Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 419845, 24 pages
doi:10.1155/2009/419845

Research Article

Sharp Hardy-Sobolev Inequalities with General Weights and Remainder Terms

Yaotian Shen and Zhihui Chen

Department of Mathematics, South China University of Technology, Guangzhou 510640, China

Correspondence should be addressed to Zhihui Chen, mazhchen@scut.edu.cn

Received 18 December 2008; Accepted 10 August 2009

Recommended by Panayiotis Siafarikas

We consider a class of sharp Hardy-Sobolev inequality, where the weights are functions of the distance from a surface. It is proved that the Hardy-Sobolev inequality can be successively improved by adding to the right-hand side a lower-order term with optimal weight and constant.

1. Introduction

The classical Hardy inequality says

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \mathrm{d}x \le \left| \frac{p}{N-p} \right|^p \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x, \quad u \in C_0^\infty\left(\mathbb{R}^N \setminus \{0\}\right), \tag{1.1}$$

where the constant $|p/(N-p)|^p$ is optimal but never attained; see, for example, [1–4]. This suggests that one might look for an error term. Brezis and Vázques [5] showed that if Ω is a bounded domain in $\mathbb{R}^N$, $N \ge 3$, with $0 \in \Omega$, then there exists a positive constant λ_Ω such that

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \ge \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} \mathrm{d}x + \lambda_\Omega \int_\Omega u^2 \mathrm{d}x, \quad u \in H_0^1(\Omega). \tag{1.2}$$

This result was extended to the L^p setting by Gazzola et al. [6]. Adimurthi et al. proved that Hardy's inequality can be successively improved by adding lower order terms; see [7, 8] for details. Abdellaoui et al. [9] obtained Hardy's inequality with the type of weight $|x|^{-p\gamma}$. See [10, 11] for the case of general weight $\phi(|x|)$.

Another type of Hardy's inequality contains weight involving the distant to the boundary of the domain. For a convex domain $\Omega \subset \mathbb{R}^N$ with smooth boundary the Hardy inequality

$$\int_\Omega \frac{|u|^p}{d^p} \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |\nabla u|^p \mathrm{d}x, \quad u \in C_0^\infty(\Omega) \tag{1.3}$$

is valid with $(p/(p-1))^p$ being the best constant, where d is the distance to the boundary $\partial\Omega$, that is, $d = d(x) = \operatorname{dist}(x, \partial\Omega)$, cf [12, 13]. Brezis and Marcus [14] proved that for bounded and convex domain Ω there holds

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \geq \frac{1}{4}\int_\Omega \frac{u^2}{d^2} \mathrm{d}x + \frac{1}{4L^2}\int_\Omega u^2 \mathrm{d}x, \quad u \in C_0^\infty(\Omega), \tag{1.4}$$

where $L = \operatorname{diam}\Omega$.

Throughout this paper, $p > 1$, Ω is a domain in $\mathbb{R}^N$, $N \geq 2$, and $K \subset \overline{\Omega}$ is a piecewise smooth closed and connected surface of codimension $k = 1, \ldots, N$. The distance from K is denoted by d, that is $d = d(x) = \operatorname{dist}(x, K)$. Then d is a Lipschitz continuous function with $|\nabla d| = 1$ a.e.

Suppose that for $p \neq k$, the following inequality holds in the weak sense:

$$\Delta_p d^{(p-k)/(p-1)} \leq 0, \quad \text{in } \Omega \setminus \mathrm{K}. \tag{C}$$

Define $X_1(t) = (1 - \log t)^{-1}$ for $t \in (0,1)$, and recursively $X_k(t) = X_1(X_{k-1}(t))$ for $k = 2, 3, \ldots$. Barbatis et al. [15] proved that if $\sup_{x\in\Omega} d(x) < \infty$, then there exists a positive constant $D_0 = D_0(k,p) \geq \sup_{x\in\Omega} d(x)$ such that for any $D \geq D_0$, $m \in \mathbb{N}$ and all $u \in W_0^{1,p}(\Omega \setminus K)$ there holds

$$\begin{aligned} &\int_\Omega |\nabla u|^p \mathrm{d}x - \left|\frac{k-p}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} \mathrm{d}x \\ &\quad \geq \frac{p-1}{2p}\left|\frac{k-p}{p}\right|^{p-2} \left(\sum_{i=1}^m \int_\Omega \frac{|u|^p}{d^p} X_1^2\left(\frac{d}{D}\right) \cdots X_i^2\left(\frac{d}{D}\right) \mathrm{d}x\right), \end{aligned} \tag{1.5}$$

and the constants in front of integrals are optimal. The authors also obtained the result for the degenerate case of $p = k$.

Let ϕ be positive and continuous in $(0, \infty)$. In this paper, we are concerned with a general class of sharp Hardy inequality with general weight $\phi(d)$. Define

$$\overline{h}(r_1, r_2) = c_0 \int_{r_1}^{r_2} \left(\phi r^{k-1}\right)^{-1/(p-1)} \mathrm{d}r \tag{1.6}$$

for $0 \leq r_1 \leq r_2 \leq \infty$, where c_0 is a positive constant. Let us consider three cases:

(A_1) $\overline{h}(r,\infty) < \infty$ and $\overline{h}(0,r) = \infty$ for $r > 0$;

(A_2) $\overline{h}(r,\infty) = \infty$ and $\overline{h}(0,r) = \infty$ for $r > 0$;

(A_3) $\overline{h}(r,\infty) = \infty$ and $\overline{h}(0,r) < \infty$ for $r > 0$.

Definition 1.1. Let $p > 1$. If (A_1) or (A_2) occurs, we denote by $W_0^{1,p}(\Omega,\phi)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,p,\phi} = \left(\int_\Omega \phi(d)|\nabla u|^p \mathrm{d}x\right)^{1/p}. \tag{1.7}$$

If (A_3) occurs, we denote by $W_0^{1,p}(\Omega \setminus K,\phi)$ the completion of $C_0^\infty(\Omega \setminus K)$ with respect to the above norm. For simplicity, we use W to denote $W_0^{1,p}(\Omega,\phi)$ or $W_0^{1,p}(\Omega \setminus K,\phi)$.

Let $r > 0$, define

$$\overline{h}(r) = \begin{cases} \overline{h}(r,\infty), & \text{if } (A_1) \text{ occurs},\\ \overline{h}(r,D), & \text{if } (A_2) \text{ occurs},\\ \overline{h}(0,r), & \text{if } (A_3) \text{ occurs},\end{cases} \tag{1.8}$$

$$h(r) = \overline{h}^{-(p-1)/p}(r) = \begin{cases} \left(c_0\displaystyle\int_r^\infty \left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)/p}, & \text{if } (A_1) \text{ occurs},\\ \left(c_0\displaystyle\int_r^D \left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)/p}, & \text{if } (A_2) \text{ occurs},\\ \left(c_0\displaystyle\int_0^r \left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)/p}, & \text{if } (A_3) \text{ occurs},\end{cases} \tag{1.9}$$

where D is a positive constant such that $\Omega \subset B_D(0)$.

Theorem 1.2. *Let $p > 1$, Ω be a bounded domain in $\mathbb{R}^N$ and K a piecewise smooth surface of codimension k, $k = 1,\ldots,N$. Assume*

$$\operatorname{div}\left(\phi(d)\left|\nabla\overline{h}(d)\right|^{p-2}\nabla\overline{h}(d)\right) \leq 0 \quad \text{in } \Omega \setminus K. \tag{C*}$$

Then for all $u \in W$,

$$\int_\Omega \psi(d)|u|^p \mathrm{d}x \leq \int_\Omega \phi(d)|\nabla u|^p \mathrm{d}x, \tag{1.10}$$

where $\psi = \phi|h'/h|^p$. Moreover, the constant 1 is optimal, that is,

$$1 = \inf_{u\in W\setminus\{0\}} \frac{\int_\Omega \psi(d)|u|^p}{\int_\Omega \phi(d)|\nabla u|^p \mathrm{d}x}. \tag{1.11}$$

Example 1.3. Let $K = \mathbb{R}^{N-k}$. Then $d = |x'| = (x_1^2 + \cdots + x_k^2)^{1/2}$. If $\phi(r) = r^\alpha$ with $\alpha > p - k$, then

$$h(r) = r^{(p-k-\alpha)/p}, \qquad \varphi(r) = \left(\frac{k+\alpha-p}{p}\right)^p r^{\alpha-p}, \tag{1.12}$$

and we have by Theorem 1.2

$$\int_{\mathbb{R}^N} |u|^p \left|x'\right|^\alpha \mathrm{d}x \le \left(\frac{p}{\alpha+k}\right)^p \int_{\mathbb{R}^N} |\nabla u|^p \left|x'\right|^{\alpha+p} \mathrm{d}x, \quad u \in C_0^\infty\left(\mathbb{R}^N\right), \tag{1.13}$$

see also Secchi et al. [16]. If $\phi(r) = r^\alpha$ with $\alpha = p - k$, then

$$h(r) = \left(\ln \frac{D}{r}\right)^{(p-1)/p}, \qquad \varphi(r) = \left(\frac{p}{p-1}\right)^p r^{\alpha-p} \left(\ln \frac{D}{r}\right)^{-p}. \tag{1.14}$$

If $\phi(r) = r^\alpha$ with $\alpha = p - k = 0$, Theorem 1.2 turns to be Theorem 4.2 in [17].

Let $r > 0$, define

$$h_1(r) = \begin{cases} \dfrac{p}{(p-1)c_0} \ln \dfrac{h(r)}{h(D)}, & \text{if } (\mathrm{A}_1) \text{ occurs}, \\ \dfrac{p}{(p-1)c_0} \ln h(r), & \text{if } (\mathrm{A}_2) \text{ occurs}, \\ \dfrac{p}{(p-1)c_0} \ln \dfrac{h(D)}{h(r)}, & \text{if } (\mathrm{A}_3) \text{ occurs}, \end{cases} \tag{1.15}$$

and $h_{i+1}(r) = \ln \mathrm{e} h_i(r)$ for $i = 1, 2, \ldots$.

For convenience, we write

$$I_{m,\phi}[u] = \int_\Omega \phi(d) |\nabla u|^p \mathrm{d}x - \int_\Omega \varphi(d) |u|^p \mathrm{d}x - \frac{p}{2(p-1)c_0^2} \sum_{i=1}^m \int_\Omega \varphi h_1^{-2}(d) \cdots h_i^{-2}(d) |u|^p \mathrm{d}x. \tag{1.16}$$

Theorem 1.4. *Let $p > 1$, Ω be a bounded domain in $\mathbb{R}^N$ and K a piecewise smooth surface of codimension k, $k = 1, \ldots, N$. Assume that (C^*) holds, then*

(1) *there exists a positive constant $D_0 = D_0(k, p) > \sup_{x\in\Omega} d(x)$ such that for all $D \ge D_0$ and $u \in W$, there holds*

$$\int_\Omega \phi(d) |\nabla u|^p \mathrm{d}x - \int_\Omega \varphi(d) |u|^p \mathrm{d}x \ge \frac{p}{2(p-1)c_0^2} \sum_{i=1}^m \int_\Omega \varphi(d) h_1^{-2}(d) \cdots h_i^{-2}(d) |u|^p \mathrm{d}x \tag{1.17}$$

where $\varphi = \phi |h'/h|^p$, if in addition $p \ge 2$ and (A_1) occurs, then one can take $D_0 = \sup_{x\in\Omega} d(x)$.

(2) *the constants in* (1.17) *are optimal, that is,*

$$\frac{p}{2(p-1)c_0^2} = \inf_{u\in W\setminus\{0\}} \frac{I_{m-1,\phi}[u]}{\int_\Omega \varphi(d)h_1^{-2}(d)\cdots h_m^{-2}(d)|u|^p \mathrm{d}x}. \tag{1.18}$$

Remark 1.5. Let $\phi(r) = r^\alpha$. Then (A_1) occurs if $k > p-\alpha$, (A_2) occurs if $k = p-\alpha$ and (A_3) occurs if $k < p-\alpha$. There are three cases for k and K: (1) $k=1$ and $K = \partial\Omega$; (2) $2 \le k \le N-1$ and $\Omega \cap K \neq \emptyset$; (3) $k = N$ and $K = \{0\} \subset \Omega$. If $\alpha = 0$ and $k = 1$, then neither (A_1) nor (A_2) occurs because of $p > 1$.

Remark 1.6. Theorem 1.4 extends the inequality (1.5) to Sobolev space with general weight $\phi(d)$. Moreover, it also includes the results of [18, 19].

Example 1.7. Let $\phi(r) = r^\alpha$. If $\alpha > p-k$, we have

$$\begin{gathered} c_0 = \frac{k+\alpha-p}{p-1}, \qquad h(r) = r^{(p-k-\alpha)/p}, \qquad \varphi(r) = \left(\frac{k+\alpha-p}{p}\right)^p r^{\alpha-p}, \\ h_1(r) = \ln\frac{D}{r}, \qquad h_{i+1}(r) = \ln \mathrm{e}h_i(r), \quad i = 1,2,\ldots. \end{gathered} \tag{1.19}$$

Then it follows from Theorem 1.4 that for all $u \in W_0^{1,p}(\Omega, d^\alpha)$,

$$\begin{aligned} &\int_\Omega d^\alpha |\nabla u|^p \mathrm{d}x - \left|\frac{k+\alpha-p}{p}\right|^p \int_\Omega d^{\alpha-p}|u|^p \mathrm{d}x \\ &\quad \ge \frac{p-1}{2p}\left|\frac{k+\alpha-p}{p}\right|^{p-2} \sum_{i=1}^m \int_\Omega d^{\alpha-p} h_1^{-2}(d)\cdots h_i^{-2}(d)|u|^p \mathrm{d}x, \end{aligned} \tag{1.20}$$

which is (1.5) [15, Theorem A], if $\alpha = 0$ (i.e., $p < k$). If $\alpha < p-k$, the above inequality holds for all $u \in W_0^{1,p}(\Omega \setminus K, d^\alpha)$. If $\alpha = p-k$, we have

$$\begin{aligned} &\int_\Omega d^\alpha |\nabla u|^p \mathrm{d}x - \left(\frac{p-1}{p}\right)^p \int_\Omega d^{\alpha-p}\left(\ln\frac{D}{d}\right)^{-p} |u|^p \mathrm{d}x \\ &\quad \ge \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \sum_{i=2}^m \int_\Omega d^{\alpha-p}\left(\ln\frac{D}{d}\right)^{-p} h_1^{-2}(d)\cdots h_i^{-2}(d)|u|^p \mathrm{d}x. \end{aligned} \tag{1.21}$$

This is the result of Theorem B in [15] if $\alpha = 0$.

2. Preliminary Lemmas

Lemma 2.1. *If (A_1) or (A_2) occurs, then*

$$\operatorname{div}\left(\phi h^{\alpha}(-h')^{p-1}\nabla d\right) = (1-\alpha)\phi h^{\alpha-1}(-h')^{p} + \phi h^{\alpha}(-h')^{p-1}\left(\Delta d - \frac{k-1}{d}\right). \tag{2.1}$$

If (A_3) occurs, then

$$\operatorname{div}\left(\phi h^{\alpha}(-h')^{p-1}\nabla d\right) = (\alpha-1)\phi h^{\alpha-1}(-h')^{p} + \phi h^{\alpha}(-h')^{p-1}\left(\Delta d - \frac{k-1}{d}\right). \tag{2.2}$$

Proof. Note that

$$h = \left(c_0\int_d^b (\phi r^{k-1})^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)/p}. \tag{2.3}$$

where $b=\infty$ for the case (A_1) and $b=D$ for the case (A_2), then

$$\phi h^{\alpha}(-h')^{p-1} = \left(\frac{p-1}{p}\right)^{p-1} c_0^{p-1}\left(c_0\int_d^b (\phi r^{k-1})^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)(\alpha-1)/p} d^{1-k}. \tag{2.4}$$

Hence

$$\begin{aligned}
&\left(\phi h^{\alpha}(-h')^{p-1}\right)' \\
&\quad= \left(\frac{p-1}{p}\right)^{p}(1-\alpha)c_0^{p}\left(c_0\int_d^b \left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)(\alpha-1)/p-1} \\
&\qquad\times\left(\phi d^{k-1}\right)^{-1/(p-1)} d^{1-k} + \phi h^{\alpha}(-h')^{p-1}(1-k)d^{-1}.
\end{aligned} \tag{2.5}$$

Then

$$\begin{aligned}
\operatorname{div}\left(\phi h^{\alpha}(-h')^{p-1}\nabla d\right) &= \left(\phi h^{\alpha}(-h')^{p-1}\right)'|\nabla d|^2 + \phi h^{\alpha}(-h')^{p-1}\Delta d \\
&= (1-\alpha)\phi h^{\alpha-1}(-h')^{p} + \phi h^{\alpha}(-h')^{p-1}\left(\Delta d - \frac{k-1}{d}\right).
\end{aligned} \tag{2.6}$$

The same argument can give the corresponding result if (A_3) occurs. □

Lemma 2.2. *Let $1 < p < N$ and $K = \{0\} \subset \Omega$. If (A_1) is satisfies, then $\overline{h}$ is the fundamental solution for the p-Laplace operator with weight ϕ, that is,*

$$-\operatorname{div}\left(\phi\left|\nabla\overline{h}\right|^{p-2}\nabla\overline{h}\right) = c_0^{p-1}\omega_N\delta(x), \tag{2.7}$$

where $\delta(x)$ is the Dirac measure, and ω_N denotes the volume of the unit sphere in $\mathbb{R}^N$.

Proof. Since $h = \left(c_0\int_r^\infty (\phi r^{N-1})^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)/p}$, we have

$$\begin{aligned}
-\operatorname{div}\left(\phi\left|\nabla\overline{h}\right|^{p-2}\nabla\overline{h}\right) &= -\operatorname{div}\left(\phi\left|\overline{h'}\right|^{p-1}\frac{x}{|x|}\right) \\
&= -c_0^{p-1}\operatorname{div}\left(\phi\left(\phi r^{N-1}\right)^{-1}\frac{x}{|x|}\right) \\
&= -c_0^{p-1}\operatorname{div}\left(\frac{x}{|x|^N}\right) = -c_0^{p-1}\omega_N\delta(x),
\end{aligned} \tag{2.8}$$

where the last equality sign is because of $-\operatorname{div}(x/|x|^N) = \omega_N\delta(x)$. □

Proposition 2.3. *Let $1 < p < N$ and $K = \{0\} \subset \Omega$. If (A_1) is satisfies, then h_i is the fundamental solution for the Laplace operator with weight ϕ_i, that is, for $i = 1, 2, \ldots$,*

$$-\operatorname{div}(\phi_i\nabla h_i) = -c_i\omega_N\delta(x). \tag{2.9}$$

Proof. The result follows by the following equalities:

$$-\operatorname{div}(\phi_i\nabla h_i) = -\operatorname{div}\left(\phi_i h_i'\frac{x}{|x|}\right) = -c_i\operatorname{div}\left(\frac{x}{|x|^N}\right) = -c_i\omega_N\delta(x). \tag{2.10}$$

□

Set $Y_1^{-1}(r) = h_1(r)$, it follows from (1.15) that, if (A_1) or (A_2) occurs,

$$-\frac{Y_1'}{Y_1^2} = (h_1)' = \frac{p}{(p-1)c_0}\frac{h'}{h}. \tag{2.11}$$

Define $Y_i^{-1}(r) = h_i(r)$, then

$$-Y_i^{-2}Y_i' = (h_i)' = (\ln h_{i-1})' = \frac{1}{h_{i-1}}(h_{i-1})' = \frac{p}{(p-1)c_0}\frac{h'}{h}Y_{i-1}\cdots Y_1, \tag{2.12}$$

that is,

$$Y_i' = -\frac{p}{(p-1)c_0}\frac{h'}{h}Y_1\cdots Y_{i-1}Y_i^2, \tag{2.13}$$

and so for any $\beta \neq -1$, $i \in \mathbb{N}$, we have

$$\left(Y_i^{\beta}\right)' = -\beta \frac{p}{(p-1)c_0} \frac{h'}{h} Y_1 \cdots Y_{i-1} Y_i^{\beta+1}. \tag{2.14}$$

Let $m \in \mathbb{N}$ and write

$$\eta(r) = \sum_{i=1}^{m} Y_1 \cdots Y_i, \qquad B(r) = \sum_{i=1}^{m} Y_1^2 \cdots Y_m^2, \tag{2.15}$$

then a simple calculation gives

$$\eta' = -\frac{p}{(p-1)c_0} \frac{h'}{h} \left(\frac{1}{2} \left(B(d) + \eta^2(d) \right) \right). \tag{2.16}$$

Similarly, if (A_3) occurs, we have

$$\begin{gathered} -\frac{Y_1'}{Y_1^2} = -\frac{p}{(p-1)c_0} \frac{h'}{h}, \\ Y_i' = \frac{p}{(p-1)c_0} \frac{h'}{h} Y_1 \cdots Y_{i-1} Y_i^2. \end{gathered} \tag{2.17}$$

Then

$$\eta' = \frac{p}{(p-1)c_0} \frac{h'}{h} \left(\frac{1}{2} \left(B(d) + \eta^2(d) \right) \right). \tag{2.18}$$

3. Proof of Theorems

Proof of Theorem 1.2. Define a C^1 vector field as

$$T = \begin{cases} \phi\left(-\dfrac{h'}{h}\right)^{p-1} \nabla d & \text{if } (A_1) \text{ or } (A_2) \text{ occurs}, \\ -\phi\left(\dfrac{h'}{h}\right)^{p-1} \nabla d & \text{if } (A_3) \text{ occurs}. \end{cases} \tag{3.1}$$

Then we can prove (1.10) analogous to the following proof of Theorem 1.4 (1). As to the best constant, we fix small positive parameter α and define the functions

$$w(x) = h^{1-(\alpha/(p-1)c_0)}. \tag{3.2}$$

The rest is similar to the following proof of Theorem 1.4(2). □

Proof of Theorem 1.4(1) . We will make use of a suitable vector field as in [15]. To proceed we now make a specific choice of T. Firstly, we consider the cases (A_1) and (A_2). We take

$$\begin{aligned} T &= \phi\left(-\frac{h'}{h}\right)^{p-1}\left(1 + c_0^{-1}\eta + a\eta^2\right)\nabla d \\ &= \phi\left(-\frac{h'}{h}\right)^{p-1}\nabla d + c_0^{-1}\phi\left(-\frac{h'}{h}\right)^{p-1}\eta\nabla d + a\phi\left(-\frac{h'}{h}\right)^{p-1}\eta^2\nabla d \\ &=: T_1 + T_2 + T_3, \end{aligned} \tag{3.3}$$

where a is a free parameter to be chosen later. In any cases a will be such that the quantity $1 + c_0^{-1}\eta + a\eta^2$ is positive on Ω. Note that T is singular at $x \in K$, but $\operatorname{div} T$ and T are integrable if (A_1) or (A_2) occurs.

Let $u \in C_0^\infty(\Omega)$ if (A_1) or (A_2) occurs. We integrate by parts to obtain, for any positive ϵ,

$$\int_{\Omega\setminus\Omega_\epsilon} \operatorname{div} T|u|^p \mathrm{d}x = -p\int_{\Omega\setminus\Omega_\epsilon}(T\cdot\nabla u)|u|^{p-2}u\,\mathrm{d}x + p\int_{\Omega\cap\{d=\epsilon\}} T\cdot|u|^p\nabla d\cdot\overrightarrow{n}\mathrm{d}S, \tag{3.4}$$

where $\Omega_\epsilon = \{x \in \Omega \mid d(x) < \epsilon\}$, and $\overrightarrow{n}$ denotes the unit outer normal to $\partial\Omega_\epsilon$. Note that

$$I \equiv \left|\int_{\Omega\cap\{d=\epsilon\}} T\cdot|u|^p\nabla d\cdot\overrightarrow{n}\mathrm{d}S\right| \le \int_{\Omega\cap\{d=\epsilon\}}|T||u|^p\mathrm{d}S = \int_{\Omega\cap\{d=\epsilon\}}\phi\left|\frac{h'}{h}\right|^{p-1}|u|^p\mathrm{d}S. \tag{3.5}$$

It follows from (1.9) that

$$\phi\left|\frac{h'}{h}\right|^{p-1} \le \left(\int_d^b\left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-(p-1)} d^{k-1}, \tag{3.6}$$

where $b = \infty$ if (A_1) occurs, or $b = D$ if (A_2) occurs. Since

$$c_1 r^{k-1} \le \int_{\Omega\cap\{d=r\}}\mathrm{d}S \le c r^{k-1} \tag{3.7}$$

for some positive constants c and c_1, (A_1) or (A_2) implies that

$$\left(\int_\epsilon^b\left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-(p-1)} \longrightarrow 0 \tag{3.8}$$

as ϵ tends to 0. Since η is bounded, we know $I \to 0$ as $\epsilon \to 0$. Hence,

$$\int_\Omega \operatorname{div} T|u|^p\mathrm{d}x = -p\int_\Omega(T\cdot\nabla u)|u|^{p-2}u\,\mathrm{d}x. \tag{3.9}$$

By Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}\int_\Omega \operatorname{div} T|u|^p \mathrm{d}x &= -p\int_\Omega (T\cdot\nabla u)|u|^{p-2}u\,\mathrm{d}x\\ &\le p\left(\int_\Omega \phi|\nabla u|^p\mathrm{d}x\right)^{1/p}\left(\int_\Omega \left|T\phi^{-1/p}\right|^{p/(p-1)}|u|^p\mathrm{d}x\right)^{(p-1)/p}\\ &\le \int_\Omega \phi|\nabla u|^p\mathrm{d}x + (p-1)\int_\Omega \left|T\phi^{-1/p}\right|^{p/p-1}|u|^p\mathrm{d}x.\end{aligned} \tag{3.10}$$

We therefore arrive at

$$\int_\Omega \phi|\nabla u|^p\mathrm{d}x \ge \int_\Omega\left(\operatorname{div} T - (p-1)\left|T\phi^{-1/p}\right|^{p/(p-1)}\right)|u|^p\mathrm{d}x. \tag{3.11}$$

If (A_3) occurs, the above inequality is obvious for $u \in C_0^\infty(\Omega \setminus K)$.

By Lemma 2.1 and condition (C^*), we have

$$\operatorname{div} T_1 = p\phi\left(-\frac{h'}{h}\right)^p + \phi h^{1-p}(-h')^{p-1}\left(\Delta d - \frac{k-1}{d}\right) \ge p\phi\left(-\frac{h'}{h}\right)^p. \tag{3.12}$$

Similarly, it follows from Lemma 2.1, condition (C^*) and (2.16)

$$\begin{aligned}\operatorname{div} T_2 &= c_0^{-1}\eta \operatorname{div}\phi\left(-\frac{h'}{h}\right)^{p-1}\nabla d + c_0^{-1}\phi\left(-\frac{h'}{h}\right)^{p-1}|\nabla d|^2\eta'\\ &\ge pc_0^{-1}\eta\phi\left(-\frac{h'}{h}\right)^p + \frac{p}{(p-1)c_0^2}\phi\left(-\frac{h'}{h}\right)^p\left(\frac{B}{2}+\frac{\eta^2}{2}\right),\end{aligned} \tag{3.13}$$

$$\operatorname{div} T_3 \ge ap\eta^2\phi\left(-\frac{h'}{h}\right)^p + \frac{p}{(p-1)c_0^2}\phi\left(-\frac{h'}{h}\right)^p\eta\left(B+\eta^2\right). \tag{3.14}$$

Combining (3.12)–(3.14), we obtain

$$\operatorname{div} T \ge \phi\left(-\frac{h'}{h}\right)^p\left[\left(p + pc_0^{-1}\eta + ap\eta^2\right) + \frac{p}{2(p-1)c_0^2}\left(B+\eta^2\right) + \frac{ap}{(p-1)c_0}\left(B\eta+\eta^3\right)\right]. \tag{3.15}$$

Next we compute $(p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)}$. We set for convenience

$$g(\eta) = \left(1 + c_0^{-1}\eta + a\eta^2\right)^{p/(p-1)}. \tag{3.16}$$

When $\eta > 0$ is small, the Taylor expansion of $g(\eta)$ about $\eta = 0$ gives

$$\begin{aligned} g(\eta) = 1 &+ \frac{p}{(p-1)c_0}\eta + \frac{1}{2}\left(\frac{p}{(p-1)^2c_0^2} + \frac{2pa}{p-1}\right)\eta^2 \\ &+ \frac{1}{6}\left(\frac{p(2-p)}{(p-1)^3c_0^3} + \frac{6pa}{(p-1)^2c_0}\right)\eta^3 + O\left(\eta^4\right), \end{aligned} \tag{3.17}$$

and so

$$\begin{aligned} (p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} = -\phi\left(-\frac{h'}{h}\right)^p\Bigg[&(p-1) + \frac{p}{c_0}\eta + \left(\frac{p}{2(p-1)c_0^2} + pa\right)\eta^2 \\ &+ \left(\frac{p(2-p)}{(p-1)^2c_0^3} + \frac{pa}{(p-1)c_0}\right)\eta^3 + O\left(\eta^4\right)\Bigg]. \end{aligned} \tag{3.18}$$

Hence

$$\begin{aligned} &\operatorname{div} T - (p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} \\ &\quad\geq \phi\left(-\frac{h'}{h}\right)^p\left[1 + \frac{pB}{2(p-1)c_0^2} + \frac{pa}{(p-1)c_0}B\eta - \frac{p(2-p)}{(p-1)^2c_0^3}\eta^3 + O\left(\eta^4\right)\right]. \end{aligned} \tag{3.19}$$

If we show

$$\frac{ap}{(p-1)c_0} \geq \left(\frac{p(2-p)}{(p-1)^2c_0^3} + O(\eta)\right)\frac{\eta^2}{B} \tag{3.20}$$

then we obtain

$$\operatorname{div} T - (p-1)\phi^{-1/p-1}|T|^{p/p-1} \geq \phi\left(-\frac{h'}{h}\right)^p\left[1 + \frac{p}{2(p-1)c_0^2}B\right]. \tag{3.21}$$

From the definition of η and B it follows easily that

$$m \geq \frac{\eta^2}{B} \geq 1. \tag{3.22}$$

(a) If $1 < p < 2$, we assume that η is small for the case (A_1). Since

$$h_1 = \frac{p}{(p-1)c_0} \ln \frac{h(r)}{h(D)} \tag{3.23}$$

and $\Omega \subset B_{r_0}(0)$ is bounded, we can choose D large enough such that $h_1^{-1}(r)$ is small enough if $r < r_0$, and then η is small. It is enough to show that we can choose a such that (3.20) holds. In view of (3.22), we see that for (3.20) to be valid, it is enough to take a to be big and positive. It is similar for the case (A_2).

(b) If $p \geq 2$, we choose $a = 0$, then

$$\left(1 + c_0^{-1}\eta\right)^{p/(p-1)} = 1 + \frac{p}{(p-1)c_0}\eta + \frac{p}{2(p-1)^2 c_0^2}\eta^2 + \frac{p(2-p)}{6(p-1)^3 c_0^3}\left(1 + c_0^{-1}\xi\right)^{(3-2p)/(p-1)}\eta^3 \tag{3.24}$$

for some $\xi \in (0, \eta)$, without any smallness assumption. Since $2 - p \leq 0$, we have

$$\left(1 + c_0^{-1}\eta\right)^{p/(p-1)} \leq 1 + \frac{p}{(p-1)c_0}\eta + \frac{p}{2(p-1)^2 c_0^2}\eta^2. \tag{3.25}$$

It follows from (3.15) that

$$\operatorname{div} T \geq \phi\left(-\frac{h'}{h}\right)^p \left[p\left(1 + c_0^{-1}\eta\right) + \frac{p(B + \eta^2)}{2(p-1)c_0^2}\right]. \tag{3.26}$$

Hence

$$\operatorname{div} T - (p-1)\phi^{1/(p-1)}|T|^{p/(p-1)} \geq \phi\left(-\frac{h'}{h}\right)^p \left(1 + \frac{Bp}{2(p-1)c_0^2}\right). \tag{3.27}$$

Then (1.17) follows by inserting the above inequality into (3.11).

Now we consider the case (A_3). In this case, $h' > 0$, that is,

$$h = \left(c_0 \int_0^d \left(\phi r^{k-1}\right)^{-1/(p-1)} \mathrm{d}r\right)^{(p-1)/p}. \tag{3.28}$$

We take

$$T = -\phi\left(\frac{h'}{h}\right)^p \nabla d\left(1 - c_0^{-1}\eta + a\eta^2\right), \tag{3.29}$$

where a is a free parameter to be chosen later. In any case a will be such that the quantity $1 - c_0^{-1}\eta + a\eta^2$ is positive on Ω. Note that T is singular at $x \in K$, but since $u \in C_0^\infty(\Omega \setminus K)$ all previous calculations are legitimate. Analogues to the calculations before, by Lemma 2.1 and (2.18), we have

$$\begin{aligned}
\operatorname{div} T &= p\phi\left(\frac{h'}{h}\right)^p\left(1-c_0^{-1}\eta + a\eta^2\right) - \left(\Delta d - \frac{k-1}{d}\right)\phi\left(\frac{h'}{h}\right)^p\left(1-c_0^{-1}\eta+a\eta^2\right) \\
&\quad + \frac{p}{2(p-1)c_0^2}\phi\left(\frac{h'}{h}\right)^p\left(\eta^2 + B\right) - \frac{ap}{(p-1)c_0}\phi\left(\frac{h'}{h}\right)^p\left(B\eta + \eta^3\right) \\
&\geq \phi\left(\frac{h'}{h}\right)^p\left[p\left(1-c_0^{-1}\eta+a\eta^2\right) + \frac{p}{2(p-2)c_0^2}\left(\eta^2+B-2ac_0\left(B\eta+\eta^3\right)\right)\right], \\
(p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} &\geq \phi\left(\frac{h'}{h}\right)^p\left[(p-1) - \frac{p}{c_0}\eta + \left(\frac{p}{2(p-1)c_0^2} + pa\right)\eta^2\right. \\
&\quad \left. + \frac{1}{6}\left(-\frac{p(2-p)}{(p-1)^2c_0^3} - \frac{6pa}{(p-1)c_0}\right)\eta^3\right] + O\left(\eta^4\right).
\end{aligned} \tag{3.30}$$

Hence

$$\begin{aligned}
&\operatorname{div} T - (p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} \\
&\quad \geq \phi\left(\frac{h'}{h}\right)^p\left(1 + \frac{pB}{2(p-1)c_0^2} - \frac{apB}{(p-1)c_0}\eta + \frac{p(2-p)}{6(p-1)^2c_0^3}\eta^3\right) + O\left(\eta^4\right).
\end{aligned} \tag{3.31}$$

If $1 < p < 2$, since

$$h_1 = \frac{p}{(p-1)c_0}\ln\frac{h(D)}{h(d)} \tag{3.32}$$

when (A_1) occurs, we can choose D large enough such that $h(D)$ is so large and $h_1^{-1}(r)$ is small, then we know that η is small. By taking $a = 0$, we obtain

$$\operatorname{div} T - (p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} \geq \phi\left(\frac{h'}{h}\right)^p\left(1 + \frac{pB}{2(p-1)c_0}\right). \tag{3.33}$$

If $p \geq 2$, taking a be negative with $|a|$ large enough, we also arrive at (3.33) by using (3.21). The result (1.17) then follows from (3.11) and (3.33). □

Proof of Theorem 1.4(2). All our analysis will be local, say, in a fixed ball of radius δ (denoted by B_δ) centered at the origin, for some fixed small δ. The proof we present works for any $k = 1, 2, \ldots, N$. We note however that for $k = N$ (distant from a point) the subsequent calculations are substantially simplified, whereas for $k = 1$ (distant from the boundary) one should replace B_δ by $B_\delta \cap \Omega$. This last change entails some minor modifications, the arguments otherwise being the same. Without any loss of generality we may assume that $0 \in K \cap \Omega$ ($k \neq 1$), or $0 \in \partial\Omega$ if $k = 1$. We divide the proof into several steps.

Step 1. Let $\theta \in C_0^\infty(B_\delta)$ be such that $0 \le \theta \le 1$ in B_δ and $\theta = 1$ in $B_{\delta/2}$. We fix small positive parameters $\alpha_0, \alpha_1, \ldots, \alpha_m$ and define the functions

$$w(x) = h^{1-\alpha_0/(p-1)c_0} h_1^{(1-\alpha_1)/p} \cdots h_m^{(1-\alpha_m)/p}(d), \qquad u(x) = \theta(x) w(x). \tag{3.34}$$

Let (A_1) or (A_2) happen. Hence $u \in W_0^{1,p}(\Omega, \phi)$. To prove the proposition we will estimate the corresponding Rayleigh quotient of u in the limit $\alpha_0 \to 0$, $\alpha_1 \to 0, \ldots, \alpha_m \to 0$ in this order.

It is easily seen that

$$\nabla w = \frac{p}{(p-1)c_0} h^{-\alpha_0/(p-1)c_0} h' \nabla d Y_1^{(-1+\alpha_1)/p} \cdots Y_m^{(-1+\alpha_m)/p} \left(\frac{(p-1)c_0}{p} + \frac{\overline{\eta}}{p} \right), \tag{3.35}$$

where $Y_i = h_i^{-1}$ and $\overline{\eta} = -\alpha_0 + (1-\alpha_1)Y_1 + \cdots + (1-\alpha_m)Y_1 \cdots Y_m$.

Now $\nabla u = \theta \nabla w + w \nabla \theta$ and hence, using the elementary inequality

$$|a+b|^p \le |a|^p + c_p\left(|a|^{p-1}|b| + |b|^p\right), \quad a, b \in \mathbb{R}^N \tag{3.36}$$

for $p > 1$, we obtain

$$\int_\Omega \phi |\nabla u|^p \mathrm{d}x \le \int_\Omega \phi \theta^p |\nabla w|^p \mathrm{d}x + c_p \int_\Omega \phi \theta^{p-1} |\nabla \theta| |w| |\nabla w|^{p-1} \mathrm{d}x + c_p \int_\Omega \phi |\nabla \theta|^p |w|^p \mathrm{d}x =: I_1 + I_2 + I_3. \tag{3.37}$$

We claim that

$$I_2, I_3 = O(1) \quad \text{uniformly as } \alpha_0, \alpha_1, \ldots, \alpha_{\mathrm{m}} \text{ tend to zero.} \tag{3.38}$$

Let us give the proof for I_2:

$$
\begin{aligned}
I_2 \le C\int_{B_\delta} & \phi h^{-\alpha_0/c_0}|\nabla h|^{p-1}\Upsilon_1^{(-1+\alpha_1)(p-1)/p}\cdots\Upsilon_m^{(-1+\alpha_m)(p-1)/p} \\
& \times\left[(p-1)c_0+\alpha_0+(1-\alpha_1)\Upsilon_1+\cdots+(1-\alpha_m)\Upsilon_1\cdots\Upsilon_m\right]^{p-1} \\
& \times h^{1-\alpha_0/(p-1)c_0}\Upsilon_1^{(-1+\alpha_1)/p}\cdots\Upsilon_m^{(-1+\alpha_m)/p}\mathrm{d}x \\
\le C\int_{B_\delta} & \phi h^{1-\alpha_0 p/(p-1)c_0}|\nabla h|^{p-1}\Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m} \\
& \times\left[(p-1)c_0+\alpha_0+(1-\alpha_1)\Upsilon_1+\cdots+(1-\alpha_m)\Upsilon_1\cdots\Upsilon_m\right]^{p-1}\mathrm{d}x.
\end{aligned}
\tag{3.39}
$$

It follows from the definition of h that

$$
h' = \frac{p-1}{p}\left(c_0\int_d^b\left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-1/p}\left(-c_0\left(\phi d^{k-1}\right)^{-1/(p-1)}\right). \tag{3.40}
$$

Then

$$
\phi(-h')^{p-1}h = \left(\frac{c_0(p-1)}{p}\right)^{p-1}d^{1-k}. \tag{3.41}
$$

Hence, by the coarea formula and the fact that

$$
c_1 r^{k-1} \le \int_{\{d=r\}\cap B_\delta}\mathrm{d}S \le c_2 r^{k-1} \tag{3.42}
$$

we have

$$
\begin{aligned}
I_2 &\le C\int_{B_\delta} d^{1-k}h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}\mathrm{d}x \\
&= C\int_0^\delta \mathrm{d}r\int_{\{d=r\}}\frac{d^{1-k}h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}}{|\nabla d|}\mathrm{d}S \\
&= C\int_0^\delta \mathrm{d}r\int_{\{d=r\}} d^{1-k}h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}\mathrm{d}S \\
&\le C\int_0^\delta h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}(r)\mathrm{d}r.
\end{aligned}
\tag{3.43}
$$

The boundedness of $h^{-1}(r)$ together with the fact $h^{-1}(0) = 0$ implies that I_2 is uniformly bounded. The integral I_3 is treated similarly.

Step 2. We will repeatedly deal with integrals of the form

$$Q = \int_\Omega \theta^p \phi h^{-\beta_0 p/(p-1)c_0} (-h')^p \Upsilon_1^{1+\beta_1} \cdots \Upsilon_m^{1+\beta_m} \mathrm{d}x. \tag{3.44}$$

By (1.9), we have

$$\begin{aligned} \phi h^{-\beta_0 p/(p-1)c_0} (-h)'^p &= \phi \left(c_0 \int_d^b \left(\phi r^{k-1}\right)^{-1/(p-1)} \mathrm{d}r \right)^{-\beta_0/c_0 - 1} \left(c_0 \phi d^{k-1}\right)^{-p/(p-1)}, \\ d^{k-1} \phi h^{-\beta_0 p/(p-1)c_0} (-h)'^p &= c \left(\int_d^b \left(\phi r^{k-1}\right)^{-1/(p-1)} \mathrm{d}r \right)^{-\beta_0/c_0 - 1} \left(\phi d^{k-1}\right)^{-1/(p-1)}. \end{aligned} \tag{3.45}$$

Using the coarea formula and (3.41), if $\beta_0 = \cdots = \beta_{m-1} = 0$ and $\beta_m > 0$, by (1.15), (2.14) and $h'/h = ((p-1)/p)(\overline{h}'/\overline{h})$, we have

$$C \int_0^{\delta/2} \left| \frac{h'}{h} \right| \Upsilon_1 \cdots \Upsilon_{m-1} \Upsilon_m^{1+\beta_m} \mathrm{d}r \le Q \le C \int_0^{\delta} \left| \frac{h'}{h} \right| \Upsilon_1 \cdots \Upsilon_{m-1} \Upsilon_m^{1+\beta_m} \mathrm{d}r = \frac{C}{\beta_m} \Upsilon_m^{\beta_m} \Big|_0^{\delta} < +\infty. \tag{3.46}$$

Analogue arguments arrive at

$$Q < \infty \Longleftrightarrow \begin{cases} \beta_0 > 0, & \text{or} \\ \beta_0 = 0, \ \beta_1 > 0, & \text{or} \\ \vdots & \\ \beta_0 = \beta_1 = \cdots = \beta_{m-1}, \ \beta_m > 0. & \end{cases} \tag{3.47}$$

Moreover, if $\beta_m \to 0$, we have

$$Q \longrightarrow \infty. \tag{3.48}$$

Step 3. We introduce some auxiliary quantities and prove some simple relations about them. For $0 \le i \le j \le m$ we define

$$\begin{aligned} A_0 &= \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p \Upsilon_1^{-1+\alpha_1} \cdots \Upsilon_m^{-1+\alpha_m} \mathrm{d}x, \\ A_i &= \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p \Upsilon_1^{1+\alpha_1} \cdots \Upsilon_i^{1+\alpha_i} \Upsilon_{i+1}^{-1+\alpha_{i+1}} \cdots \Upsilon_m^{-1+\alpha_m} \mathrm{d}x, \\ \Gamma_{0i} &= \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p \Upsilon_1^{\alpha_1} \cdots \Upsilon_i^{\alpha_i} \Upsilon_{i+1}^{-1+\alpha_{i+1}} \cdots \Upsilon_m^{-1+\alpha_m} \mathrm{d}x, \\ \Gamma_{ij} &= \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p \Upsilon_1^{1+\alpha_1} \cdots \Upsilon_i^{1+\alpha_i} \Upsilon_{i+1}^{\alpha_{i+1}} \cdots \Upsilon_j^{\alpha_j} \Upsilon_{j+1}^{-1+\alpha_{j+1}} \cdots \Upsilon_m^{-1+\alpha_m} \mathrm{d}x. \end{aligned} \tag{3.49}$$

with $\Gamma_{ii} = A_i$. We have the following two identities. Let $0 \leq i \leq m-1$ be given and assume that $\alpha_0 = \alpha_1 = \cdots = \alpha_{i-1} = 0$, then

$$\alpha_i A_i = \sum_{j=i+1}^{m} (1-\alpha_j)\Gamma_{ij} + O(1), \tag{3.50}$$

$$\alpha_i \Gamma_{ij} = -\sum_{k=i+1}^{j} \alpha_k \Gamma_{kj} + \sum_{k=j+1}^{m} (1-\alpha_k)\Gamma_{jk} + O(1), \tag{3.51}$$

where the $O(1)$ is uniform as the α_i's tend to zero. Let us give the proof for (3.50). Firstly, we discuss the case of $i = 0$. By Lemma 2.1, we have

$$\begin{aligned} &\phi h^{-\alpha_0 p/(p-1)c_0}(-h')^p \\ &\quad= \frac{(p-1)c_0}{p\alpha_0}\left[\operatorname{div}\left(\phi h^{1-\alpha_0\, p/(p-1)c_0}(-h')^{p-1}\nabla d\right) - \phi h^{1-\alpha_0 p/(p-1)c_0}(-h')^{p-1}\left(\Delta d - \frac{k-1}{d}\right)\right]. \end{aligned} \tag{3.52}$$

Multiplying the above equality by $\theta^p \Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}$ and integrating over Ω, we obtain

$$\begin{aligned} A_0 &= \frac{(p-1)c_0}{p\alpha_0}\int_\Omega \operatorname{div}\left(\phi h^{1-\alpha_0 p/(p-1)c_0}(-h')^{p-1}\nabla d\right)\theta^p \Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}\,\mathrm{d}x \\ &\quad - \frac{(p-1)c_0}{p\alpha_0}\int_\Omega \phi h^{1-\alpha_0 p/(p-1)c_0}(-h')^{p-1}\left(\Delta d - \frac{k-1}{d}\right)\theta^p \Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}\,\mathrm{d}x \\ &=: A_{01} + A_{02}. \end{aligned} \tag{3.53}$$

Let us estimate A_{01}. Using integration by parts, we obtain

$$\begin{aligned} A_{01} &= -\frac{(p-1)c_0}{p\alpha_0}\int_\Omega \phi h^{1-\alpha_0 p/(p-1)c_0}(-h')^{p-1}\nabla d \nabla\left(\theta^p \Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}\right)\mathrm{d}x \\ &= -\frac{(p-1)c_0}{p\alpha_0}\int_\Omega \phi h^{1-\alpha_0 p/(p-1)c_0}(-h')^{p-1} \\ &\quad\times\left\{\theta^p\left[\left(\Upsilon_1^{-1+\alpha_1}\right)'\Upsilon_2^{-1+\alpha_2}\cdots\Upsilon_m^{-1+\alpha_m} + \cdots + \Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_{m-1}^{-1+\alpha_{m-1}}\left(\Upsilon_m^{-1+\alpha_m}\right)'\right]\right. \\ &\qquad \left. + \Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}\cdot p\nabla\theta^p\nabla d\right\}\mathrm{d}x. \end{aligned} \tag{3.54}$$

It follows from (2.14) that

$$\left(Y_i^{-1+\alpha_i}\right)' = (1-\alpha_i)\frac{p}{(p-1)c_0}\frac{h'}{h}Y_1\cdots Y_{i-1}Y_i^{\alpha_i}. \tag{3.55}$$

Then we have

$$\begin{aligned} A_{01} &= (1-\alpha_0)\Gamma_{01}+\cdots+(1-\alpha_m)\Gamma_{0m} \\ &\quad -\frac{p(p-1)c_0}{p\alpha_0}\int_\Omega \phi h^{1-\alpha_0 p/(p-1)c_0}(-h')^{p-1}\nabla\theta^p\nabla d Y_1^{-1+\alpha_1}\cdots Y_m^{-1+\alpha_m}\mathrm{d}x. \end{aligned} \tag{3.56}$$

Hence, by (3.41), (3.42) and condition (A_1) (or (A_2)), we obtain

$$A_{01} = (1-\alpha_0)\Gamma_{01}+\cdots+(1-\alpha_m)\Gamma_{0m}+O(1). \tag{3.57}$$

For A_{02}, note that it is a direct consequence of [20, Theorem 3.2], that

$$d\Delta d + 1 - k = O(d) \tag{3.58}$$

as d tends to zero, a similar argument as before, we can obtain $A_{02} = O(1)$.

Now we assume that $\alpha_0=\alpha_1=\cdots=\alpha_{i-1}=0$. By Lemma 2.1 and (2.14), we have

$$\begin{aligned} &\mathrm{div}\left(\phi h(-h')^{p-1}Y_i^{\alpha_i}\nabla d\right) \\ &\quad = Y_i^{\alpha_i}\,\mathrm{div}\left(\phi h(-h')^{p-1}\nabla d\right)+\phi h(-h')^{p-1}(Y_i^{\alpha_i})'|\nabla d|^2 \\ &\quad = Y_i^{\alpha_i}\phi h(-h')^{p-1}\left(\Delta d-\frac{k-1}{d}\right)-\alpha_i\phi h(-h')^{p-1}Y_1\cdots Y_{i-1}Y_i^{1+\alpha_i}\cdot\frac{p}{(p-1)c_0}\frac{h'}{h}, \end{aligned} \tag{3.59}$$

that is,

$$\begin{aligned} &\alpha_i\phi(-h')^pY_1\cdots Y_{i-1}Y_i^{1+\alpha_i} \\ &\quad = \frac{(p-1)c_0}{p}\,\mathrm{div}\left(\phi h(-h')^{p-1}Y_i^{\alpha_i}\nabla d\right)-\frac{(p-1)c_0}{p}\phi h(-h')^{p-1}Y_i^{\alpha_i}\left(\Delta d-\frac{k-1}{d}\right). \end{aligned} \tag{3.60}$$

Hence, we have

$$\begin{aligned}\alpha_i A_i &= \alpha_i \int_\Omega \theta^p \phi(-h')^p Y_1 \cdots Y_{i-1} Y_i^{1+\alpha_i} Y_{i+1}^{-1+\alpha_{i+1}} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x \\ &= \int_\Omega \theta^p \left[\frac{(p-1)c_0}{p} \operatorname{div}\left(\phi h (-h')^{p-1} Y_i^{\alpha_i} \nabla d \right) - \frac{(p-1)c_0}{p} \phi h (-h')^{p-1} Y_i^{\alpha_i} \left(\Delta d - \frac{k-1}{d} \right) \right] \\ &\quad \times Y_{i+1}^{-1+\alpha_{i+1}} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x \\ &=: E_1 + E_2.\end{aligned} \tag{3.61}$$

Integration by parts gives

$$\begin{aligned}E_1 &= -\frac{(p-1)c_0}{p} \int_\Omega \theta^p \phi h (-h')^{p-1} Y_i^{\alpha_i} \nabla d \cdot \nabla \left(Y_{i+1}^{-1+\alpha_{i+1}} \cdots Y_m^{-1+\alpha_m} \right) \mathrm{d}x \\ &\quad + \frac{(p-1)c_0}{p} \int_\Omega \phi h (-h')^{p-1} Y_i^{\alpha_i} \nabla d \nabla \theta^p Y_{i+1}^{-1+\alpha_{i+1}} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x \\ &=: E_{11} + E_{22}.\end{aligned} \tag{3.62}$$

Since

$$\left(Y_{i+1}^{-1+\alpha_{i+1}} \cdots Y_m^{-1+\alpha_m} \right)' = -\sum_{j=i+1}^{m} (-1+\alpha_j) \frac{ph'}{(p-1)c_0 h} \cdot Y_1 \cdots Y_i Y_{i+1}^{\alpha_{i+1}} \cdots Y_j^{\alpha_j} Y_{j+1}^{-1+\alpha_{j+1}} \tag{3.63}$$

we have

$$\begin{aligned}E_{11} &= \sum_{j=i+1}^{m} (1-\alpha_j) \int_\Omega \theta^p \phi (-h)^p Y_1 \cdots Y_i Y_{i+1}^{\alpha_{i+1}} \cdots Y_j^{\alpha_j} Y_{j+1}^{-1+\alpha_{j+1}} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x \\ &= \sum_{j=i+1}^{m} (1-\alpha_j) \Gamma_{ij}.\end{aligned} \tag{3.64}$$

A similar argument as the estimation of I_2 in Step 1 shows that

$$E_{12} = O(1). \tag{3.65}$$

For E_2, since

$$d\Delta d + (1-k) = O(d) \tag{3.66}$$

as $d \to 0$, by (3.41) and (3.42), we can know that E_2 is bounded uniformly in $\alpha_0, \ldots, \alpha_m$. Hence (3.50) has been proved. To prove (3.51), we use (3.59) once more and proceed similarly, we omit the details.

Step 4. We proceed to estimate I_1:

$$I_1 = \int_\Omega \phi\theta^p |\nabla w|^p \mathrm{d}x \le \left(\frac{p}{(p-1)c_0}\right)^p \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m} \left(\frac{(p-1)c_0}{p} + \frac{\overline{\eta}}{p}\right)^p \mathrm{d}x, \tag{3.67}$$

where $\overline{\eta} = -\alpha_0 + (1-\alpha_1)Y_1 + \cdots + (1-\alpha_m)Y_1 \cdots Y_m$. Since $\overline{\eta}$ is small compared to $(p-1)c_0/p$ we may use Taylor's expansion to obtain

$$\left(\frac{(p-1)c_0}{p} + \frac{\overline{\eta}}{p}\right)^p \le \left(\frac{(p-1)c_0}{p}\right)^p + \left(\frac{(p-1)c_0}{p}\right)^{p-1} \overline{\eta} + \frac{p-1}{2p}\left(\frac{(p-1)c_0}{p}\right)^{p-2} \overline{\eta}^2 + C\overline{\eta}^3, \tag{3.68}$$

Using this inequality we can bound I_1 by

$$I_1 \le I_{10} + I_{11} + I_{12} + I_{13}, \tag{3.69}$$

where

$$\begin{aligned} I_{10} &= \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x \\ &= \int_\Omega \theta^p \psi h^{p-\alpha_0 p/(p-1)c_0} Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x \\ &= \int_\Omega \theta^p \psi |w|^p \mathrm{d}x = \int_\Omega \psi |u|^p \mathrm{d}x, \\ I_{12} &= \left(\frac{p}{(p-1)c_0}\right)^p \frac{p-1}{2p}\left(\frac{(p-1)c_0}{p}\right)^{p-2} \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m} \overline{\eta}^2 \mathrm{d}x \\ &= \frac{p-1}{2p}\left(\frac{p}{(p-1)c_0}\right)^2 \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m} \overline{\eta}^2 \mathrm{d}x \\ &= \frac{p}{2(p-1)c_0^2} \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m} \overline{\eta}^2 \mathrm{d}x. \end{aligned} \tag{3.70}$$

We will prove that

$$I_{11}, I_{13} = O(1) \quad \text{uniformly in } \alpha_0, \alpha_1, \ldots, \alpha_{\mathrm{m}}. \tag{3.71}$$

Firstly,

$$
\begin{aligned}
I_{11} &= \left(\frac{p}{(p-1)c_0}\right)^p \left(\frac{(p-1)c_0}{p}\right)^{p-1} \int_\Omega \theta^p \phi (-h')^p h^{-\alpha_0 p/(p-1)c_0} \Upsilon_1^{-1+\alpha_1} \cdots \Upsilon_m^{-1+\alpha_m} \overline{\eta}\, \mathrm{d}x \\
&= \frac{p}{(p-1)c_0} \left[-\alpha_0 \int_\Omega \theta^p \phi (-h')^p h^{-\alpha_0 p/(p-1)c_0} \Upsilon_1^{-1+\alpha_1} \cdots \Upsilon_m^{-1+\alpha_m} \mathrm{d}x \right. \\
&\qquad + (1-\alpha_1) \int_\Omega \theta^p \phi (-h')^p h^{-\alpha_0 p/(p-1)c_0} \Upsilon_1^{\alpha_1} \cdots \Upsilon_m^{-1+\alpha_m} \mathrm{d}x + \cdots \\
&\qquad \left. + (1-\alpha_m) \int_\Omega \theta^p \phi (-h')^p h^{-\alpha_0 p/(p-1)c_0} \Upsilon_1^{\alpha_1} \cdots \Upsilon_m^{\alpha_m} \mathrm{d}x \right] + O(1) \\
&= \frac{p}{(p-1)c_0} \left(-\alpha_0 A_0 + (1-\alpha_1)\Gamma_{01} + \cdots + (1-\alpha_m)\Gamma_{0m} \right) + O(1).
\end{aligned}
\tag{3.72}
$$

To estimate I_{13}, by (1.15) and $\Upsilon_i = h_i^{-1}$, we have

$$
\Upsilon_1 \cdots \Upsilon_i \le C \Upsilon_1 \quad \text{for some C} > 0 \tag{3.73}
$$

and thus obtain

$$
\begin{aligned}
I_{13} &\le C\alpha_0^3 \int_\Omega \theta^p \phi (-h')^p h^{-\alpha_0 p/(p-1)c_0} \Upsilon_1^{-1+\alpha_1} \cdots \Upsilon_m^{-1+\alpha_m} \mathrm{d}x \\
&\quad + C \int_\Omega \theta^p \phi (-h')^p h^{-\alpha_0 p/(p-1)c_0} \Upsilon_1^{2+\alpha_1} \cdots \Upsilon_m^{-1+\alpha_m} \mathrm{d}x \\
&=: I'_{13} + I''_{13}.
\end{aligned}
\tag{3.74}
$$

Note that

$$
\Upsilon_1^{-2} = \begin{cases} \dfrac{p^2}{(p-1)^2 c_0^2} \left[\ln \dfrac{h(r)}{h(D)} \right]^2 & \text{if } (\mathrm{A}_1) \text{ occurs}, \\ \dfrac{p^2}{(p-1)^2 c_0^2} \ln h(d) & \text{if } (\mathrm{A}_2) \text{ occurs}. \end{cases}
\tag{3.75}
$$

hence, if (A_1) occurs, by the coarea formula and (3.42) we have

$$
\begin{aligned}
I'_{13} &\le C\alpha_0^3 \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p \Upsilon_1^{-2} \mathrm{d}x \\
&\le C\alpha_0^3 \int_\Omega \theta^p \phi \left(\int_d^\infty \left(\phi r^{k-1} \right)^{-1/(p-1)} \mathrm{d}r \right)^{-\alpha_0/c_0} \\
&\quad \times \left(\int_d^\infty (\phi r^{k-1})^{-1/(p-1)} \mathrm{d}r \right)^{-1} \left(\phi d^{k-1} \right)^{-p/(p-1)} \left[\ln \frac{h(d)}{h(D)} \right]^2 d^{k-1} \mathrm{d}x
\end{aligned}
$$

$$\begin{aligned}
&\leq C\alpha_0^3\int_0^\delta \left(\phi r^{k-1}\right)^{1-p/(p-1)}\left(\int_r^\infty \left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-1-\alpha_0/c_0}\left[\ln\frac{h(r)}{h(D)}\right]^2\mathrm{d}r \\
&= C\alpha_0^3\int_0^\delta \left(\int_r^\infty \left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-1-\alpha_0/c_0}\left[\ln\frac{h(r)}{h(D)}\right]^2\mathrm{d}\left(\int_r^\infty \left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right) \\
&\leq C\alpha_0^2 c_0\int_0^\delta \left[\ln\frac{h(r)}{h(D)}\right]^2\mathrm{d}\left(\int_r^\infty \left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-\alpha_0/c_0}.
\end{aligned} \tag{3.76}$$

Denote

$$s = \left(\int_d^\infty \left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-\alpha_0/c_0} \tag{3.77}$$

and then we have

$$I_{13}' \leq C\alpha_0^2\int_0^\delta \left[C - \frac{(p-1)c_0}{p\alpha_0}\ln s\right]^2\mathrm{d}s \leq O(1). \tag{3.78}$$

The boundedness of $Y_1^{2+\alpha_1}Y_2^{-1+\alpha_2}\cdots Y_m^{-1+\alpha_m}$ implies that I_{13}'' is bounded uniformly in the α_i's. Hence we conclude that

$$\int_\Omega \phi|\nabla u|^p\mathrm{d}x - \int_\Omega \psi|u|^p\mathrm{d}x \leq I_{12} + O(1) \tag{3.79}$$

uniformly in the α_i's. If (A_2) occurs, we can also obtain the above estimate by similar arguments with ∞ being replaced by D.

Step 5. Recalling the definition of $I_{m-1,\phi}[\cdot]$ we obtain from (3.79)

$$\begin{aligned}
I_{m-1,\phi}[u] &\leq \frac{p}{2(p-1)c_0^2}\int_\Omega \theta^p\phi(-h')^p h^{-\alpha_0 p/(p-1)c_0}Y_1^{-1+\alpha_1}\cdots Y_m^{-1+\alpha_m} \\
&\quad\times\left(\overline{\eta}^2 - \sum_{i=1}^{m-1}Y_1^2\cdots Y_i^2\right)\mathrm{d}x + O(1) \\
&= \frac{p}{2(p-1)c_0^2}J + O(1),
\end{aligned} \tag{3.80}$$

where

$$J = \int_{\Omega} \theta^p \phi(-h')^p h^{-\alpha_0 p/(p-1)c_0} Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m}$$
$$\times \left(\alpha_0^2 + \sum_{i=1}^{m} (1-\alpha_i)^2 Y_1^2 \cdots Y_i^2 - \sum_{i=1}^{m-1} Y_1^2 \cdots Y_i^2 - 2\alpha_0 \sum_{j=1}^{m} (1-\alpha_j) Y_1 \cdots Y_j \right.$$
$$\left. + 2\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} (1-\alpha_i)(1-\alpha_j) Y_1^2 \cdots Y_i^2 Y_{i+1} \cdots Y_j \right) \mathrm{d}x \quad (3.81)$$
$$= \alpha_0^2 A_0 + A_m + \sum_{i=1}^{m} \left(\alpha_i^2 - 2\alpha_i\right) A_i - 2\alpha_0 \sum_{j=1}^{m} (1-\alpha_j) \Gamma_{0j}$$
$$+ \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} 2(1-\alpha_i)(1-\alpha_j) \Gamma_{ij}.$$

Step 6. We intend to take the limit $\alpha_0 \to 0$ in (3.81). By (3.50) and (3.51), analogues to [15, Step 7], we have

$$\alpha_0^2 A_0 - 2\alpha_0 \sum_{j=1}^{m} (1-\alpha_j)\Gamma_{0j} = \sum_{i=1}^{m} \left(\alpha_i - \alpha_i^2\right) A_i + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} (2\alpha_i - 1)(1-\alpha_j)\Gamma_{ij} + O(1). \quad (3.82)$$

All the terms in the last expression remain bounded as $\alpha_0 \to 0$, taking the limit in (3.81) we obtain

$$J = A_m - \sum_{i=1}^{m} \alpha_i A_i + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} (1-\alpha_j)\Gamma_{ij} + O(1) \quad (\alpha_0 = 0), \quad (3.83)$$

where the $O(1)$ is uniform with respect to $\alpha_1, \ldots, \alpha_m$. Next taking $\alpha_i \to 0, \ldots, \alpha_{m-1} \to 0$ in order, the same argument as before gives

$$J = (1-\alpha_m) A_m + O(1) \quad (\alpha_0 = \alpha_1 = \cdots = \alpha_{m-1} = 0) \quad (3.84)$$

uniformly in α_m. Combing (3.80) and (3.84), we conclude that

$$\frac{I_{m-1,\phi}[u]}{\int_{\Omega} \psi h_1^{-2} \cdots h_m^{-2} |u|^p \mathrm{d}x} \le \frac{p}{2(p-1)c_0^2} \frac{(1-\alpha_m)A_m + O(1)}{A_m} \longrightarrow \frac{p}{2(p-1)c_0^2} \quad (3.85)$$

as $\alpha_m \to 0$, since $A_m \to \infty$ as $\alpha_m \to 0$ by (3.48). This completes the proof. □

Acknowledgment

This project was supported by the NSFC (no. 10771074, 10726060) and the NSF (no. 04020077).

References

[1] E. B. Davies and A. M. Hinz, "Explicit constants for Rellich inequalities in $L_p(\Omega)$," *Mathematische Zeitschrift*, vol. 227, no. 3, pp. 511–523, 1998.
[2] J. P. García Azorero and I. Peral Alonso, "Hardy inequalities and some critical elliptic and parabolic problems," *Journal of Differential Equations*, vol. 144, no. 2, pp. 441–476, 1998.
[3] B. Opic and A. Kufner, *Hardy-Type Inequalities*, vol. 219 of *Pitman Research Notes in Mathematics Series*, Longman Scientific & Technical, Harlow, UK, 1990.
[4] Y. T. Shen, "On the Dirichlet problem for quasilinear elliptic equation with strongly singular coefficients," in *Proceedings of the Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980)*, pp. 1407–1417, Science Press, Beijing, China, 1982.
[5] H. Brezis and J. L. Vázquez, "Blow-up solutions of some nonlinear elliptic problems," *Revista Matemática de la Universidad Complutense de Madrid*, vol. 10, no. 2, pp. 443–469, 1997.
[6] F. Gazzola, H.-C. Grunau, and E. Mitidieri, "Hardy inequalities with optimal constants and remainder terms," *Transactions of the American Mathematical Society*, vol. 356, no. 6, pp. 2149–2168, 2004.
[7] Adimurthi, N. Chaudhuri, and M. Ramaswamy, "An improved Hardy-Sobolev inequality and its application," *Proceedings of the American Mathematical Society*, vol. 130, no. 2, pp. 489–505, 2002.
[8] Adimurthi and M. J. Esteban, "An improved Hardy-Sobolev inequality in $W^{1,p}$ and its application to Schrödinger operators," *Nonlinear Differential Equations and Applications*, vol. 12, no. 2, pp. 243–263, 2005.
[9] B. Abdellaoui, E. Colorado, and I. Peral, "Some improved Caffarelli-Kohn-Nirenberg inequalities," *Calculus of Variations and Partial Differential Equations*, vol. 23, no. 3, pp. 327–345, 2005.
[10] Y. T. Shen, "The Dirichlet problem for degenerate or singular elliptic equation of high order," *Journal of China University of Science and Technology*, vol. 10, no. 2, pp. 1–11, 1980 (Chinese).
[11] Y. T. Shen and X. K. Guo, "Weighted Poincaré inequalities on unbounded domains and nonlinear elliptic boundary value problems," *Acta Mathematica Scientia*, vol. 4, no. 3, pp. 277–286, 1984.
[12] M. Marcus, V. J. Mizel, and Y. Pinchover, "On the best constant for Hardy's inequality in $\mathbb{R}^n$," *Transactions of the American Mathematical Society*, vol. 350, no. 8, pp. 3237–3255, 1998.
[13] T. Matskewich and P. E. Sobolevskii, "The best possible constant in generalized Hardy's inequality for convex domain in $\mathbb{R}^n$," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 28, no. 9, pp. 1601–1610, 1997.
[14] H. Brezis and M. Marcus, "Hardy's inequalities revisited," *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV*, vol. 25, no. 1-2, pp. 217–237, 1997.
[15] G. Barbatis, S. Filippas, and A. Tertikas, "Series expansion for L^p Hardy inequalities," *Indiana University Mathematics Journal*, vol. 52, no. 1, pp. 171–190, 2003.
[16] S. Secchi, D. Smets, and M. Willem, "Remarks on a Hardy-Sobolev inequality," *Comptes Rendus Mathématique. Académie des Sciences. Paris*, vol. 336, no. 10, pp. 811–815, 2003.
[17] G. Barbatis, S. Filippas, and A. Tertikas, "A unified approach to improved L^p Hardy inequalities with best constants," *Transactions of the American Mathematical Society*, vol. 356, no. 6, pp. 2169–2196, 2004.
[18] S. Yaotian and C. Zhihui, "General Hardy inequalities with optimal constants and remainder terms," *Journal of Inequalities and Applications*, vol. 2005, no. 3, pp. 207–219, 2005.
[19] Y. T. Shen and Z. H. Chen, "Sobolev-Hardy space with general weight," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 2, pp. 675–690, 2006.
[20] L. Ambrosio and H. M. Soner, "Level set approach to mean curvature flow in arbitrary codimension," *Journal of Differential Geometry*, vol. 43, no. 4, pp. 693–737, 1996.