Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 617185, 16 pages
doi:10.1155/2009/617185

Research Article

Generalized Vector Complementarity Problems with Moving Cones

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Received 3 October 2009; Accepted 30 November 2009

Recommended by Charles E. Chidume

We introduce and discuss a class of generalized vector complementarity problems with moving cones. We discuss the existence results for the generalized vector complementarity problem under inclusive type condition. We obtain equivalence results between the generalized vector complementarity problem, the generalized vector variational inequality problem, and other related problems. The theorems presented here improve, extend, and develop some earlier and very recent results in the literature.

1. Introduction and Preliminaries

It is well known that vector variational inequalities were initially studied by Giannessi [1] and ever since have been widely studied in infinite-dimensional spaces see, for example, [2–8] and the references therein.

Very recently, Huang et al. [9] considered a class of vector complementarity problems with moving cones. They established existence results of a solution for this class of vector complementarity problems under an inclusive type condition. They also obtained some equivalence results among a vector complementarity problem, a vector variational inequality problem, a vector optimization problem, a weak minimal element problem, and a vector unilateral optimization problem in ordered Banach spaces. Their results generalized the main results in [4].

The purpose of this paper is to introduce and discuss a class of generalized vector complementarity problems with moving cones which is a variable ordering relation. We derive existence of a solution for this class of generalized vector complementarity problems under an inclusive type condition. This inclusive condition requires that any two of the family of closed and convex cones satisfy an inclusion relation so long as their corresponding variables satisfy certain conditions. We also obtain some equivalence results among a generalized vector complementarity problem, a generalized vector variational inequality problem, a generalized vector optimization problem, a generalized weak minimal element problem, and a generalized vector unilateral optimization problem under some monotonicity conditions and some inclusive type conditions in ordered Banach spaces. The theorems presented in this paper improve, extend, and develop some earlier and very recent results in the literature including [4, 9].

Let X be a Banach space, and A a subset of X. The topological interior of a subset A in X is denoted by $\operatorname{int} A$. A nonempty subset P in X is called a convex cone if $P + P \subset P$, and $\lambda P \subset P$ for any $\lambda > 0$. The relations $\leq_P$ and $\nleq_P$ in X are defined as $x \leq_P y$ if $y - x \in P$ and $x \nleq_P y$ if $y - x \notin P$, for any $x, y \in X$. Similarly, we can define the relations $\leq_{\operatorname{int} P}$ and $\nleq_{\operatorname{int} P}$ if we replace the set P by $\operatorname{int} P$. P is called a pointed cone if P is a cone and $P \cap (-P) = \{0\}$.

Let $L(X, Y)$ be the space of all continuous linear mappings from X to Y. We denote the value of $l \in L(X, Y)$ at $x \in X$ by (l, x).

Let $X,\ Y$ be two Banach spaces, and $P : K \to 2^Y$ a set-valued mapping such that, for each $x \in K$, $P(x)$ is a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} P(x) \neq \emptyset$, and $T : X \to L(X, Y)$. Very recently, Huang et al. [9] introduced the following three kinds of vector complementarity problems.

(Weak) vector complementarity problem (VCP): finding $x \in K$ such that

$$\langle Tx, x\rangle \ngeq_{\operatorname{int} P(x)} 0, \quad \langle Tx, y\rangle \nleq_{\operatorname{int} P(x)} 0, \quad \forall y \in K. \tag{1.1}$$

Positive vector complementarity problem (PVCP): finding $x \in K$ such that

$$\langle Tx, x\rangle \ngeq_{\operatorname{int} P(x)} 0, \quad \langle Tx, y\rangle \geq_{P(x)} 0, \quad \forall y \in K. \tag{1.2}$$

Strong vector complementarity problem (SVCP): finding $x \in K$ such that

$$\langle Tx, x\rangle = 0, \quad \langle Tx, y\rangle \geq_{P(x)} 0, \quad \forall y \in K. \tag{1.3}$$

We remark that if $P(x) = P$ for all $x \in K$, where P is a closed, pointed, and convex cone in Y with nonempty interior $\operatorname{int} P(x)$, then (VCP), (PVCP) and (SVCP) reduce to the problems considered in Chen and Yang [4]. In [9], they actually only studied the first two kinds complementarity problems. For the existence results of (SVCP), we refer the reader to our recent results [Submitted, On the F-implicit vector complementarity problem].

Motivated and inspired by the above three kinds of vector complementarity problems, in this paper we introduce three kinds of generalized vector complementarity problems. Let $X,\ Y$ be two Banach spaces, and $P : K \to 2^Y$ a set-valued mapping such that, for each $x \in K$, $P(x)$ is a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} P(x) \neq \emptyset$, let $A : L(X, Y) \to L(X, Y)$ be a single-valued mapping, and $T : X \to 2^{L(X,Y)}$ a set-valued mapping, where $2^{L(X,Y)}$ is a collection of all nonempty subsets of $L(X, Y)$. We consider the following three kinds of generalized vector complementarity problems.

(Weak) generalized vector complementarity problem (GVCP): finding $x \in K$ and $u \in Tx$ such that

$$\langle Au, x\rangle \not\geq_{\operatorname{int} P(x)} 0, \quad \langle Au, y\rangle \not\leq_{\operatorname{int} P(x)} 0, \quad \forall y \in K. \tag{1.4}$$

Generalized positive vector complementarity problem (GPVCP): finding $x \in K$ and $u \in Tx$ such that

$$\langle Au, x\rangle \not>_{\operatorname{int} P(x)} 0, \quad \langle Au, y\rangle \geq_{P(x)} 0, \quad \forall y \in K. \tag{1.5}$$

Generalized strong vector complementarity problem (GSVCP): finding $x \in K$ and $u \in Tx$ such that

$$\langle Au, x\rangle = 0, \quad \langle Au, y\rangle \geq_{P(x)} 0, \quad \forall y \in K. \tag{1.6}$$

We remark that if $A = I$ the identity mapping of $L(X, Y)$, and $T : K = X \to 2^{L(X,Y)}$ is a single-valued mapping, then three kinds of generalized vector complementarity problems reduce to three kinds of vector complementarity problems in Huang et al. [9], respectively.

2. Existence of a Solution for GVCP

Huang et al. [9] established some equivalence results between the positive vector complementarity problem and the vector extremum problem and also sufficient conditions for the existence of a solution of the vector extremum problem. In this section, we extend their results to the cases involving the set-valued mappings.

Let X be an arbitrary real Hausdorff topological vector spaces, and Y a Banach space. $L(X, Y)$ denotes the space of all continuous linear mappings from X to Y. Let K be a nonempty set of X, and $P : K \to 2^Y$ a set-valued mapping such that, for each $x \in K$, $P(x)$ is a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} P(x) \neq \emptyset$. Let A be a subset of Y. For each $x \in K$, a point $z \in A$ is called a *minimal point* of A with respect to the cone $P(x)$ if $A \cap (z - P(x)) = \{z\}$; $\operatorname{Min}^{P(x)} A$ is the set of all minimal points of A with respect to the cone $P(x)$; a point $z \in A$ is called a *weakly minimal point* of A with respect to the cone $P(x)$ if $A \cap (z - \operatorname{int} P(x)) = \emptyset$; $\operatorname{Min}_w^{P(x)} A$ is the set of all weakly minimal points of A with respect to the cone $P(x)$, we refer the reader to [10] for more detail.

Let $A : L(X, Y) \to L(X, Y)$ be a single-valued mapping and let $T : X \to 2^{L(X,Y)}$ be a set-valued mapping. Now, we consider the following generalized vector complementarity problem (GVCP). Find $x \in K$ and $u \in Tx$ such that

$$\langle Au, x\rangle \not\geq_{\operatorname{int} P(x)} 0, \quad \langle Au, y\rangle \not\leq_{\operatorname{int} P(x)} 0, \quad \forall y \in K. \tag{2.1}$$

A feasible set of (GVCP) is

$$\mathfrak{F} = \left\{(x, u) \in K \times TK : u \in Tx, \ \langle Au, y\rangle \not\leq_{\operatorname{int} P(x)} 0, \ \forall y \in K\right\}. \tag{2.2}$$

We consider the following generalized vector optimization problem (GVOP):

$$\mathrm{Min}_P \langle Au, x\rangle \quad \text{subject to } (x,u) \in \mathfrak{F}. \tag{2.3}$$

A point $(x,u) \in \mathfrak{F}$ is called a weakly minimal solution of (GVOP) with respect to the cone $P(x)$, if $\langle Au, x\rangle$ is a weakly minimal point of (GVOP) with respect to the cone $P(x)$, that is, $\langle Au, x\rangle \in \mathrm{Min}_w^{P(x)}\{\langle Au, x\rangle : (x,u) \in \mathfrak{F}\}$. We denote the set of all weakly minimal solutions of (GVOP) with respect to the cone $P(x)$ by $\Omega_w^{P(x)}$ and the set of all weakly minimal solutions of (GVOP) by Ω_w, that is,

$$\Omega_w = \bigcup_{x\in K} \Omega_w^{P(x)}. \tag{2.4}$$

Theorem 2.1. *If $\Omega_w \neq \emptyset$ and, for some $x \in K$, there exists $(x,u) \in \Omega_w^{P(x)}$ such that $\langle Au, x\rangle \ngeq_{\mathrm{int}\,P(x)} 0$, then the generalized vector complementarity problem (GVCP) is solvable.*

Proof. Let $(x,u) \in \Omega_w^{P(x)}$ and $\langle Au, x\rangle \ngeq_{\mathrm{int}\,P(x)} 0$. Then $x \in K$, $u \in Tx$, and

$$\langle Au, x\rangle \ngeq_{\mathrm{int}\,P(x)} 0, \quad \langle Au, y\rangle \nleq_{\mathrm{int}\,P(x)} 0, \quad \forall y \in K. \tag{2.5}$$

It follows that x is a solution of (GVCP). This completes the proof. □

We remark that if $A = I$ the identity mapping of $L(X,Y)$ and T is a single-valued mapping from $K = X$ to $L(X,Y)$, then Theorem 2.1 coincides with Theorem 2.1 in Huang et al. [9].

Definition 2.2. Let $T : K \to 2^{L(X,Y)}$, $P : K \to 2^Y$ be two set-valued mappings with $\mathrm{int}\,P(x) \neq \emptyset$ for every $x \in K$, $A : L(X,Y) \to L(X,Y)$ a single-valued mapping, and $\mathfrak{F}$ a subset of $K \times TK$. We say that P is inclusive with respect to $\mathfrak{F}$ if for any $(x,u), (y,v) \in \mathfrak{F}$,

$$\langle Au, x\rangle \leq_{\mathrm{int}\,P(y)} \langle Av, y\rangle \text{ implies that } P(x) \subset P(y). \tag{2.6}$$

It is easy to see that, if $P(x) = P$ for all $x \in K$, where P is a closed, pointed, and convex cone in Y, then P is inclusive with respect to $\mathfrak{F}$.

Example 2.3. Let $X = Y = R^2$, $K = [0,1] \times [0,1]$, $A = I$ be the identity mapping of $L(X,Y)$. For each $x = (x_1, x_2) \in K$, define

$$P(x) = \left\{(z_1, z_2) \in R^2 : 0 \leq z_2 \leq (1 + x_1) z_1\right\}, \tag{2.7}$$

and, for each $x \in K$,

$$T(x) = \left\{ \begin{bmatrix} 3 + \dfrac{2}{1+x_1} & 0 \\ 0 & 3 + \dfrac{2}{1+x_2} \end{bmatrix} \right\} \subset L(X,Y). \tag{2.8}$$

Also, define

$$\langle u, x \rangle = \left(3x_1 + \frac{2x_1}{1+x_1}, 3x_2 + \frac{2x_2}{1+x_2}\right), \quad \forall (x,u) \in \mathfrak{F}. \tag{2.9}$$

Then it is easy to see that P is inclusive with respect to $\mathfrak{F}$. Indeed, for any $(x,u), (y,v) \in \mathfrak{F}$ with $x = (x_1, x_2)$, $y = (y_1, y_2)$, $u = Tx$, and $v = Ty$, if $\langle u, x\rangle \leq_{\operatorname{int} P(y)} \langle v, y\rangle$, then $\langle v, y\rangle - \langle u, x\rangle \in \operatorname{int} P(y)$ and so $x_1 < y_1$. Therefore, $P(x) \subset P(y)$ and P is inclusive with respect to $\mathfrak{F}$.

Theorem 2.4. *Suppose that P is inclusive with respect to $\mathfrak{F}$. If there exist at most a finite number of solutions for (GVCP), then (GVCP) is solvable if and only if $\Omega_w \neq \emptyset$, and there exists $(x,u) \in \Omega_w^{P(x)}$ such that $\langle Au, x\rangle \not\geq_{\operatorname{int} P(x)} 0$.*

Proof. Let η_1 be a solution of (GVCP). Then there exists $u_1 \in T\eta_1$ such that

$$\langle Au_1, \eta_1\rangle \not\geq_{\operatorname{int} P(\eta_1)} 0, \quad \langle Au_1, y\rangle \not\leq_{\operatorname{int} P(\eta_1)} 0, \quad \forall y \in K. \tag{2.10}$$

If $(\eta_1, u_1) \in \Omega_w^{P(\eta_1)}$, then

$$\langle Au_1, \eta_1\rangle \not\geq_{\operatorname{int} P(\eta_1)} 0, \tag{2.11}$$

and hence the conclusion holds. If $(\eta_1, u_1) \notin \Omega_w^{P(\eta_1)}$, by the definition of a weakly minimal solution, there exists $(\eta_2, u_2) \in \mathfrak{F}$ such that

$$\begin{gathered} \langle Au_2, y\rangle \not\leq_{\operatorname{int} P(\eta_2)} 0, \quad \forall y \in K, \\ \langle Au_2, \eta_2\rangle \leq_{\operatorname{int} P(\eta_1)} \langle Au_1, \eta_1\rangle \not\geq_{\operatorname{int} P(\eta_1)} 0. \end{gathered} \tag{2.12}$$

This implies that

$$\langle Au_2, \eta_2\rangle \not\geq_{\operatorname{int} P(\eta_1)} 0. \tag{2.13}$$

Since $\langle Au_2, \eta_2\rangle \leq_{\operatorname{int} P(\eta_1)} \langle Au_1, \eta_1\rangle$, and P is inclusive with respect to $\mathfrak{F}$, it follows that $P(\eta_2) \subset P(\eta_1)$ and this implies that

$$\langle Au_2, \eta_2\rangle \not\geq_{\operatorname{int} P(\eta_2)} 0. \tag{2.14}$$

Thus, η_2 is a solution of (GVCP) and $\eta_2 \neq \eta_1$. Continuing this process, there exists $(\eta_n, u_n) \in \mathfrak{F}$ such that η_n is a solution of (GVCP) and $(\eta_n, u_n) \in \Omega_w^{P(\eta_n)}$, since (GVCP) has at most a finite number of solutions. Thus, $\langle Au_n, \eta_n\rangle \in \operatorname{Min}_w^{P(\eta_n)}\{\langle Au, \eta_n\rangle : (\eta_n, u) \in \mathfrak{F}\}$ and

$$\langle Au_n, \eta_n\rangle \not\geq_{\operatorname{int} P(\eta_n)} 0. \tag{2.15}$$

Combining this result and Theorem 2.1, we have the conclusion of the theorem. □

Remark 2.5. (1) If $A = I$ the identity mapping of $L(X, Y)$, T is a single-valued mapping from X to $L(X, Y)$, and $P(x) = P$ for all $x \in X$, where P is a closed, pointed, and convex cone in Y, then $P(x)$ satisfies the inclusive condition with respect to $\mathfrak{F}$ and Theorem 2.4 reduces to Theorem 3.2 of Chen and Yang [4].

(2) If $A = I$ the identity mapping of $L(X, Y)$ and T is a single-valued mapping from $K = X$ to $L(X, Y)$, then Theorem 2.4 coincides with Theorem 2.2 of Huang et al. [9].

We next consider the generalized positive vector complementarity problem (GPVCP). Finding $x \in K$ and $u \in Tx$ such that

$$\langle Au, x\rangle \not\geq_{\operatorname{int} P(x)} 0, \quad \langle Au, y\rangle \geq_{P(x)} 0, \quad \forall y \in K. \tag{2.16}$$

Let

$$\mathfrak{G} = \{(x, u) \in K \times TK : u \in Tx,\ \langle Au, y\rangle \geq_{P(x)} 0,\ \forall y \in K\}. \tag{2.17}$$

Consider the following generalized vector optimization problem $(\text{GVOP})_0$ to be

$$\operatorname{Min}_P \langle Au, x\rangle \quad \text{subject to } (x, u) \in \mathfrak{G}. \tag{2.18}$$

We denote the set of all minimal points of $(\text{GVOP})_0$ with respect to the cone $P(x)$ by $\Gamma^{P(x)}$, that is, $\Gamma^{P(x)} = \operatorname{Min}^{P(x)}\{\langle Au, x\rangle : (x, u) \in \mathfrak{G}\}$, and denote the set of all minimal points of $(\text{GVOP})_0$ by

$$\Gamma = \bigcup_{x \in K} \Gamma^{P(x)}. \tag{2.19}$$

Using a similar argument of Theorem 2.1, we have the following results of solvability for (GPVCP).

Theorem 2.6. *If $\Gamma \neq \emptyset$ and there exists $(x, u) \in \Gamma^{P(x)}$ such that $\langle Au, x\rangle \not\geq_{\operatorname{int} P(x)} 0$, then (GPVCP) is solvable.*

Theorem 2.7. *Suppose that P is inclusive with respect to $\mathfrak{G}$. If there exist at most a finite number of solutions of (GPVCP), then (GPVCP) is solvable if and only if $\Gamma \neq \emptyset$, and there exists $(x, u) \in \Gamma^{P(x)}$ such that $\langle Au, x\rangle \not\geq_{\operatorname{int} P(x)} 0$.*

One remarks that If $A = I$ the identity mapping of $L(X, Y)$ and T is a single-valued mapping from $K = X$ to $L(X, Y)$, then Theorems 2.6 and 2.7 coincide with Theorems 2.3 and 2.4 of Huang et al. [9], respectively.

3. Equivalences between Generalized Vector Complementarity

3.1. *Problems and Generalized Weak Minimal Element Problems*

Let X, Y be two Banach spaces, and $P : K \to 2^Y$ a set-valued mapping such that, for each $x \in K$, $P(x)$ is a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} P(x) \neq \emptyset$, let $A : L(X,Y) \to L(X,Y)$ be a single-valued mapping, and $T : X \to 2^{L(X,Y)}$ a set-valued mapping, where $2^{L(X,Y)}$ is a collection of all nonempty subsets of $L(X,Y)$, and $f : X \to Y$ a given operator.

Define the feasible set associated with T and A

$$\widetilde{\mathfrak{F}} = \{x \in K : \text{there is } u \in Tx \text{ such that } \langle Au, y\rangle \nleq_{\operatorname{int} P(x)} 0,\ \forall y \in K\}. \tag{3.1}$$

We now consider the following five problems.

(i) The generalized vector optimization problem $(\text{GVOP})_l$: for a given $l \in L(X,Y)$, finding $x \in \widetilde{\mathfrak{F}}$ such that

$$l(x) \in \operatorname{Min}_w^{P(x)} l\left(\widetilde{\mathfrak{F}}\right). \tag{3.2}$$

(ii) The generalized weak minimal element problem (GWMEP): finding $x \in \widetilde{\mathfrak{F}}$ such that $x \in \operatorname{Min}_w^K \widetilde{\mathfrak{F}}$.

(iii) The generalized vector complementarity problem (GVCP): finding $x \in \widetilde{\mathfrak{F}}$ such that $\langle Au, x\rangle \ngeq_{\operatorname{int} P(x)} 0$ where $u \in Tx$ is associated with x in the definition of $\widetilde{\mathfrak{F}}$.

(iv) The generalized vector variational inequality problem (GVVIP): finding $x \in K$ and $u \in Tx$ such that

$$\langle Au, y - x\rangle \nleq_{\operatorname{int} P(x)} 0, \quad \forall y \in K. \tag{3.3}$$

(v) The generalized vector unilateral optimization problem (GVUOP): finding $x \in K$ such that $f(x) \in \operatorname{Min}_w^{P(x)} f(K)$.

We remark that if $A = I$ the identity mapping of $L(X,Y)$ and T is a single-valued mapping from X to $L(X,Y)$, then the $(\text{GVOP})_l$, (GWMEP), (GVCP), (GVVIP), and (GVUOP) reduce to Huang, et al.'s problems $(\text{VOP})_l$, (WMEP), (VCP), (VVIP), and (VUOP), respectively; see [9] for more details.

Definition 3.1 (see [4]). A linear operator $l : X \to Y$ is called weakly positive if, for any $x, y \in X$, $x \ngeq_{\operatorname{int} C} y$ implies that $l(x) \ngeq_{\operatorname{int} P(x)} l(y)$.

Definition 3.2. Let X and Y be two Banach spaces and l a linear operator from X to Y. If the image of any bounded set in X is a self-sequentially compact set in Y, then l is called completely continuous.

A mapping $f : X \to Y$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq_{P(x)} \lambda f(x) + (1-\lambda) f(y) \tag{3.4}$$

for all $x, y \in X$ and $0 \leq \lambda \leq 1$.

Definition 3.3. Let $A : L(X,Y) \to L(X,Y)$ and $f : X \to Y$ be two mappings. f is said to be A-subdifferentiable at $x_0 \in X$ if there exists $u_0 \in L(X,Y)$ such that

$$f(x) - f(x_0) \geq_{P(x_0)} \langle Au_0, x - x_0 \rangle, \quad \forall x \in X. \tag{3.5}$$

If f is A-subdifferentiable at $x_0 \in X$, then we define the A-subdifferential of f at x_0 as follows:

$$\partial_A f(x_0) := \{u \in L(X,Y) : f(x) - f(x_0) \geq_{P(x_0)} \langle Au, x - x_0 \rangle, \ \forall x \in X\}. \tag{3.6}$$

If f is A-subdifferentiable at each $x \in X$, then we say that f is A-subdifferentiable on X.

Remark 3.4. We note that as the mentions in [9], if X and Y are two Banach spaces, a mapping $f : X \to Y$ is Fréchet differentiable at $x_0 \in X$ if there exists a linear bounded operator $Df(x_0)$ such that

$$\lim_{x \to 0} \frac{\| f(x_0 + x) - f(x_0) - \langle Df(x_0), x \rangle \|}{\|x\|} = 0, \tag{3.7}$$

where $Df(x_0)$ is said to be the Fréchet derivative of f at x_0. The mapping f is said to be Fréchet differentiable on X if f is Fréchet differentiable at each point of X. If $f : X \to Y$ is convex and Fréchet differentiable on X, then

$$f(y) - f(x) \geq_{P(x)} \langle Df(x), y - x \rangle, \quad \forall x, y \in X. \tag{3.8}$$

If f is Fréchet differentiable at $x_0 \in X$, then f is I-subdifferentiable at $x_0 \in X$ and $Df(x_0) \in \partial_I f(x_0)$.

If f is A-subdifferentiable on X, then for each $x, y \in X$ we have

$$f(y) - f(x) \geq_{P(x)} \langle Au, y - x \rangle, \quad \forall u \in \partial_A f(x). \tag{3.9}$$

Definition 3.5. Let X be a Banach space, $K \subset X$ a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} K \neq \emptyset$. The norm $\| \cdot \|$ in X is called strictly monotonically increasing on K [9] if, for each $y \in K$,

$$x \in (\{y\} - \operatorname{int} K) \cap K \quad \text{only implies } \|x\| < \|y\|. \tag{3.10}$$

For the example of the strictly monotonically increasing property, we refer the reader to [9, Example 3.1].

Theorem 3.6. *Let X, Y be two Banach spaces, $K \subset X$ a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} K \neq \emptyset$, and $P : K \to 2^Y$ a set-valued mapping with closed, convex, and pointed cones values such that $\operatorname{int} P(x) \neq \emptyset$ for all $x \in K$. Suppose that*

(1) *$T = \partial_A f$ is the A-subdifferential of a convex operator $f : X \to Y$;*

(2) *l is a weakly positive linear operator;*

(3) *there exists* $x \in \widetilde{\mathfrak{F}}$ *such that* Au *is one to one and completely continuous, where* $u \in Tx$ *is associated with* x *in the definition of* $\widetilde{\mathfrak{F}}$*;*

(4) X *is a topological dual space of a real normed space and the norm* $\|\cdot\|$ *in* X *is strictly monotonically increasing on* K.

If (GVVIP) is solvable, then (GVOP)$_l$*, (GWMEP), (GVCP), and (GVUOP) are also solvable.*

Corollary 3.7 (see [9, Theorem 3.1]). *Let* X, Y *be two Banach spaces,* $K \subset X$ *a proper closed convex and pointed cone with apex at the origin and* $\operatorname{int} K \neq \emptyset$*, and* $\{P(x) : x \in X\}$ *that a family of closed, pointed, and convex cones in* Y *such that* $\operatorname{int} P(x) \neq \emptyset$ *for all* $x \in X$*. Suppose that*

(1) $T = Df$ *is the Fréchet derivative of a convex operator* $f : X \to Y$*;*

(2) l *is a weakly positive linear operator;*

(3) *there exists* $x \in \widetilde{\mathfrak{F}}$ *such that* Tx *is one to one and completely continuous, where* $\widetilde{\mathfrak{F}} = \{x \in K : \langle Tx, y\rangle \nleq_{\operatorname{int} P(x)} 0,\ \forall y \in K\}$*;*

(4) X *is a topological dual space of a real normed space and the norm* $\|\cdot\|$ *in* X *is strictly monotonically increasing on* K.

If (VVIP) is solvable, then (VOP)$_l$*, (WMEP), (VCP), and (VUOP) are also solvable.*

Proof. Since $A = I$ the identity mapping of $L(X, Y)$ and T is a single-valued mapping from X to $L(X, Y)$, we have

$$\begin{aligned} \widetilde{\mathfrak{F}} &= \left\{x \in K : \text{there is } u \in Tx \text{ such that } \langle Au, y\rangle \nleq_{\operatorname{int} P(x)} 0,\ \forall y \in K\right\} \\ &= \left\{x \in K : \langle Tx, y\rangle \nleq_{\operatorname{int} P(x)} 0,\ \forall y \in K\right\}. \end{aligned} \tag{3.11}$$

Utilizing Theorem 3.6, we immediately obtain the desired conclusion. □

Remark 3.8. If $P(x) = P$ for all $x \in X$, where P is a closed, pointed, and convex cone in Y, then Corollary 3.7 coincides with Theorem 3.1 of Chen and Yang [4].

We need the following propositions to prove Theorem 3.6.

Proposition 3.9. *Let* $A : L(X, Y) \to L(X, Y)$ *and* $f : X \to Y$ *be two mappings, and let* $T = \partial_A f$ *be the* A*-subdifferential of* f*. Then* x *solves (GVUOP) which implies that* x *solves (GVVIP). If in addition,* f *is a convex mapping, then conversely,* x *solves (GVVIP) which implies that* x *solves (GVUOP).*

Proof. Let x be a solution of (GVUOP). Then $x \in K$ and $f(x) \in \operatorname{Min}_w^{P(x)} f(K)$, that is, $f(x) \ngeq_{\operatorname{int} P(x)} f(y)$ for all $y \in K$. Since K is a convex cone,

$$f(x) \ngeq_{\operatorname{int} P(x)} f(x + t(w - x)), \quad 0 < t < 1,\ w \in K. \tag{3.12}$$

Also, since f is A-subdifferentiable on X, it follows that for all $u \in Tx = \partial_A f(x)$

$$f(x) \ngeq_{\operatorname{int} P(x)} f(x + t(w - x)) \geq_{P(x)} f(x) + \langle Au, t(w - x)\rangle, \quad 0 < t < 1,\ w \in K. \tag{3.13}$$

This implies that

$$\langle Au, t(w-x)\rangle \nleq_{\operatorname{int} P(x)} 0, \quad 0 < t < 1, \ w \in K, \tag{3.14}$$

and hence

$$\langle Au, w-x\rangle \nleq_{\operatorname{int} P(x)} 0, \quad \forall w \in K. \tag{3.15}$$

Thus, x solves (GVVIP).

Conversely, let x solve (GVVIP). Then there exists $\widehat{u} \in Tx = \partial_A f(x)$ such that

$$\langle A\widehat{u}, w-x\rangle \nleq_{\operatorname{int} P(x)} 0, \quad \forall w \in K. \tag{3.16}$$

Since f is A-subdifferentiable on X, we have for all $u \in Tx = \partial_A f(x)$

$$f(w) - f(x) \geq_{P(x)} \langle Au, w-x\rangle, \quad \forall w \in K, \tag{3.17}$$

and hence

$$f(w) - f(x) \geq_{P(x)} \langle A\widehat{u}, w-x\rangle \nleq_{\operatorname{int} P(x)} 0, \quad \forall w \in K. \tag{3.18}$$

This implies that

$$f(w) \nleq_{\operatorname{int} P(x)} f(x), \quad \forall w \in K. \tag{3.19}$$

Consequently, x solves (GVUOP). This completes the proof. □

Proposition 3.10. *If x solves (GVVIP), then x also solves (GVCP). Conversely, if $\langle Au, x\rangle \leq_{P(x)} 0$, for all $x \in K$, $u \in Tx$, then x solves (GVCP) which implies that x solves (GVVIP).*

Proof. Let x be a solution of (GVVIP). Then there exists $u \in Tx$ such that

$$\langle Au, y-x\rangle \nleq_{\operatorname{int} P(x)} 0, \quad \forall y \in K. \tag{3.20}$$

Letting $y = 0$, we get $\langle Au, x\rangle \ngeq_{\operatorname{int} P(x)} 0$. For $y = w + x$ with any $w \in K$, we have

$$\langle Au, w\rangle \nleq_{\operatorname{int} P(x)} 0, \quad \forall w \in K. \tag{3.21}$$

Thus, x is a solution of the (GVCP).

Conversely, let x solve the (GVCP). Then there exists $u \in Tx$ such that

$$\langle Au, x\rangle \leq_{P(x)} 0 \ngeq_{\operatorname{int} P(x)} \langle Au, y\rangle, \quad \forall y \in K. \tag{3.22}$$

This implies

$$\langle Au, x\rangle \ngeq_{\operatorname{int} P(x)} \langle Au, y\rangle, \quad \forall y \in K, \tag{3.23}$$

and so

$$\langle Au, y - x\rangle \ngeq_{\operatorname{int} P(x)} 0, \quad \forall y \in K. \tag{3.24}$$

This completes the proof. □

Proposition 3.11. *Let l be a weakly positive linear operator. Then x solves (GWMEP) which implies that x solves (GVOP)$_l$.*

Proof. Let x be a solution of (GWMEP). Then $x \in \widetilde{\mathfrak{F}}$ and $x \in \operatorname{Min}_w^K \widetilde{\mathfrak{F}}$, that is, for any $z \in \widetilde{\mathfrak{F}}$, $x \ngeq_{\operatorname{int} K} z$. Since l is a weakly positive linear operator, it follows that $l(x) \ngeq_{\operatorname{int} P(x)} l(z)$ and so

$$l(x) \in \operatorname{Min}_w^{P(x)} l\left(\widetilde{\mathfrak{F}}\right), \tag{3.25}$$

hence x solves (GVOP)$_l$. This completes the proof. □

Definition 3.12 (see [9]). Let X be a Banach space, $K \subset X$ a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} K \neq \emptyset$, E a nonempty subset of X.

(1) If, for some $x \in X$, $E_x = (\{x\} - K) \cap E \neq \emptyset$, then E_x is called a section of the set E.

(2) E is called weakly closed if $\{x_n\} \subset E$, $x \in X$, $\langle x^*, x_n\rangle \to \langle x^*, x\rangle$ for all $x^* \in X^*$, then $x \in E$.

(3) E is called bounded below if there exists a point p in X such that $E \subset p + K$.

Lemma 3.13 (see [11]). *Let X be a Banach space, $K \subset X$ a proper closed convex and pointed cone with apex at the origin and* $\operatorname{int} K \neq \emptyset$*, E a nonempty subset of X and X the topological dual space of a real normed space $(Z, \|\cdot\|_Z)$. Suppose there exists $x \in X$ such that the section E_x is weakly closed and bounded below and the norm $\|\cdot\|$ in X is strictly monotonically increasing, then the set E has at least one weakly minimal point.*

Lemma 3.14. *If (GVVIP) is solvable, then the feasible set $\widetilde{\mathfrak{F}}$ is nonempty.*

Proof. Let x be a solution of (GVVIP). Then there exists $u \in Tx$ such that

$$\langle Au, y - x\rangle \nleq_{\operatorname{int} P(x)} 0, \quad \forall y \in K. \tag{3.26}$$

Taking $y = z + x$ with any $z \in K$, we know that $y \in C$ and

$$\langle Au, z\rangle \nleq_{\operatorname{int} P(x)} 0, \quad \forall z \in K. \tag{3.27}$$

Thus, $x \in \widetilde{\mathfrak{F}}$. This completes the proof. □

Let X be a Banach space, $K \subset X$ a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} K \neq \emptyset$. For any $x, y \in X$, $[x, y] = (x+K) \cap (y-K)$ is called an order interval.

Lemma 3.15 (see [4]). *Let X be a Banach space, $K \subset X$ a proper closed convex and pointed cone with apex at the origin and* $\operatorname{int} K \neq \emptyset$. *If the norm $\|\cdot\|$ in X is strictly monotonically increasing, then the order intervals in X are bounded.*

Proposition 3.16. *Suppose that (GVVIP) is solvable and*

(1) *there exists x in $\widetilde{\mathfrak{F}}$ such that Au is one to one and completely continuous, where $u \in Tx$ is associated with x in the definition of $\widetilde{\mathfrak{F}}$;*

(2) *X is the topological dual space of a real normed space $(Z, \|\cdot\|_Z)$ and the norm $\|\cdot\|$ in X is strictly monotonically increasing.*

Then (GWMEP) has at least one solution.

Proof. By the assumption and Lemma 3.14, $\widetilde{\mathfrak{F}} \neq \emptyset$. Let $x \in \widetilde{\mathfrak{F}}$ be a point such that Au is one to one and completely continuous, where $u \in Tx$ is associated with x in the definition of $\widetilde{\mathfrak{F}}$, and let $\{y_n\} \subset \widetilde{\mathfrak{F}}$ with $y_n \to y$ (weakly). Since

$$\widetilde{\mathfrak{F}}_x = (\{x\} - C) \cap \widetilde{\mathfrak{F}} \subset (\{x\} - C) \cap C = [0, x], \tag{3.28}$$

by Lemma 3.15, $[0, x]$ is bounded and so is $\widetilde{\mathfrak{F}}_x$. Since Au is completely continuous, $\langle Au, \widetilde{\mathfrak{F}}_x \rangle$ is a self-sequentially compact set and so $\{\langle Au, y_n \rangle\} \subset \langle Au, \widetilde{\mathfrak{F}}_x \rangle$ implies that there exists a subsequence $\{\langle Au, y_{n_k} \rangle\}$ which converges to $z \in \langle Au, \widetilde{\mathfrak{F}}_x \rangle$. We get a point $y_0 \in \widetilde{\mathfrak{F}}_x$ such that

$$\langle Au, y_{n_k} \rangle \longrightarrow \langle Au, y_0 \rangle \quad \text{(strongly)}. \tag{3.29}$$

On the other hand, since $y_n \to y$ (weakly) and Au is completely continuous,

$$\langle Au, y_n \rangle \longrightarrow \langle Au, y \rangle \quad \text{(strongly)}. \tag{3.30}$$

By the uniqueness of the limit, we get $\langle Au, y \rangle = \langle Au, y_0 \rangle$. Since Au is one to one, $y = y_0$, and so $y \in \widetilde{\mathfrak{F}}_x$. Since $\widetilde{\mathfrak{F}}_x$ is weakly closed, it follows from Lemma 3.13 that $\widetilde{\mathfrak{F}}$ has a weakly minimal point p such that $p \not\geq_{\operatorname{int} P(p)} x$ for all $x \in \widetilde{\mathfrak{F}}$. Therefore, (GWMEP) has at least one solution. This completes the proof. □

Definition 3.17. Let X, Y be two Banach spaces, $K \subset X$ a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} K \neq \emptyset$, and $P : K \to 2^Y$ a set-valued mapping with closed, convex and pointed cones values such that $\operatorname{int} P(x) \neq \emptyset$ for all $x \in K$. Let $A : L(X, Y) \to L(X, Y)$ be a single-valued mapping and $T : X \to 2^{L(X,Y)}$ a set-valued mapping. T is called A-positive if

$$\langle Au, y \rangle \geq_{P(x)} 0, \quad \forall x, y \in K, \; u \in Tx. \tag{3.31}$$

We now consider the generalized positive vector complementarity problem (GPVCP). Finding $x \in K$ and $u \in Tx$ such that

$$\langle Au, x\rangle \not\geq_{\operatorname{int} P(x)} 0, \quad \langle Au, y\rangle \geq_{P(x)} 0, \quad \forall y \in K. \tag{3.32}$$

The feasible set related to (GPVCP) is defined as

$$\widetilde{\mathfrak{F}}_0 = \{x \in K : \text{there is } u \in Tx \text{ such that } \langle Au, y\rangle \geq_{P(x)} 0,\ \forall y \in K\}. \tag{3.33}$$

Let us consider the following problems.

The generalized vector optimization problem $(\text{GVOP})_{l0}$: finding $x \in \widetilde{\mathfrak{F}}_0$ such that $l(x) \in \operatorname{Min}_w^P l(\widetilde{\mathfrak{F}}_0)$.

The generalized weak minimal element problem $(\text{GWMEP})_0$: finding $x \in \widetilde{\mathfrak{F}}_0$ such that $x \in \operatorname{Min}_w^K \widetilde{\mathfrak{F}}_0$.

The generalized positive vector complementarity problem (GPVCP): finding $x \in \widetilde{\mathfrak{F}}_0$ such that

$$\langle Au, x\rangle \not\geq_{\operatorname{int} P(x)} 0, \tag{3.34}$$

where $u \in Tx$ is associated with x in the definition of $\widetilde{\mathfrak{F}}_0$.

The generalized vector variational inequality problem (GVVIP): finding $x \in K$ and $u \in Tx$ such that

$$\langle Au, y - x\rangle \not\leq_{\operatorname{int} P(x)} 0, \quad \forall y \in K. \tag{3.35}$$

The generalized vector unilateral optimization problem (GVUOP): for a given mapping $f : X \to Y$, finding $x \in K$ such that $f(x) \in \operatorname{Min}_w^P f(K)$.

Definition 3.18. A set-valued mapping $T : X \to 2^{L(X,Y)}$ is said to be A-strictly monotone where $A : L(X,Y) \to L(X,Y)$ is single-valued, if

$$\langle Au - Av, x - y\rangle \geq_{\operatorname{int} P(x)} 0, \quad \forall x, y \in X,\ x \neq y,\ u \in Tx,\ v \in Ty. \tag{3.36}$$

Definition 3.19 (see [9]). We say that $P(x)$ satisfies an inclusive condition if, for any $x, y \in X$,

$$x \leq_{\operatorname{int} C} y \quad \text{only implies that } P(x) \subset P(y). \tag{3.37}$$

It is easy to see that, if $P(x) = P$ for all $x \in X$, where P is a closed, pointed, and convex cone in Y, then $P(x)$ satisfies the inclusive condition.

Example 3.20. Let $X = (-\infty, +\infty)$, $C = [0, +\infty)$, $Y = R^2$, and

$$P(x) = \begin{cases} \{(z_1, z_2) \in R^2 : 0 \le z_2 \le 2z_1\}, & x \in (-\infty, 2), \\ \{(z_1, z_2) \in R^2 : 0 \le z_2 \le xz_1\}, & x \in [2, 5), \\ \{(z_1, z_2) \in R^2 : 0 \le z_2 \le 5z_1\}, & x \in [5, +\infty) \end{cases} \tag{3.38}$$

for all $x \in X$. Then it is easy to check that $P(x)$ satisfies the inclusive condition.

Proposition 3.21. *Let T be A-strictly monotone and x a solution of (GPVCP). If P satisfies the inclusive condition, then x is a weakly minimal point of $\widetilde{\mathfrak{F}}_0$ (i.e., x solves (GWMEP)$_0$).*

Proof. It is easy to see that $x \in \widetilde{\mathfrak{F}}_0 \subset K$. If $x \in bd(K)$ (where $bd(K)$ denotes the boundary of K), then x solves (GWMEP)$_0$. Otherwise, there exists $x' \in \widetilde{\mathfrak{F}}_0$ such that $x \ge_{\operatorname{int} K} x'$ and so

$$x = x - x' + x' \in \operatorname{int} K + K \subset \operatorname{int} K, \tag{3.39}$$

which is a contradiction. If $x \in \operatorname{int} K$, by the A-strict monotonicity of T,

$$\langle Au, x - y \rangle \ge_{\operatorname{int} P(x)} \langle Av, x - y \rangle, \quad \forall y \in \widetilde{\mathfrak{F}}_0,\ y \ne x,\ v \in Ty. \tag{3.40}$$

Suppose $x \ge_{\operatorname{int} K} y$. Since T is A-positive, $\langle Av, x - y \rangle \ge_{P(y)} 0$ and

$$\langle Au, x - y \rangle \ge_{\operatorname{int} P(x)} \langle Av, x - y \rangle \ge_{P(y)} 0. \tag{3.41}$$

By the assumption, we get $P(y) \subset P(x)$ and so

$$\langle Au, x - y \rangle \in \langle Av, x - y \rangle + \operatorname{int} P(x) \subset P(y) + \operatorname{int} P(x) \subset P(x) + \operatorname{int} P(x) = \operatorname{int} P(x). \tag{3.42}$$

It follows that

$$\langle Au, x - y \rangle \ge_{\operatorname{int} P(x)} 0, \tag{3.43}$$

and thus

$$0 \not\le_{\operatorname{int} P(x)} \langle Au, x \rangle \ge_{P(x)} \langle Au, y \rangle + k \tag{3.44}$$

for some $k \in \operatorname{int} P(x)$. This implies

$$\langle Au, y \rangle + k \not\ge_{\operatorname{int} P(x)} 0. \tag{3.45}$$

Since $k \in \operatorname{int} P(x)$ and $x \in \widetilde{\mathfrak{F}}_0$,

$$\langle Au, y \rangle + k \in P(x) + \operatorname{int} P(x) \subset \operatorname{int} P(x), \tag{3.46}$$

and so

$$\langle Au, y\rangle + k \geq_{\operatorname{int} P(x)} 0, \tag{3.47}$$

which leads to a contradiction. Therefore, $x \geq_{\operatorname{int} K} y$ does not hold, that is, $x \not\geq_{\operatorname{int} K} y$ for all $y \in \widetilde{\mathfrak{F}}_0$. It follows that x solves (GWMEP)$_0$. This completes the proof. □

Proposition 3.22. *If x solves (GPVCP), then x also solves (GVVIP).*

Proof. Suppose that x solves (GPVCP). Then $x \in K$ and there exists $u \in Tx$ such that

$$\langle Au, x\rangle \not\geq_{\operatorname{int} P(x)} 0, \quad \langle Au, y\rangle \geq_{P(x)} 0, \quad \forall y \in K. \tag{3.48}$$

If $\langle Au, y - x\rangle \leq_{\operatorname{int} P(x)} 0$, then

$$\langle Au, x\rangle = -\langle Au, y - x\rangle + \langle Au, y\rangle \in \operatorname{int} P(x) + P(x) \subset \operatorname{int} P(x), \tag{3.49}$$

and so

$$\langle Au, x\rangle \geq_{\operatorname{int} P(x)} 0, \tag{3.50}$$

which is a contradiction. It follows that

$$\langle Au, y - x\rangle \not\leq_{\operatorname{int} P(x)} 0, \tag{3.51}$$

and x solves (GVVIP). This completes the proof. □

Similarly, we can obtain other equivalence conditions. We have the following theorem.

Theorem 3.23. *Let X, Y be two Banach spaces, $K \subset X$ a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} K \neq \emptyset$, and $\{P(x) : x \in X\}$ a family of closed, pointed, and convex cones in Y such that $\operatorname{int} P(x) \neq \emptyset$ for all $x \in X$. Suppose that P satisfies the inclusive condition and*

(1) *$T = \partial_A f$ is the A-subdifferential of the convex operator $f : X \to Y$;*

(2) *l is a weakly positive linear operator;*

(3) *T is A-strictly monotone.*

If (GPVCP) is solvable, then (GVOP)$_{l0}$, (GWMEP)$_0$, (GPVCP), (GVVIP), and (GVUOP) have at least a common solution.

Corollary 3.24 (see [9, Theorem 3.2]). *Let X, Y be two Banach spaces, $K \subset X$ a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} K \neq \emptyset$, and $\{P(x) : x \in X\}$ a family of closed, pointed, and convex cones in Y such that $\operatorname{int} P(x) \neq \emptyset$ for all $x \in X$. Suppose that P satisfies the* *inclusive condition and*

(1) $T = Df$ *is the Fréchet derivative of the convex operator* $f : X \to Y$*;*

(2) l *is a weakly positive linear operator;*

(3) T *is strictly monotone.*

If (PVCP) is solvable, then (VOP)$_{l0}$*, (WMEP)*$_0$*, (PVCP), (VVIP), and (VUOP) have at least a common solution.*

Proof. Note that $A = I$ the identity mapping of $L(X, Y)$ and T is a single-valued mapping from X to $L(X, Y)$. From Theorem 3.23, we immediately obtain the desired conclusion. □

Remark 3.25. If $P(x) = P$ for all $x \in X$, where P is a closed, pointed, and convex cone in Y, then Corollary 3.24 coincides with Theorem 4.1 of Chen and Yang [4].

Acknowledgments

This research of the first author was partially supported by the National Science Foundation of China (10771141), Ph.D. Program Foundation of Ministry of Education of China (20070270004), and Science and Technology Commission of Shanghai Municipality Grant (075105118). The second author would like to thank the support of NSC97-2115-M-039-001-Grant from the National Science Council of Taiwan.

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