a nonempty closed convex subset of H. Let $N : H \times H \to H, \eta : K \times K \to H$, and let $T, A : K \to H$ be mappings. Now we consider the following generalized strongly nonlinear mixed variational-like inequality problem: find $u \in K$ such that

$$\langle N(Tu, Au), \eta(v, u) \rangle + b(u, v) - b(u, u) - a(u, v - u) \geq 0, \quad \forall v \in K, \tag{2.1}$$

where $a : K \times K \to \mathbb{R}$ is a coercive continuous bilinear form, that is, there exist positive constants c and d such that

(C1) $a(v, v) \geq c\|v\|^2, \forall v \in K$;

(C2) $a(u, v) \leq d\|u\|\|v\|, \forall u, v \in K$.
Clearly, $c \leq d$.

Let $b : K \times K \to \mathbb{R}$ satisfy the following conditions:

(C3) for each $v \in K$, $b(\cdot, v)$ is linear in the first argument;

(C4) b is bounded, that is, there exists a constant $r > 0$ such that $b(u, v) \leq r\|u\|\|v\|, \forall u, v \in K$;

(C5) $b(u, v) - b(u, w) \leq b(u, v - w), \forall u, v, w \in K$;

(C6) for each $u \in K$, $b(u, \cdot)$ is convex in the second argument.

Remark 2.1. It is easy to verify that

(m1) $b(u, 0) = 0,\ b(0, v) = 0, \forall u, v \in K$;

(m2) $|b(u, v) - b(u, w)| \leq r\|u\|\|v - w\|$,

where (m2) implies that for each $u \in K$, $b(u, \cdot)$ is continuous in the second argument on K.

Special Cases

(m3) If $N(Tu, Au) = Tu - Au, a(u, v - u) = 0$ and $b(u, v) = f(v)$ for all $u, v \in K$, where $f : K \to \mathbb{R}$, then the generalized strongly nonlinear mixed variational-like inequality (2.1) collapses to seeking $u \in K$ such that

$$\langle Tu - Au, \eta(v, u) \rangle + f(v) - f(u) \geq 0, \quad \forall v \in K, \tag{2.2}$$

which was introduced and studied by Ansari and Yao [9], Ding [11] and Zeng [13], respectively.

(m4) If $\eta(v,u) = g(v) - g(u)$ for all $u, v \in K$, where $g : K \to H$, then the problem (2.2) reduces to the following problem: find $u \in K$ such that

$$\langle Tu - Au, g(v) - g(u) \rangle + f(v) - f(u) \geq 0, \quad \forall v \in K, \tag{2.3}$$

which was introduced and studied by Yao [12].

In brief, for suitable choices of the mappings N, T, A, η, a and b, one can obtain a number of known and new variational inequalities and variational-like inequalities as special cases of (2.1). Furthermore, there are a wide classes of problems arising in optimization, economics, structural analysis and fluid dynamics, which can be studied in the general framework of the generalized strongly nonlinear mixed variational-like inequality, which is the main motivation of this paper.

Definition 2.2. Let $T, A : K \to H, g : H \to H, N : H \times H \to H$ and $\eta : K \times K \to H$ be mappings.

(1) g is said to be *relaxed Lipschitz* with constant r if there exists a constant $r > 0$ such that

$$\langle g(u) - g(v), u - v \rangle \leq -r\|u - v\|^2, \quad \forall u, v \in H. \tag{2.4}$$

(2) T is said to be *cocoercive* with constant r with respect to N in the first argument if there exists a constant $r > 0$ such that

$$\langle N(Tu, x) - N(Tv, x), u - v \rangle \geq r\|N(Tu, x) - N(Tv, x)\|^2, \quad \forall x \in H, u, v \in K. \tag{2.5}$$

(3) T is said to be *g-cocoercive* with constant r with respect to N in the first argument if there exists a constant $r > 0$ such that

$$\langle N(Tu, x) - N(Tv, x), g(u) - g(v) \rangle \geq r\|N(Tu, x) - N(Tv, x)\|^2, \quad \forall x \in H, u, v \in K. \tag{2.6}$$

(4) T is said to be *relaxed (p, q)-cocoercive* with respect to N in the first argument if there exist constants $p > 0, q > 0$ such that

$$\begin{aligned} &\langle N(Tu, x) - N(Tv, x), u - v \rangle \\ &\quad \geq -p\|N(Tu, x) - N(Tv, x)\|^2 + q\|u - v\|^2, \quad \forall x \in H, u, v \in K. \end{aligned} \tag{2.7}$$

(5) A is said to be *Lipschitz continuous* with constant r if there exists a constant $r > 0$ such that

$$\|A(u) - A(v)\| \leq r\|u - v\|, \quad \forall u, v \in K. \tag{2.8}$$

(6) A is said to be *relaxed Lipschitz* with constant r with respect to N in the second argument if there exists a constant $r > 0$ such that

$$\langle N(x, Au) - N(x, Av), u - v \rangle \leq -r\|u - v\|^2, \quad \forall x \in H, u, v \in K. \tag{2.9}$$

(7) A is said to be *g-relaxed Lipschitz* with constant r with respect to N in the second argument if there exists a constant $r > 0$ such that

$$\langle N(x, Au) - N(x, Av), g(u) - g(v) \rangle \leq -r\|u - v\|^2, \quad \forall x \in H, u, v \in K. \tag{2.10}$$

(8) A is said to be *g-generalized pseudocontractive* with constant r with respect to N in the second argument if there exists a constant $r > 0$ such that

$$\langle N(x, Au) - N(x, Av), g(u) - g(v) \rangle \leq r\|u - v\|^2, \quad \forall x \in H, u, v \in K. \tag{2.11}$$

(9) η is said to be *strongly monotone* with constant r if there exists a constant $r > 0$ such that

$$\langle \eta(u, v), u - v \rangle \geq r\|u - v\|^2, \quad \forall u, v \in K. \tag{2.12}$$

(10) η is said to be *relaxed Lipschitz* with constant r if there exists a constant $r > 0$ such that

$$\langle \eta(u, v), u - v \rangle \leq -r\|u - v\|^2, \quad \forall u, v \in K. \tag{2.13}$$

(11) η is said to be *cocoercive* with constant r if there exists a constant $r > 0$ such that

$$\langle \eta(u, v), u - v \rangle \geq r\|\eta(u, v)\|^2, \quad \forall u, v \in K. \tag{2.14}$$

(12) η is said to be *Lipschitz continuous* with constant r if there exists a constant $r > 0$ such that

$$\|\eta(u, v)\| \leq r\|u - v\|, \quad \forall u, v \in K. \tag{2.15}$$

(13) N is said to be *Lipschitz continuous* in the first argument if there exists a constant $r > 0$ such that

$$\|N(u, x) - N(v, x)\| \leq r\|u - v\|, \quad \forall u, v, x \in H. \tag{2.16}$$

Similarly, we can define the Lipschitz continuity of N in the second argument.

Definition 2.3. Let D be a nonempty convex subset of H, and let $f : D \to \mathbb{R} \cup \{+\infty\}$ be a functional.

(d1) f is said to be *convex* if for any $x, y \in D$ and any $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y); \tag{2.17}$$

(d2) f is said to be *concave* if $-f$ is convex;

(d3) f is said to be *lower semicontinuous* on D if for any $t \in \mathbb{R} \cup \{+\infty\}$, the set $\{x \in D : f(x) \le t\}$ is closed in D;

(d4) f is said to be *upper semicontinuous* on D, if $-f$ is lower semicontinuous on D.

In order to gain our results, we need the following assumption.

Assumption 2.4. The mappings $T, A : K \to H, N : H \times H \to H, \eta : K \times K \to H$ satisfy the following conditions:

(d5) $\eta(v,u) = -\eta(u,v), \forall u,v \in K$;

(d6) for given $x,u \in K$, the mapping $v \mapsto \langle N(Tx, Ax), \eta(u,v)\rangle$ is concave and upper semicontinuous on K.

Remark 2.5. It follows from (d5) and (d6) that

(m5) $\eta(u,u) = 0$, $\forall u \in K$;

(m6) for any given $x,v \in K$, the mapping $u \mapsto \langle N(Tx, Ax), \eta(u,v)\rangle$ is convex and lower semicontinuous on K.

Proposition 2.6 (see [9]). *Let K be a nonempty convex subset of H. If $f : K \to \mathbb{R}$ is lower semicontinuous and convex, then f is weakly lower semicontinuous.*

Proposition 2.6 yields that if $f : K \to \mathbb{R}$ is upper semicontinuous and concave, then f is weakly upper semicontinuous.

Lemma 2.7 (see [10]). *Let X be a nonempty closed convex subset of a Hausdorff linear topological space E, and let $\phi, \psi : X \times X \to \mathbb{R}$ be mappings satisfying the following conditions:*

(a) $\psi(x,y) \le \phi(x,y)$, $\forall x,y \in X$, *and* $\psi(x,x) \ge 0$, $\forall x \in X$;

(b) *for each* $x \in X, \phi(x,\cdot)$ *is upper semicontinuous on* X;

(c) *for each* $y \in X$, *the set* $\{x \in X : \psi(x,y) < 0\}$ *is a convex set;*

(d) *there exists a nonempty compact set* $Y \subseteq X$ *and* $x_0 \in Y$ *such that* $\psi(x_0,y) < 0$, $\forall y \in X \setminus Y$.

Then there exists $\hat{y} \in Y$ *such that* $\phi(x,\hat{y}) \ge 0$, $\forall x \in X$.

3. Auxiliary Problem and Algorithm

In this section, we use the auxiliary principle technique to suggest and analyze an iterative algorithm for solving the generalized strongly nonlinear mixed variational-like inequality (2.1). To be more precise, we consider the following auxiliary problem associated with the generalized strongly nonlinear mixed variational-like inequality (2.1): given $u \in K$, find $z \in K$ such that

$$\begin{aligned} &\langle g(u) - g(z), v - z\rangle \\ &\quad \ge -\rho\langle N(Tu, Au), \eta(v,z)\rangle + \rho b(u,z) - \rho b(u,v) + \rho a(u, v - z), \quad \forall v \in K, \end{aligned} \tag{3.1}$$

where $\rho > 0$ is a constant, $g : H \to H$ is a mapping. The problem is called a *auxiliary problem* for the generalized strongly nonlinear mixed variational-like inequality (2.1).

Theorem 3.1. *Let K be a nonempty closed convex subset of the Hilbert space H. Let $a : K \times K \to \mathbb{R}$ be a coercive continuous bilinear form with (C1) and (C2), and let $b : K \times K \to \mathbb{R}$ be a functional with (C3)–(C6). Let $g : H \to H$ be Lipschitz continuous and relaxed Lipschitz with constants ζ and λ, respectively. Let $\eta : K \times K \to H$ be Lipschitz continuous with constant δ, $T, A : K \to H$, and let $N : H \times H \to H$ satisfy Assumption 2.4. Then the auxiliary problem* (3.1) *has a unique solution in K.*

Proof. For any $u \in K$, define the mappings $\phi, \psi : K \times K \to \mathbb{R}$ by

$$\begin{aligned}\phi(v,z) &= \langle g(u) - g(v), v - z\rangle + \rho\langle N(Tu, Au), \eta(v,z)\rangle \\ &\quad - \rho b(u,z) + \rho b(u,v) - \rho a(u, v - z), \quad \forall v, z \in K, \\ \psi(v,z) &= \langle g(u) - g(z), v - z\rangle + \rho\langle N(Tu, Au), \eta(v,z)\rangle \\ &\quad - \rho b(u,z) + \rho b(u,v) - \rho a(u, v - z), \quad \forall v, z \in K.\end{aligned} \tag{3.2}$$

We claim that the mappings ϕ and ψ satisfy all the conditions of Lemma 2.7 in the weak topology. Note that

$$\phi(v,z) - \psi(v,z) = -\langle g(v) - g(z), v - z\rangle \geq \lambda\|v - z\|^2 \geq 0, \tag{3.3}$$

and $\psi(v,v) \geq 0$ for any $v, z \in K$. Since b is convex in the second argument and a is a coercive continuous bilinear form, it follows from Remark 2.1 and Assumption 2.4 that for each $v \in K$, $\phi(v,\cdot)$ is weakly upper semicontinuous on K. It is easy to show that the set $\{v \in K : \psi(v,z) < 0\}$ is a convex set for each fixed $z \in K$. Let $v_0 \in K$ be fixed and put

$$\begin{gathered}\omega = \lambda^{-1}\left(\zeta\|u - v_0\| + \rho\delta\|N(Tu, Au)\| + \rho r\|u\| + \rho d\|u\|\right), \\ Y = \{z \in K : \|z - v_0\| \leq \omega\}.\end{gathered} \tag{3.4}$$

Clearly, Y is a weakly compact subset of K. From Assumption 2.4, the continuity of η and g, and the properties of a and b, we gain that for any $z \in K \setminus Y$

$$\begin{aligned}\psi(v_0, z) &= \langle g(z) - g(v_0), z - v_0\rangle + \langle g(v_0) - g(u), z - v_0\rangle \\ &\quad + \rho\langle N(Tu, Au), \eta(v_0, z)\rangle - \rho b(u,z) + \rho b(u, v_0) - \rho a(u, v_0 - z) \\ &\leq -\lambda\|z - v_0\|\left[\|z - v_0\| - \lambda^{-1}\left(\zeta\|u - v_0\| + \rho\delta\|N(Tu, Au)\| + \rho r\|u\| + \rho d\|u\|\right)\right] < 0.\end{aligned} \tag{3.5}$$

Thus the conditions of Lemma 2.7 are satisfied. It follows from Lemma 2.7 that there exists a $\widehat{z} \in Y \subseteq K$ such that $\phi(v, \widehat{z}) \geq 0$ for any $v \in K$, that is,

$$\langle g(u) - g(v), v - \widehat{z}\rangle + \rho\langle N(Tu, Au), \eta(v, \widehat{z})\rangle - \rho b(u, \widehat{z}) + \rho b(u, v) - \rho a(u, v - \widehat{z}) \geq 0, \quad \forall v \in K. \tag{3.6}$$

Let $t \in (0,1]$ and $v \in K$. Replacing v by $x_t = tv + (1-t)\widehat{z}$ in (3.6) we gain that

$$\begin{aligned}0 &\leq \langle g(u) - g(x_t), x_t - \widehat{z}\rangle + \rho\langle N(Tu, Au), \eta(x_t, \widehat{z})\rangle \\ &\quad - \rho b(u, \widehat{z}) + \rho b(u, x_t) - \rho a(u, x_t - \widehat{z}) \\ &= t\langle g(u) - g(x_t), v - \widehat{z}\rangle - \rho\langle N(Tu, Au), \eta(\widehat{z}, tv + (1-t)\widehat{z})\rangle \\ &\quad - \rho b(u, \widehat{z}) + \rho b(u, tv + (1-t)\widehat{z}) - t\rho a(u, v - \widehat{z}) \\ &\leq t\langle g(u) - g(x_t), v - \widehat{z}\rangle + \rho t\langle N(Tu, Au), \eta(v, \widehat{z})\rangle \\ &\quad + t\rho(b(u, v) - b(u, \widehat{z})) - t\rho a(u, v - \widehat{z}).\end{aligned} \tag{3.7}$$

Letting $t \to 0^+$ in (3.7), we get that

$$\begin{aligned}&\langle g(u) - g(\widehat{z}), v - \widehat{z}\rangle \\ &\quad \geq -\rho\big\langle N(Tu, Au), \eta(v, \widehat{z})\big\rangle - \rho b(u, v) + \rho b(u, \widehat{z}) + \rho a(u, v - \widehat{z}), \quad \forall v \in K,\end{aligned} \tag{3.8}$$

which means that $\widehat{z}$ is a solution of (3.1).

Suppose that $z_1, z_2 \in K$ are any two solutions of the auxiliary problem (3.1). It follows that

$$\begin{aligned}&\langle g(u) - g(z_1), v - z_1\rangle \\ &\quad \geq -\rho\big\langle N(Tu, Au), \eta(v, z_1)\big\rangle - \rho b(u, v) + \rho b(u, z_1) + \rho a(u, v - z_1), \quad \forall v \in K,\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\langle g(u) - g(z_2), v - z_2\rangle \\ &\quad \geq -\rho\big\langle N(Tu, Au), \eta(v, z_2)\big\rangle - \rho b(u, v) + \rho b(u, z_2) + \rho a(u, v - z_2), \quad \forall v \in K.\end{aligned} \tag{3.10}$$

Taking $v = z_2$ in (3.9) and $v = z_1$ in (3.10) and adding these two inequalities, we get that

$$\langle g(z_2) - g(z_1), z_2 - z_1\rangle \geq 0. \tag{3.11}$$

Since g is relaxed Lipschitz, we find that

$$0 \leq \langle g(z_2) - g(z_1), z_2 - z_1\rangle \leq -\lambda\|z_2 - z_1\|^2 \leq 0, \tag{3.12}$$

which implies that $z_1 = z_2$. That is, the auxiliary problem (3.1) has a unique solution in K. This completes the proof. □

Applying Theorem 3.1, we construct an iterative algorithm for solving the generalized strongly nonlinear mixed variational-like inequality (2.1).

Algorithm 3.2. (i) At step 0, start with the initial value $u_0 \in K$.

(ii) At step n, solve the auxiliary problem (3.1) with $u = u_n \in K$. Let $u_{n+1} \in K$ denote the solution of the auxiliary problem (3.1). That is,

$$\begin{aligned}&\langle g(u_n) - g(u_{n+1}), v - u_{n+1}\rangle \\ &\quad \geq -\rho\langle N(Tu_n, Au_n), \eta(v, u_{n+1})\rangle + \rho b(u_n, u_{n+1}) - \rho b(u_n, v) + \rho a(u_n, v - u_{n+1}), \quad \forall v \in K,\end{aligned} \tag{3.13}$$

where $\rho > 0$ is a constant.

(iii) If, for given $\varepsilon > 0$, $\|x_{n+1} - x_n\| < \varepsilon$, stop. Otherwise, repeat (ii).

4. Existence of Solutions and Convergence Analysis

The goal of this section is to prove several existence of solutions and convergence of the sequence generated by Algorithm 3.2 for the generalized strongly nonlinear mixed variational-like inequality (2.1).

Theorem 4.1. *Let K be a nonempty closed convex subset of the Hilbert space H. Let $a : K \times K \to \mathbb{R}$ be a coercive continuous bilinear form with (C1) and (C2), and let $b : K \times K \to \mathbb{R}$ be a functional with (C3)–(C6). Let $N : H \times H \to H$ be Lipschitz continuous with constants i, j in the first and second arguments, respectively. Let $T, A : K \to H, g : H \to H$ and $\eta : K \times K \to H$ be Lipschitz continuous with constants ξ, μ, ζ, δ, respectively, let T be cocoercive with constant β with respect to N in the first argument, let g be relaxed Lipschitz with constant λ, and let η be strongly monotone with constant α. Assume that Assumption 2.4 holds. Let*

$$\begin{aligned}&L = \delta^{-1}\left(\lambda - \sqrt{1 - 2\lambda + \zeta^2} - \sqrt{1 - 2\alpha + \delta^2}\right), \qquad F = 1 - L^2, \\ &E = i^2\xi^2\beta - L\left(j\mu + \delta^{-1}(r + d)\right), \qquad D = i^2\xi^2 - \left(j\mu + \delta^{-1}(r + d)\right)^2.\end{aligned} \tag{4.1}$$

If there exists a constant ρ satisfying

$$2\beta \leq \rho < \frac{\delta L}{j\mu\delta + r + d} \tag{4.2}$$

and one of the following conditions:

$$D > 0, \quad E^2 > DF, \quad \left|\rho - \frac{E}{D}\right| < \frac{\sqrt{E^2 - DF}}{D}, \tag{4.3}$$

$$D < 0, \quad E^2 > DF, \quad \left|\rho - \frac{E}{D}\right| > \frac{-\sqrt{E^2 - DF}}{D}, \tag{4.4}$$

$$D = 0, \quad E > 0, \quad F > 0, \quad \rho > \frac{F}{2E}, \tag{4.5}$$

$$D = 0, \quad E < 0, \quad F < 0, \quad \rho < \frac{F}{2E}, \tag{4.6}$$

then the generalized strongly nonlinear mixed variational-like inequality (2.1) *possesses a solution* $u \in K$ *and the sequence* $\{u_n\}_{n\geq 0}$ *defined by Algorithm 3.2 converges to* u.

Proof. It follows from (3.13) that

$$\begin{aligned}
&\langle g(u_{n-1}) - g(u_n), u_{n+1} - u_n \rangle \\
&\quad \geq -\rho\langle N(Tu_{n-1}, Au_{n-1}), \eta(u_{n+1}, u_n)\rangle + \rho b(u_{n-1}, u_n) - \rho b(u_{n-1}, u_{n+1}) \\
&\qquad + \rho a(u_{n-1}, u_{n+1} - u_n), \quad \forall n \geq 1, \\
&\langle g(u_n) - g(u_{n+1}), u_n - u_{n+1} \rangle \\
&\quad \geq -\rho\langle N(Tu_n, Au_n), \eta(u_n, u_{n+1})\rangle + \rho b(u_n, u_{n+1}) - \rho b(u_n, u_n) \\
&\qquad + \rho a(u_n, u_n - u_{n+1}), \quad \forall n \geq 0.
\end{aligned} \tag{4.7}$$

Adding (4.7), we obtain that

$$\begin{aligned}
&-\langle g(u_n) - g(u_{n+1}), u_n - u_{n+1} \rangle \\
&\quad \leq \langle u_n - u_{n-1} + g(u_n) - g(u_{n-1}), u_n - u_{n+1} \rangle \\
&\qquad + \langle u_{n-1} - u_n - \rho(N(Tu_{n-1}, Au_{n-1}) - N(Tu_n, Au_{n-1})), \eta(u_n, u_{n+1})\rangle \\
&\qquad - \rho\langle N(Tu_n, Au_{n-1}) - N(Tu_n, Au_n), \eta(u_n, u_{n+1})\rangle \\
&\qquad + \langle u_{n-1} - u_n, u_n - u_{n+1} - \eta(u_n, u_{n+1})\rangle + \rho b(u_n - u_{n-1}, u_n) \\
&\qquad - \rho b(u_n - u_{n-1}, u_{n+1}) + \rho a(u_{n-1} - u_n, u_n - u_{n+1}) \\
&\quad \leq \|u_n - u_{n-1} + g(u_n) - g(u_{n-1})\| \|u_n - u_{n+1}\| \\
&\qquad + \|u_{n-1} - u_n - \rho(N(Tu_{n-1}, Au_{n-1}) - N(Tu_n, Au_{n-1}))\| \|\eta(u_n, u_{n+1})\| \\
&\qquad + \rho\|N(Tu_n, Au_{n-1}) - N(Tu_n, Au_n)\| \|\eta(u_n, u_{n+1})\| \\
&\qquad + \|u_{n-1} - u_n\| \|u_n - u_{n+1} - \eta(u_n, u_{n+1})\| \\
&\qquad + \rho r\|u_n - u_{n-1}\| \|u_n - u_{n+1}\| + \rho d\|u_{n-1} - u_n\| \|u_n - u_{n+1}\|, \quad \forall n \geq 1.
\end{aligned} \tag{4.8}$$

Since g is relaxed Lipschitz and Lipschitz continuous with constants λ and ζ, and η is strongly monotone and Lipschitz continuous with constants α and δ, respectively, we get that

$$\begin{aligned}
\|u_n - u_{n-1} + g(u_n) - g(u_{n-1})\|^2 &\leq \left(1 - 2\lambda + \zeta^2\right)\|u_n - u_{n-1}\|^2, \quad \forall n \geq 1, \\
\|u_n - u_{n+1} - \eta(u_n, u_{n+1})\|^2 &\leq \left(1 - 2\alpha + \delta^2\right)\|u_n - u_{n+1}\|^2, \quad \forall n \geq 0.
\end{aligned} \tag{4.9}$$

Notice that N is Lipschitz continuous in the first and second arguments, T and A are both Lipschitz continuous, and T is cocoercive with constant r with respect to N in the first argument. It follows that

$$\begin{aligned}
&\left\|u_{n-1} - u_n - \rho(N(Tu_{n-1}, Au_{n-1}) - N(Tu_n, Au_{n-1}))\right\|^2 \\
&\qquad \le \left(1 + i^2\xi^2\left(\rho^2 - 2\rho\beta\right)\right)\|u_{n-1} - u_n\|^2, \quad \forall n \ge 1, \\
&\|N(Tu_n, Au_{n-1}) - N(Tu_n, Au_n)\|\|\eta(u_n, u_{n+1})\| \\
&\qquad \le j\mu\delta\|u_{n-1} - u_n\|\|u_n - u_{n+1}\|, \quad \forall n \ge 1.
\end{aligned} \tag{4.10}$$

Let

$$\theta = \lambda^{-1}\left[\sqrt{1 - 2\lambda + \zeta^2} + \sqrt{1 - 2\alpha + \delta^2} + \delta\sqrt{1 + i^2\xi^2(\rho^2 - 2\rho\beta)} + \rho(j\mu\delta + r + d)\right]. \tag{4.11}$$

It follows from (4.8)–(4.10) that

$$\|u_n - u_{n+1}\| \le \theta\|u_{n-1} - u_n\|, \quad \forall n \ge 1. \tag{4.12}$$

From (4.2) and one of (4.3)–(4.6), we know that $\theta < 1$. It follows from (4.12) that $\{u_n\}_{n\ge 0}$ is a Cauchy sequence in K. By the closedness of K there exists $u \in K$ satisfying $\lim_{n\to\infty} u_n = u$. In term of (3.13) and the Lipschitz continuity of g, we gain that

$$\begin{aligned}
&\langle g(u_n) - g(u_{n+1}), v - u_{n+1}\rangle + \rho\langle N(Tu_n, Au_n), \eta(v, u_{n+1})\rangle \\
&\qquad + \rho(b(u_n, v) - b(u_n, u_{n+1})) - \rho a(u_n, v - u_{n+1}) \ge 0, \quad \forall n \ge 0, \\
&\left|\langle g(u_n) - g(u_{n+1}), v - u_{n+1}\rangle\right| \\
&\qquad \le \zeta\|u_n - u_{n+1}\|\|v - u_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.13}$$

By Assumption 2.4, we deduce that

$$\langle N(Tu, Au), \eta(v, u)\rangle \ge \limsup_{n\to\infty}\langle N(Tu, Au), \eta(v, u_{n+1})\rangle. \tag{4.14}$$

Since $N(Tu_n, Au_n) \to N(Tu, Au)$ as $n \to \infty$ and $\{\eta(v, u_{n+1})\}_{n\geq 0}$ is bounded, it follows that

$$\begin{aligned}
0 &\leq \langle N(Tu, Au), \eta(v, u)\rangle - \limsup_{n\to\infty}\langle N(Tu, Au), \eta(v, u_{n+1})\rangle \\
&= \liminf_{n\to\infty}\{\langle N(Tu, Au), \eta(v, u)\rangle - \langle N(Tu, Au), \eta(v, u_{n+1})\rangle\} \\
&= \liminf_{n\to\infty}\{\langle N(Tu, Au), \eta(v, u)\rangle - \langle N(Tu, Au), \eta(v, u_{n+1})\rangle \\
&\qquad + \langle N(Tu, Au) - N(Tu_n, Au_n), \eta(v, u_{n+1})\rangle\} \\
&= \liminf_{n\to\infty}\{\langle N(Tu, Au), \eta(v, u)\rangle - \langle N(Tu_n, Au_n), \eta(v, u_{n+1})\rangle\},
\end{aligned} \tag{4.15}$$

which implies that

$$\langle N(Tu, Au), \eta(v, u)\rangle \geq \limsup_{n\to\infty}\langle N(Tu_n, Au_n), \eta(v, u_{n+1})\rangle. \tag{4.16}$$

In light of (C3) and (m2), we get that

$$\begin{aligned}
|b(u_n, u_{n+1}) - b(u, u)| &\leq |b(u_n, u_{n+1}) - b(u_n, u)| + |b(u_n, u) - b(u, u)| \\
&\leq r\|u_n\|\|u_{n+1} - u\| + r\|u_n - u\|\|u\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{4.17}$$

which means that $b(u_n, u_{n+1}) \to b(u, u)$ as $n \to \infty$. Similarly, we can infer that $b(u_n, v) \to b(u, v)$ as $n \to \infty$. Therefore,

$$\langle N(Tu, Au), \eta(v, u)\rangle + b(u, v) - b(u, u) - a(u, v - u) \geq 0, \quad \forall v \in K. \tag{4.18}$$

This completes the proof. □

Theorem 4.2. *Let K, H, g, a, b, N, F, and L be as in Theorem 4.1. Assume that $T, A : K \to H$, $\eta : K \times K \to H$ are Lipschitz continuous with constants ξ, μ, and δ, respectively, η is relaxed Lipschitz with constant α, and A is relaxed Lipschitz with constant β with respect to N in the second argument. Let*

$$D = j^2\mu^2 - \left(i\xi + \frac{r + d}{\delta}\right)^2, \qquad E = \beta - Li\xi - \frac{L(r + d)}{\delta}. \tag{4.19}$$

If there exists a constant ρ satisfying

$$0 < \rho < \frac{\delta L}{i\xi\delta + r + d} \tag{4.20}$$

and one of (4.3)–(4.6)*, then the generalized strongly nonlinear mixed variational-like inequality* (2.1) *possesses a solution $u \in K$ and the sequence $\{u_n\}_{n\geq 0}$ defined by Algorithm 3.2 converges to u.*

Proof. As in the proof of Theorem 4.1, we deduce that

$$
\begin{aligned}
&- \langle g(u_n) - g(u_{n+1}), u_n - u_{n+1} \rangle \\
&\quad \le \langle u_n - u_{n-1} + g(u_n) - g(u_{n-1}), u_n - u_{n+1} \rangle \\
&\qquad + \langle u_n - u_{n-1} + \rho(N(Tu_{n-1}, Au_n) - N(Tu_{n-1}, Au_{n-1})), \eta(u_n, u_{n+1}) \rangle \\
&\qquad - \rho \langle N(Tu_{n-1}, Au_n) - N(Tu_n, Au_n), \eta(u_n, u_{n+1}) \rangle \\
&\qquad + \langle u_{n-1} - u_n, u_n - u_{n+1} + \eta(u_n, u_{n+1}) \rangle - \rho b(u_{n-1} - u_n, u_n) \\
&\qquad + \rho b(u_{n-1} - u_n, u_{n+1}) + \rho a(u_{n-1} - u_n, u_n - u_{n+1}), \quad \forall n \ge 1.
\end{aligned}
\tag{4.21}
$$

Because η is relaxed Lipschitz and Lipschitz continuous, A is relaxed Lipschitz with respect to N in the second argument and Lipschitz continuous, and N is Lipschitz continuous in the second argument, we conclude that

$$
\begin{aligned}
&\left\| u_n - u_{n+1} + \eta(u_n, u_{n+1}) \right\|^2 \\
&\quad \le \left(1 - 2\alpha + \delta^2\right) \|u_n - u_{n+1}\|^2, \quad \forall n \ge 0, \\
&\left\| u_n - u_{n-1} + \rho(N(Tu_{n-1}, Au_n) - N(Tu_{n-1}, Au_{n-1})) \right\|^2 \\
&\quad \le \left(1 - 2\rho\beta + \rho^2 j^2 \mu^2\right) \|u_{n-1} - u_n\|^2, \quad \forall n \ge 1.
\end{aligned}
\tag{4.22}
$$

The rest of the argument is the same as in the proof of Theorem 4.1 and is omitted. This completes the proof. □

Theorem 4.3. *Let K, H, g, a, b, A, N, D, and E be as in Theorem 4.1, and let η be as in Theorem 4.2. Assume that T is g-cocoercive with constant β with respect to N in the first argument and Lipschitz continuous with constant ξ. Let*

$$
L = \delta^{-1}\left[\lambda - \left(1 + \sqrt{1 - 2\lambda + \zeta^2}\right)\sqrt{1 - 2\alpha + \delta^2}\right], \qquad F = \zeta^2 - L^2.
\tag{4.23}
$$

If there exists a constant ρ satisfying (4.2) *and one of* (4.3)–(4.6), *then the generalized strongly nonlinear mixed variational-like inequality* (2.1) *possesses a solution $u \in K$ and the sequence $\{u_n\}_{n\ge0}$ defined by Algorithm 3.2 converges to u.*

Proof. As in the proof of Theorem 4.1, we derive that

$$\begin{aligned}
&-\langle g(u_n)-g(u_{n+1}),u_n-u_{n+1}\rangle \\
&\quad\le \langle g(u_n)-g(u_{n-1})+u_n-u_{n-1},u_n-u_{n+1}+\eta(u_n,u_{n+1})\rangle \\
&\qquad+\langle g(u_{n-1})-g(u_n)-\rho(N(Tu_{n-1},Au_{n-1})-N(Tu_n,Au_{n-1})),\eta(u_n,u_{n+1})\rangle \\
&\qquad-\rho\langle N(Tu_n,Au_{n-1})-N(Tu_n,Au_n),\eta(u_n,u_{n+1})\rangle \\
&\qquad+\langle u_{n-1}-u_n,u_n-u_{n+1}+\eta(u_n,u_{n+1})\rangle+\rho b(u_n-u_{n-1},u_n) \\
&\qquad+\rho b(u_{n-1}-u_n,u_{n+1})+\rho a(u_{n-1}-u_n,u_n-u_{n+1}),\quad \forall n\ge 1.
\end{aligned} \tag{4.24}$$

Because g is Lipschitz continuous, N is Lipschitz continuous in the first argument, and T is g-cocoercive with with respect to N in the first argument and Lipschitz continuous, we gain that

$$\begin{aligned}
&\left\| g(u_{n-1})-g(u_n)-\rho(N(Tu_{n-1},Au_{n-1})-N(Tu_n,Au_{n-1}))\right\|^2 \\
&\quad\le \left(\zeta^2+\left(\rho^2-2\rho\beta\right)i^2\xi^2\right)\|u_{n-1}-u_n\|^2,\quad \forall n\ge 1.
\end{aligned} \tag{4.25}$$

The rest of the proof is identical with the proof of Theorem 4.1 and is omitted. This completes the proof. □

Theorem 4.4. *Let K, H, g, a, b, and N be as in Theorem 4.1. Let D and F be as in Theorems 4.2 and 4.3, respectively. Assume that $T,A:K\to H$, $\eta:K\times K\to H$ are Lipschitz continuous with constants ξ, μ, and δ, respectively, A is g-generalized pseudocontractive with constant β with respect to N in the second argument, and η is cocoercive with constant $\alpha\in(0,1/2]$. Let*

$$L=\delta^{-1}\left[\lambda-\sqrt{1+(1-2\alpha)\delta^2}\left(1+\sqrt{1-2\lambda+\zeta^2}\right)\right],\qquad E=-\beta-L\left(i\xi+\frac{r+d}{\delta}\right). \tag{4.26}$$

If there exists a constant ρ satisfying

$$0<\rho<\frac{\delta L}{\delta i\xi+r+d} \tag{4.27}$$

and one of (4.3), (4.4), *and* (4.6), *then the generalized strongly nonlinear mixed variational-like inequality* (2.1) *possesses a solution $u\in K$ and the sequence $\{u_n\}_{n\ge 0}$ defined by Algorithm 3.2 converges to u.*

Proof. By a similar argument used in the proof of Theorem 4.1, we conclude that

$$\begin{aligned}
&-\langle g(u_n)-g(u_{n+1}),u_n-u_{n+1}\rangle\\
&\quad\le \langle g(u_n)-g(u_{n-1})+u_n-u_{n-1},u_n-u_{n+1}-\eta(u_n,u_{n+1})\rangle\\
&\qquad-\langle g(u_{n-1})-g(u_n)+\rho\langle N(Tu_{n-1},Au_{n-1})-N(Tu_{n-1},Au_n),\eta(u_n,u_{n+1})\rangle\\
&\qquad-\rho\langle N(Tu_{n-1},Au_n)-N(Tu_n,Au_n),\eta(u_n,u_{n+1})\rangle\\
&\qquad+\langle u_{n-1}-u_n,u_n-u_{n+1}-\eta(u_n,u_{n+1})\rangle+\rho b(u_n-u_{n-1},u_n)\\
&\qquad+\rho b(u_{n-1}-u_n,u_{n+1})+\rho a(u_{n-1}-u_n,u_n-u_{n+1}),\quad \forall n\ge 1.
\end{aligned}\tag{4.28}$$

Since A is g-generalized pseudocontractive with respect to N in the second argument and Lipschitz continuous, g is Lipschitz continuous and N is Lipschitz continuous in the second argument, η is cocoercive and Lipschitz continuous, it follows that

$$\begin{aligned}
&\left\|g(u_{n-1})-g(u_n)+\rho(N(Tu_{n-1},Au_{n-1})-N(Tu_{n-1},Au_n))\right\|^2\\
&\qquad\le\left(\zeta^2+2\rho\beta+\rho^2j^2\mu^2\right)\|u_{n-1}-u_n\|^2,\quad\forall n\ge 1,\\
&\left\|u_n-u_{n+1}-\eta(u_n,u_{n+1})\right\|^2\\
&\qquad\le\left(1+(1-2\alpha)\delta^2\right)\|u_n-u_{n+1}\|^2,\quad\forall n\ge 0.
\end{aligned}\tag{4.29}$$

The rest of the argument follows as in the proof of Theorem 4.1 and is omitted. This completes the proof. □

Theorem 4.5. *Let K, H, g, η, a, b, N, and F be as in Theorem 4.1. Assume that $T,A:K\to H$ are Lipschitz continuous with constants ξ, μ, respectively, T is relaxed (p,q)-cocoercive with respect to N in the first argument, A is g-relaxed Lipschitz with constant β with respect to N in the second argument. Let*

$$\begin{aligned}
J&=\frac{\delta\sqrt{\zeta^2-2\beta+j^2\mu^2}+\delta\sqrt{1-2q+(2p+1)i^2\xi^2}+\delta+r+d}{1+\sqrt{1-2\lambda+\zeta^2}},\\
L&=\frac{\lambda}{1+\sqrt{1-2\lambda+\zeta^2}},\qquad D=\delta^2-J^2,\qquad E=\alpha-JL.
\end{aligned}\tag{4.30}$$

If there exists a constant ρ satisfying

$$0<\rho<\frac{L}{J}\tag{4.31}$$

and one of (4.3)–(4.6), *then the generalized strongly nonlinear mixed variational-like inequality* (2.1) *possesses a solution $u\in K$ and the sequence $\{u_n\}_{n\ge 0}$ defined by Algorithm 3.2 converges to u.*

Proof. Notice that

$$
\begin{aligned}
&-\langle g(u_n)-g(u_{n+1}),u_n-u_{n+1}\rangle\\
&\quad\le \langle g(u_{n-1})-g(u_n)+u_{n-1}-u_n,\rho\eta(u_n,u_{n+1})-(u_n-u_{n+1})\rangle\\
&\qquad+\langle u_n-u_{n-1},\rho\eta(u_n,u_{n+1})-(u_n-u_{n+1})\rangle\\
&\qquad-\langle g(u_{n-1})-g(u_n)+N(Tu_{n-1},Au_{n-1})-N(Tu_{n-1},Au_n),\rho\eta(u_n,u_{n+1})\rangle\\
&\qquad+\langle u_{n-1}-u_n-(N(Tu_{n-1},Au_n)-N(Tu_n,Au_n)),\rho\eta(u_n,u_{n+1})\rangle\\
&\qquad+\langle u_n-u_{n-1},\rho\eta(u_n,u_{n+1})\rangle+\rho b(u_n-u_{n-1},u_n)\\
&\qquad+\rho b(u_{n-1}-u_n,u_{n+1})+\rho a(u_{n-1}-u_n,u_n-u_{n+1}),\quad \forall n\ge 1,\\
&\left\|\rho\eta(u_n,u_{n+1})-(u_n-u_{n+1})\right\|^2\\
&\quad\le \left(1-2\rho\alpha+\rho^2\delta^2\right)\|u_n-u_{n+1}\|^2,\quad \forall n\ge 0,\\
&\left\|g(u_{n-1})-g(u_n)+N(Tu_{n-1},Au_{n-1})-N(Tu_{n-1},Au_n)\right\|^2\\
&\quad\le \left(\zeta^2-2\beta+j^2\mu^2\right)\|u_{n-1}-u_n\|^2,\quad \forall n\ge 1,\\
&\|u_{n-1}-u_n-(N(Tu_{n-1},Au_n)-N(Tu_n,Au_n))\|^2\\
&\quad\le \left(1-2q+(2p+1)i^2\xi^2\right)\|u_{n-1}-u_n\|^2,\quad \forall n\ge 1.
\end{aligned}
\tag{4.32}
$$

The rest of the proof is similar to the proof of Theorem 4.1 and is omitted. This completes the proof. □

Remark 4.6. Theorems 4.1–4.5 extend, improve, and unify the corresponding results in [9, 12, 13].

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