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Research Article

On the Connection between Kronecker and Hadamard Convolution Products of Matrices and Some Applications

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We are concerned with Kronecker and Hadamard convolution products and present some important connections between these two products. Further we establish some attractive inequalities for Hadamard convolution product. It is also proved that the results can be extended to the finite number of matrices, and some basic properties of matrix convolution products are also derived.

1. Introduction

There has been renewed interest in the Convolution Product of matrix functions that is very useful in some applications; see for example [1–6]. The importance of this product stems from the fact that it arises naturally in divers areas of mathematics. In fact, the convolution product plays very important role in system theory, control theory, stability theory, and, other fields of pure and applied mathematics. Further the technique has been successfully applied in various fields of matrix algebra such as, in matrix equations, matrix differential equations, matrix inequalities, and many other subjects; for details see [1, 7, 8]. For example, in [2], Nikolaos established some inequalities involving convolution product of matrices and presented a new method to obtain closed form solutions of transition probabilities and dependability measures and then solved the renewal matrix equation by using the convolution product of matrices. In [6], Sumita established the matrix Laguerre transform to calculate matrix convolutions and evaluated a matrix renewal function, similarly, in [9], Boshnakov showed that the entries of the autocovariances matrix function can be expressed in terms of the Kronecker convolution product. Recently in [1], Kiliçman and Al Zhour presented the iterative solution of such coupled matrix equations based on the Kronecker convolution structures.

In this paper, we consider Kronecker and Hadamard convolution products for matrices and define the so-called Dirac identity matrix $D_n(t)$ which behaves like a group identity element under the convolution matrix operation. Further, we present some results which includes matrix equalities as well as inequalities related to these products and give attractive application to the inequalities that involves Hadamard convolution product. Some special cases of this application are also considered. First of all, we need the following notations. The notation $M_{m,n}^I$ is the set of all $m \times n$ absolutely integrable matrices for all $t \geq 0$, and if $m = n$, we write M_n^I instead of $M_{m,n}^I$. The notation $A^T(t)$ is the transpose of matrix function $A(t)$. The notations $\delta(t)$ and $D_n(t) = \delta(t)I_n$ are the Dirac delta function and Dirac identity matrix, respectively; here, the notation I_n is the scalar identity matrix of order $n \times n$. The notations $A(t) * B(t)$, $A(t) \odot B(t)$, and $A(t) \bullet B(t)$ are convolution product, Kronecker convolution product and Hadamard convolution product of matrix functions $A(t)$ and $B(t)$, respectively.

2. Matrix Convolution Products and Some Properties

In this section, we introduce Kronecker and Hadamard convolution products of matrices, obtain some new results, and establish connections between these products that will be useful in some applications.

Definition 2.1. Let $A(t) = [f_{ij}(t)] \in M_{m,n}^I$, $B(t) = [g_{jr}(t)] \in M_{n,p}^I$, and $C(t) = [z_{ij}(t)] \in M_{m,n}^I$. The convolution, Kronecker convolution and Hadamard convolution products are matrix functions defined for $t \geq 0$ as follows (whenever the integral is defined).

(i) Convolution product

$$A(t) * B(t) = (h_{ir}(t)) \quad \text{with } h_{ir}(t) = \sum_{k=1}^{n} \int_0^t f_{ik}(t-x) g_{kr}(x) dx = \sum_{k=1}^{n} f_{ik}(t) * g_{kr}(t). \tag{2.1}$$

(ii) Kronecker convolution product

$$A(t) \odot B(t) = \left[f_{ij}(t) * B(t) \right]_{ij}. \tag{2.2}$$

(iii) Hadamard convolution product

$$A(t) \bullet C(t) = \left[f_{ij}(t) * z_{ij}(t) \right]_{ij}. \tag{2.3}$$

where $f_{ij}(t) * B(t)$ is the ijth submatrix of order $n \times p$; thus $A(t) \odot B(t)$ is of order $mn \times np$, $A(t) * B(t)$ is of order $m \times p$, and similarly, the product $A(t) \bullet C(t)$ is of order $m \times n$.

The following two theorems are easily proved by using the definition of the convolution product and Kronecker product of matrices, respectively.

Theorem 2.2. *Let* $A(t)$, $B(t)$, $C(t) \in M_n^I$, *and let* $D_n(t) = \delta(t) I_n \in M_n^I$. *Then for scalars* α *and* β

(i)

$$\left(\alpha A(t) + \beta B(t)\right) * C(t) = \alpha(A(t) * C(t)) + \beta(B(t) * C(t)), \tag{2.4}$$

(ii)

$$(A(t) * B(t)) * C(t) = A(t) * (B(t) * C(t)), \tag{2.5}$$

(iii)

$$A(t) * D_n(t) = D_n(t) * A(t) = A(t), \tag{2.6}$$

(iv)

$$(A(t) * B(t))^T = B^T(t) * A^T(t). \tag{2.7}$$

Theorem 2.3. *Let* $A(t)$, $C(t) \in M_{m,n}^I$, $B(t) \in M_{p,q}^I$, *and let* $D_n(t) = \delta(t) I_n \in M_n^I$. *Then*

(i)

$$D_n(t) \odot A(t) = \operatorname{diag}(A(t), A(t), \ldots, A(t)), \tag{2.8}$$

(ii)

$$D_n(t) \odot D_m(t) = D_{nm}(t), \tag{2.9}$$

(iii)

$$(A(t) + C(t)) \odot B(t) = A(t) \odot B(t) + C(t) \odot B(t), \tag{2.10}$$

(iv)

$$(A(t) \odot B(t))^T = A^T(t) \odot B^T(t), \tag{2.11}$$

(v)

$$(A(t) \odot B(t)) * (C(t) \odot D(t)) = (A(t) * C(t)) \odot (B(t) * D(t)), \tag{2.12}$$

(vi)

$$(A(t) \odot D_m(t)) * (D_n(t) \odot B(t)) = (D_n(t) \odot B(t)) * (A(t) \odot D_m(t)) = A(t) \odot B(t). \tag{2.13}$$

The above results can easily be extended to the finite number of matrices as in the following corollary.

Corollary 2.4. *Let $A_i(t)$ and $B_i(t) \in M_n^I (1 \le i \le k)$ be matrices. Then*

(i)

$$\prod_{i=1}^{k} * (A_i(t) \odot B_i(t)) = \left(\prod_{i=1}^{k} * A_i(t)\right) \odot \left(\prod_{i=1}^{k} * B_i(t)\right), \tag{2.14}$$

(ii)

$$\prod_{i=1}^{k} \odot (A_i(t) * B_i(t)) = \left(\prod_{i=1}^{k} \odot A_i(t)\right) * \left(\prod_{i=1}^{k} \odot B_i(t)\right). \tag{2.15}$$

Proof. (i) The proof is a consequence of Theorem 2.3(v). Now we can proceed by induction on k. Assume that Corollary 2.4 holds for products of $k-1$ matrices. Then

$$\begin{aligned}
&(A_1(t) \odot B_1(t)) * (A_2(t) \odot B_2(t)) * \cdots * (A_k(t) \odot B_k(t)) \\
&\quad = \{(A_1(t) \odot B_1(t)) * (A_2(t) \odot B_2(t)) * \cdots * (A_{k-1}(t) \odot B_{k-1}(t))\} * (A_k(t) \odot B_k(t)) \\
&\quad = \{(A_1(t) * A_2(t) * \cdots * A_{k-1}(t)) \odot (B_1(t) * B_2(t) * \cdots * B_{k-1}(t))\} * (A_k(t) \odot B_k(t)) \\
&\quad = \{(A_1(t) * A_2(t) * \cdots * A_{k-1}(t) * A_k(t))\} \odot \{(B_1(t) * B_2(t) * \cdots * B_{k-1}(t) * B_k(t))\} \\
&\quad = \left(\prod_{i=1}^{k} * A_i(t)\right) \odot \left(\prod_{i=1}^{k} * B_i(t)\right).
\end{aligned} \tag{2.16}$$

Similarly we can prove (ii). □

Theorem 2.5. *Let $A(t) = [f_{ij}(t)]$, and let $B(t) = [g_{ij}(t)] \in M_{m,n}^I$. Then*

$$A \bullet B(t) = P_m^T(t) * (A \odot B)(t) * P_n(t). \tag{2.17}$$

Here, $P_n(t) = (\mathrm{Vec}\ E_{11}^{(n)}(t), \ldots, \mathrm{Vec}\ E_{nn}^{(n)}(t)) \in M_{n^2,n}$ and $E_{ij}(t) = e_i(t) * e_j^T(t)$ of order $n \times n$, $e_i(t)$ is the ith column of Dirac identity matrix $D_n(t) = \delta(t) I_n \in M_n$ with property $P_n^T(t) * P_n(t) = D_n(t)$. In particular, if $m = n$, then we have

$$A \bullet B(t) = P_n^T(t) * (A \odot B)(t) * P_n(t). \tag{2.18}$$

Proof. Compute

$$
\begin{aligned}
P_m^T(t) * (A \odot B)(t) * P_n(t) &= \left(\operatorname{Vec} E_{11}^{(m)}(t), \ldots, \operatorname{Vec} E_{mm}^{(m)}(t)\right)^T * (A \odot B)(t) \\
&\quad * \left(\operatorname{Vec} E_{11}^{(n)}(t), \ldots, \operatorname{Vec} E_{nn}^{(n)}(t)\right) \\
&= \sum_{k=1}^{n} \operatorname{diag}\ \left(f_{ik}(t), f_{2k}(t), \ldots, f_{mk}(t)\right) * B(t) * E_{kk}^{(n)}(t) \\
&= \left(\sum_{k=1}^{n} f_{ik}(t) * g_{ij}(t) * \delta_{jk}(t)\right) = \left(f_{ij}(t) * g_{ij}(t)\right) = A\bullet B(t).
\end{aligned}
\tag{2.19}
$$

This completes the proof of Theorem 2.5. □

Corollary 2.6. *Let $A_i(t) \in M_{m,n}^I (1 \le i \le k,\ k \ge 2)$. Then there exist two matrices $P_{km}(t)$ of order $m^k \times m$ and $P_{kn}(t)$ of order $n^k \times n$ such that*

$$
\prod_{i=1}^{k} \bullet A_i(t) = P_{km}^T(t) * \left(\prod_{i=1}^{k} \odot A_i(t)\right) * P_{kn}(t), \tag{2.20}
$$

where

$$
P_{km}^T(t) = \left(E_{11}^{(m)}(t), 0^{(m)}, \ldots, 0^{(m)}, E_{22}^{(m)}(t), 0^{(m)}, \ldots, 0^{(m)}, E_{mm}^{(m)}(t)\right) \tag{2.21}
$$

is of order $m \times m^k$, $0^{(m)}$ is an $m \times m$ matrix with all entries equal to zero, $E_{ij}^{(m)}(t)$ is an $m \times m$ matrix of zeros except for a $\delta(t)$ in the ijth position, and there are $\sum_{s=1}^{k-2} m^s$ zero matrices $0^{(m)}$ between $E_{ii}^{(m)}(t)$ and $E_{i+1,i+1}^{(m)}(t)$ $(1 \le i \le m-1)$. In particular, if $m = n$, then we have

$$
\prod_{i=1}^{k} \bullet A_i(t) = P_{km}^T(t) * \left(\prod_{i=1}^{k} \odot A_i(t)\right) * P_{km}(t). \tag{2.22}
$$

Proof. The proof is by induction on k. If $k = 2$, then the result is true by using (2.17). Now suppose that corollary holds for the Hadamard convolution product of k matrices. Then we have

$$
\begin{aligned}
\prod_{i=1}^{k+1} \bullet A_i(t) &= A_1(t) \bullet \left(\prod_{i=1}^{k+1} \bullet A_i(t)\right) = P_m^T(t) * \left(A_1(t) \odot \left(\prod_{i=1}^{k+1} \bullet A_i(t)\right)\right) * P_n(t) \\
&= P_m^T(t) * \left(\left(D_m(t) \odot P_{km}^T(t)\right) * \left(\prod_{i=1}^{k+1} \odot A_i(t)\right) * \left(D_n(t) \odot P_{kn}(t)\right)\right) * P_n(t) \\
&= \left(P_m^T(t) * \left(D_m(t) \odot P_{km}^T(t)\right)\right) * \left(\prod_{i=1}^{k+1} \odot A_i(t)\right) * \left(\left(D_n(t) \odot P_{kn}(t)\right) * P_n(t)\right),
\end{aligned}
\tag{2.23}
$$

which is based on the fact that

$$P_m^T(t) * \left(D_m(t) \odot P_{km}^T(t)\right) = P_{(k+1)m}^T(t), \qquad (D_n(t) \odot P_{kn}(t)) * P_n(t) = P_{(k+1)n}(t), \tag{2.24}$$

and thus the inductive step is completed. □

Corollary 2.7. *Let $A(t)$, $B(t) \in M_m^I$ and $P_m(t)$ be a matrix of zeros and $D_m(t)$ that satisfies the (2.17). Then $P_m^T(t) * P_m(t) = D_m(t)$ and $P_m * P_m^T$ is a diagonal $m^2 \times m^2$ matrix of zeros, and then the following inequality satisfied*

$$0 \le P_m(t) * P_m^T(t) \le D_{m^2}. \tag{2.25}$$

Proof. It follows immediately by the definition of matrix $P_m(t)$. □

Theorem 2.8. *Let $A(t)$ and $B(t) \in M_{m,n}^I$. Then for any $m^2 \times n^2$ matrix $L(t)$,*

$$P_m^T(t) * L(t) * L^T(t) * P_m(t) \ge \left(P_m^T(t) * L(t) * P_n(t)\right) * \left(P_m^T(t) * L(t) * P_n(t)\right)^T \ge 0. \tag{2.26}$$

Proof. By Corollary 2.7, it is clear that $D_{n^2}(t) \ge P_n(t) * P_n^T(t) \ge 0$ and so

$$\begin{aligned} P_m^T(t) * L(t) * D_{n^2}(t) * L^T(t) * P_m(t) &= P_m^T(t) * L(t) * L^T(t) * P_m(t) \\ &\ge P_m^T(t) * L(t) * P_n(t) * P_n^T(t) * L^T(t) * P_m(t) \\ &= \left(P_m^T(t) * L(t) * P_n(t)\right) * \left(P_m^T(t) * L(t) * P_n(t)\right)^T \ge 0. \end{aligned} \tag{2.27}$$

This completes the proof of Theorem 2.8. □

We note that Hadamard convolution product differs from the convolution product of matrices in many ways. One important difference is the commutativity of Hadamard convolution multiplication

$$A \bullet B(t) = B \bullet A(t). \tag{2.28}$$

Similarly, the diagonal matrix function can be formed by using Hadamard convolution multiplication with Dirac identity matrix. For example, if $A(t)$, $B(t) \in M_n^I$, and $D_n(t)$ Dirac identity then we have

(i) $A \bullet B(t) = A * B(t)$ if and only if $A(t)$ and $B(t)$ are both diagonal matrices;

(ii) $(A \bullet B(t)) \bullet D_n(t) = (A \bullet D_n(t)) * (B \bullet D_n(t))$.

3. Some New Applications

Now based on inequality (2.26) in the previous section we can easily make some different inequalities on using the commutativity of Hadamard convolution product. Thus we have the following theorem.

Theorem 3.1. *For matrices* $A(t)$ *and* $B(t) \in M_{m,n}^{I}$ *and for* $s \in [-1,1]$, *we have* $(A(t) * A^T(t)) \bullet (B(t) * B^T(t)) + s((A(t) * B^T(t) \bullet B(t) * A^T(t)))$

$$\geq (1+s)\Big((A(t)\bullet B(t)) * (A(t)\bullet B(t))^T\Big). \tag{3.1}$$

In particular, if $s = 0$, *then we have*

$$\Big(A(t) * A^T(t)\Big)\bullet\Big(B(t) * B^T(t)\Big) \geq (A(t)\bullet B(t)) * (A(t)\bullet B(t))^T. \tag{3.2}$$

Proof. Choose $L(t) = \alpha A(t) \odot B(t) + \beta B(t) \odot A(t)$, where $A(t)$, and $B(t) \in M_{m,n}^{I}$ and α, β are real scalars not both zero. Since

$$L(t) * L^T(t) = \Big\{(\alpha A(t) \odot B(t) + \beta B(t) \odot A(t)) * (\alpha A(t) \odot B(t) + \beta B(t) \odot A(t))^T\Big\}, \tag{3.3}$$

on using Theorem 2.5 we can easily obtain that

$$\begin{aligned} P_m^T(t) * L(t) * L^T(t) * P_m(t) &= \Big(\alpha^2\Big(A(t) * A^T(t)\Big)\bullet\Big(B(t) * B^T(t)\Big)\Big) \\ &\quad + \Big(\alpha\beta\Big(A(t) * B^T(t)\Big)\bullet\Big(B(t) * A^T(t)\Big)\Big) \\ &\quad + \Big(\alpha\beta\Big(B(t) * A^T(t)\Big)\bullet\Big(A(t) * B^T(t)\Big)\Big) \\ &\quad + \Big(\beta^2\Big(B(t) * B^T(t)\Big)\bullet\Big(A(t) * A^T(t)\Big)\Big) \\ &= \Big(\alpha^2+\beta^2\Big)\Big(\Big(A(t) * A^T(t)\Big)\bullet\Big(B(t) * B^T(t)\Big)\Big) \\ &\quad + 2\alpha\beta\Big(\Big(A(t) * B^T(t)\Big)\bullet\Big(B(t) * A^T(t)\Big)\Big). \end{aligned} \tag{3.4}$$

Now one can also easily show that

$$\Big(P_m^T(t) * L(t) * P_n(t)\Big) * \Big(P_m^T(t) * L(t) * P_n(t)\Big)^T = (\alpha+\beta)^2 (A(t)\bullet B(t)) * (A(t)\bullet B(t))^T. \tag{3.5}$$

By setting $s = 2\alpha\beta/(\alpha^2+\beta^2)$, then it follows that $s+1 = (\alpha+\beta)^2/(\alpha^2+\beta^2)$; further the arithmetic-geometric mean inequality ensures that $|s| \leq 1$ and the choices $\beta = 1$ and $\alpha \in [-1,1]$ thus s takes all values in $[-1,1]$. Now by using (3.4), (3.5) and inequality (2.26) we can establish Theorem 3.1. □

Further, Theorem 3.1 can be extended to the case of Hadamard convolution products which involves finite number of matrices as follows.

Theorem 3.2. *Let* $A_i \in M^I_{m,n}(1 \le i \le k,\ k \ge 2)$*. Then for real scalars* $\alpha_1,\ \alpha_2, \dots,\ \alpha_k$*, which are not all zero*

$$\left(\sum_{i=1}^{k}\alpha_i^2\right)\left(\prod_{i=1}^{k}\bullet\left(A_i(t) * A_i^T(t)\right)\right) + \left(\sum_{r=1}^{k-1}\mu_r\prod_{w=1}^{k}\bullet\left(A_w(t) * A^T_{(w+r)'}(t)\right)\right) \ge \left(\sum_{i=1}^{k}\alpha_i\right)^2\left(\prod_{i=1}^{k}\bullet A_i(t)\right)\left(\prod_{i=1}^{k}\bullet A_i(t)\right)^T, \tag{3.6}$$

where $\mu_r = \sum_{w=1}^{k}\alpha_w\alpha_{(w+r)'}$ *and* $w + r \equiv (w+r)' \bmod k$ *with* $1 \le (w+r)' \le k$.

Proof. Let

$$L(t) = \alpha_1(A_1(t) \odot A_2(t) \odot \cdots \odot A_k(t)) + \alpha_2(A_2(t) \odot \cdots \odot A_k(t) \odot A_1(t)) + \cdots + \alpha_k(A_k(t) \odot A_1(t) \odot \cdots \odot A_{k-1}(t)). \tag{3.7}$$

By taking indices "mod$k$" and using (2.20) of Corollary 2.6 follows that

$$\begin{aligned} L(t) * L^T(t) = {} & \alpha_1^2\left(A_1(t) * A_1^T(t)\right) \odot \cdots \odot \left(A_k(t) * A_k^T(t)\right) \\ & + \cdots + \alpha_k^2\left(A_k(t) * A_k^T(t)\right) \odot \left(A_1(t) * A_1^T(t)\right) \\ & \odot \cdots \odot \left(A_{k-1}(t) * A_{k-1}^T(t)\right) \\ & + \sum_{i \ne j}^{k}\alpha_i\alpha_j\left\{\left(A_i(t) * A_j^T(t)\right) \odot \left(A_{j+1}(t) * A_{j+1}^T(t)\right)\right. \\ & \left.\odot \cdots \odot \left(A_{j-1}(t) * A_{j-1}^T(t)\right)\right\}. \end{aligned} \tag{3.8}$$

Now on using Corollary 2.6 and the commutativity of Hadamard convolution product yields

$$\begin{aligned} P_{km}^T(t) * L(t) * L^T(t) * P_{km}(t) = {} & \left(\sum_{i=1}^{k}\alpha_i^2\right)\left(\prod_{i=1}^{k}\bullet\left(A_i(t) * A_i^T(t)\right)\right) \\ & + \left(\sum_{r=1}^{k-1}\mu_r\prod_{w=1}^{k}\bullet\left(A_w(t) * A^T_{(w+r)'}(t)\right)\right) \end{aligned} \tag{3.9}$$

where $\mu_r = \sum_w^k \alpha_w \alpha_{(w+r)'}$ and $w + r \equiv (w+r)'$ mod k with $1 \leq (w+r)' \leq k$ then

$$\begin{aligned}\left(P_{km}^T(t) * L(t) * P_{kn}(t)\right) &= \alpha_1 P_{km}^T(t) * (A_1(t) \odot A_2(t) \odot \cdots \odot A_k(t)) * P_{kn}(t) \\ &\quad + \alpha_2 P_{km}^T(t) * (A_2(t) \odot \cdots \odot A_k(t) \odot A_1(t)) * P_{kn}(t) \\ &\quad + \cdots + \alpha_k P_{km}^T(t) * (A_k(t) \odot A_1(t) \odot \cdots \odot A_{k-1}(t)) * P_{kn}(t) \\ &= \left(\sum_{i=1}^k \alpha_i\right)\left(\prod_{i=1}^k \bullet A_i(t)\right).\end{aligned} \tag{3.10}$$

Thus it follows that

$$\begin{aligned}\left(P_{km}^T(t) * L(t) * P_{kn}(t)\right)^T &= \left(\sum_{i=1}^k \alpha_i\right)\left(\prod_{i=1}^k \bullet A_i(t)\right)^T, \\ \left(P_{km}^T(t) * L(t) * P_{kn}(t)\right) * \left(P_{km}^T(t) * L(t) * P_{kn}(t)\right)^T &= \left(\sum_{i=1}^k \alpha_i\right)^2 \left(\prod_{i=1}^k \bullet A_i(t)\right) * \left(\prod_{i=1}^k \bullet A_i(t)\right)^T.\end{aligned} \tag{3.11}$$

Now by applying inequality (2.26), and (3.6) and (3.7) thus we establish Theorem 3.2. □

We note that many special cases can be derived from Theorem 3.2. For example, in order to see that inequality (3.6) is an extension of inequality (3.2) we set $\alpha_1 = 1$ and $\alpha_2 = \cdots = \alpha_k = 0$. Next, we recover inequality (3.1) of Theorem 3.1, by letting $k = 2$, then $\mu_1 = \sum_{w=1}^2 \alpha_w \alpha_{(w+1)'}$ with $w + 1 \equiv (w+1)'$ mod 2, that is, $\mu_1 = 2\alpha_1\alpha_2$ then we have

$$\begin{aligned}&\left(\alpha_1^2 + \alpha_2^2\right)\left(\left(A_1(t) * A_1^T(t)\right)\bullet\left(A_2(t) * A_2^T(t)\right)\right) + 2\alpha_1\alpha_2\left(\left(A_1(t) * A_2^T(t)\right)\bullet\left(A_2(t) * A_1^T(t)\right)\right) \\ &\quad \geq (\alpha_1 + \alpha_2)^2 (A_1(t)\bullet A(t)) * (A_1(t)\bullet A_2(t))^T.\end{aligned} \tag{3.12}$$

By simplification we have

$$\begin{aligned}&A\left({}_1(t) * A_1^T(t)\right)\bullet\left(A_2(t) * A_2^T(t)\right) + s\left(\left(A_1(t) * A_2^T(t)\right)\bullet\left(A_2(t) * A_1^T(t)\right)\right) \\ &\quad \geq (1+s)(A_1(t)\bullet A_2(t)) * (A_1(t)\bullet A_2(t))^T\end{aligned} \tag{3.13}$$

for every $s \in [-1,1]$, just as required. Finally, if we let $k=3$, $\alpha_1 = 1$, and $\alpha_2 = \alpha_3 = -1/2$, then on using Theorem 3.2 we have an attractive inequality as follows.

$$\begin{aligned}
&\left(A_1(t) * A_1^T(t)\right)\bullet A\left({}_2(t) * A_2^T(t)\right)\bullet A_3(t) * A_3^T(t) \\
&\quad\geq \frac{1}{2}\Big\{A_1\left((t) * A_2^T(t)\right)\bullet\left(A_2(t) * A_3^T(t)\right)\bullet\left(A_3(t) * A_1^T(t)\right) \\
&\qquad + \left(A_2(t) * A_1^T(t)\right)\bullet\left(A_3(t) * A_2^T(t)\right)\bullet\left(A_1(t) * A_3^T(t)\right)\Big\}.
\end{aligned} \tag{3.14}$$

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