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Research Article

Ostrowski Type Inequalities for Higher-Order Derivatives

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This paper has shown some new Ostrowski type inequalities involving higher-order derivatives. The results generalized the Ostrowski type inequalities. Applications of the inequalities are also given.

1. Main Result and Introduction

The following inequality is well known in literature as Ostrowski's integral inequality.

Let $f : [a,b] \to R$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to R$ is bounded on (a,b), that is, $\|f'\|_\infty = \sup_{t\in(a,b)}|f'(t)| < \infty$. Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left\{ \frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2} \right\}(b-a)\|f'\|_\infty. \tag{1.1}$$

Moreover the constant $1/4$ is the best possible. Because Ostrowski's integral inequality is useful in some fields, many generalizations, extensions, and variants of this inequality have appeared in the literature; see [1–9] and the references given therein. The main aim of this paper is to establish some new Ostrowski type inequalities involving higher-order derivatives. The analysis used in the proof is elementary. The main result of this paper is the following inequality.

Theorem 1.1. *Suppose*

(1) $f : [a,b] \to R$ *to be continuous on* $[a,b]$;

(2) $f : [a,b] \to R$ *to be nth order differentiable on* (a,b) *whose nth order derivative* $f^{(n)} : (a,b) \to R$ *is bounded on* (a,b), *that is,* $\|f^{(n)}\|_\infty = \sup_{t\in(a,b)}|f^{(n)}(t)| < \infty$;

(3) *there exists* $x_0 \in (a,b)$ *such that* $f^{(k)}(x_0) = 0,\ k = 1,2,\ldots,n-1$.

Then for any $x \in [a,b]$, *we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{\|f^{(n)}\|_\infty}{n!}\left\{\left(\left|x - \frac{a+b}{2}\right| + \frac{b-a}{2}\right)^n + \frac{(b-a)^n}{n+1}\right\}. \tag{1.2}$$

As applications of the inequality (1.2), we give more Ostrowski type inequalities.

2. The Proof of Theorem 1.1

In this section, we use the Taylor expansion to prove Theorem 1.1. Before the proof, we need the following lemmas.

Lemma 2.1. *Suppose* $a \le x \le b$ *and* $a < t < b$, *then we have*

$$(x-t)^2 \le \left(\left|x - \frac{a+b}{2}\right| + \frac{b-a}{2}\right)^2 \tag{2.1}$$

Proof. When $a \le x \le (a+b)/2$, then

$$(x-t)^2 \le (x-b)^2 = \left(\left|x - \frac{a+b}{2}\right| + \frac{b-a}{2}\right)^2. \tag{2.2}$$

When $(a+b)/2 \le x \le b$, then

$$(x-t)^2 \le (x-a)^2 = \left(\left|x - \frac{a+b}{2}\right| + \frac{b-a}{2}\right)^2. \tag{2.3}$$

From (2.2) and (2.3), we know that (2.1) holds. □

Lemma 2.2. *Suppose $a \le t \le b$, then for $n \ge 1$ we have*

$$(b-t)^n + (t-a)^n \le (b-a)^n. \tag{2.4}$$

Proof. It is obvious that (2.4) is true for $n = 1$. When $n \ge 2$, let

$$g(t) = (b-t)^n + (t-a)^n, \quad a \le t \le b, \tag{2.5}$$

then

$$g'(t) = n\left[(t-a)^{n-1} - (b-t)^{n-1}\right]. \tag{2.6}$$

The only real root of $g'(t) = 0$ is $t = (a+b)/2$. Notice

$$g\left(\frac{a+b}{2}\right) = \frac{(b-a)^n}{2^{n-1}} \le (b-a)^n = g(a) = g(b). \tag{2.7}$$

Therefore we get the inequality (2.4). □

Now, we give the proof of Theorem 1.1.

Proof. Using the Taylor expansion of $f(x)$ at x_0 gives

$$f(x) = f(x_0) + \frac{f^{(n)}(x_0 + \theta(x-x_0))}{n!}(x-x_0)^n, \quad 0 \le \theta \le 1. \tag{2.8}$$

Taking the integral on both sides of (2.8) with respect to variable x over $[a,b]$, we have

$$\frac{1}{b-a}\int_a^b f(x)dx = f(x_0) + \frac{1}{n!(b-a)}\int_a^b f^{(n)}(x_0+\theta(x-x_0))(x-x_0)^n dx, \tag{2.9}$$

where the parameter θ is not a constant but depends on x. From (2.8) and (2.9) one gets

$$\begin{aligned} f(x) - \frac{1}{b-a}\int_a^b f(x)dx = {} & \frac{f^{(n)}(x_0+\theta(x-x_0))}{n!}(x-x_0)^n \\ & - \frac{1}{n!(b-a)}\int_a^b f^{(n)}(x_0+\theta(x-x_0))(x-x_0)^n dx. \end{aligned} \tag{2.10}$$

So we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\quad=\left|\frac{f^{(n)}(x_0+\theta(x-x_0))}{n!}(x-x_0)^n\right. \\
&\qquad\left.-\frac{1}{n!(b-a)}\int_a^b f^{(n)}(x_0+\theta(x-x_0))(x-x_0)^n dx\right| \\
&\quad\le\left|\frac{f^{(n)}(x_0+\theta(x-x_0))}{n!}(x-x_0)^n\right| \\
&\qquad+\left|\frac{1}{n!(b-a)}\int_a^b f^{(n)}(x_0+\theta(x-x_0))(x-x_0)^n dx\right| \\
&\quad\le\frac{\|f^{(n)}\|_\infty}{n!}\left\{\left(|x-x_0|^n+\frac{1}{b-a}\right)\int_a^b|x-x_0|^n dx\right\} \\
&\quad=\frac{\|f^{(n)}\|_\infty}{n!}\left\{\left(|x-x_0|^n+\frac{1}{(n+1)(b-a)}\right)\left[(b-x_0)^{n+1}+(x_0-a)^{n+1}\right]\right\}.
\end{aligned} \tag{2.11}$$

Using Lemmas 2.1 and 2.2 gives (1.2). Thus, we complete the proof. □

3. Some Applications

In this section, we show some applications of the inequality (1.2). In fact, we can use (1.2) to derive some new Ostrowski type inequalities.

Theorem 3.1. *Suppose*

(1) $f:[a,b]\to R$ *to be continuous on* $[a,b]$;

(2) $f:[a,b]\to R$ *to be second order differentiable on* (a,b) *whose second derivative* $f'':(a,b)\to R$ *is bounded on* (a,b), *that is,* $\|f''\|_\infty=\sup_{t\in(a,b)}|f''(t)|<\infty$;

(3) $f(a)=f(b)$.

Then for any $x\in[a,b]$, *we have*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\quad\le\frac{1}{2}\|f''\|_\infty(b-a)^2\left\{\frac{(x-(a+b)/2)^2}{(b-a)^2}+\frac{|x-(a+b)/2|}{b-a}+\frac{7}{12}\right\}.
\end{aligned} \tag{3.1}$$

Proof. From Rolle's mean value theorem, we know that there exists $x_0\in(a,b)$ such that $f'(x_0)=0$. Let $n=2$ in the inequality (1.2), then we have (3.1). □

Corollary 3.2. *With the assumptions in Theorem 3.1, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{1}{2}\|f''\|_\infty (b-a)^2\left\{\frac{(x-(a+b)/2)^2}{(b-a)^2}+\frac{13}{12}\right\}. \tag{3.2}$$

Proof. For any $x \in [a,b]$, we have

$$\left|x-\frac{a+b}{2}\right| \le \frac{b-a}{2}. \tag{3.3}$$

Consequently, (3.1) gives

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\quad\le \frac{1}{2}\|f''\|_\infty (b-a)^2\left\{\frac{(x-(a+b)/2)^2}{(b-a)^2}+\frac{|x-(a+b)/2|}{b-a}+\frac{7}{12}\right\} \\
&\quad\le \frac{1}{2}\|f''\|_\infty (b-a)^2\left\{\frac{(x-(a+b)/2)^2}{(b-a)^2}+\frac{b-a}{2(b-a)}+\frac{7}{12}\right\} \\
&\quad= \frac{1}{2}\|f''\|_\infty (b-a)^2\left\{\frac{(x-(a+b)/2)^2}{(b-a)^2}+\frac{13}{12}\right\}.
\end{aligned} \tag{3.4}$$

□

Corollary 3.3. *With the assumptions in Theorem 3.1, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{1}{2}\|f''\|_\infty (b-a)^2\left\{\frac{|x-(a+b)/2|}{b-a}+\frac{5}{6}\right\}. \tag{3.5}$$

Proof. For any $x \in [a,b]$, we have

$$\left(x-\frac{a+b}{2}\right)^2 \le \frac{(b-a)^2}{4}. \tag{3.6}$$

Substituting (3.6) into (3.1) gives (3.5). □

Theorem 3.4. *Suppose*

(1) *$f : [a,b] \to R$ to be continuous on $[a,b]$;*

(2) *$f : [a,b] \to R$ to be nth order differentiable on (a,b) whose nth order derivative $f^{(n)} : (a,b) \to R$ is bounded on (a,b), that is, $\|f^{(n)}\|_\infty = \sup_{t\in(a,b)}|f^{(n)}(t)| < \infty$.*

Then for any $x_0 \in (a,b)$ *and* $x \in [a,b]$, *we have*

$$\begin{aligned}
&\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\quad \le \frac{\|f^{(n)}\|_\infty}{n!}\left\{\left(\left|x - \frac{a+b}{2}\right| + \frac{b-a}{2}\right)^n + \frac{(b-a)^n}{n+1}\right\} \\
&\qquad + (b-a)\left(\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2}\right)\sum_{k=1}^{n-1}\frac{|f^{(k)}(x_0)|}{(k-1)!}\left(\left|x_0 - \frac{a+b}{2}\right| + \frac{b-a}{2}\right)^{k-1}.
\end{aligned} \tag{3.7}$$

Proof. Let

$$\begin{gathered}
p(x) = \sum_{k=1}^{n-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \\
F(x) = f(x) - p(x).
\end{gathered} \tag{3.8}$$

Then we have

$$\begin{gathered}
F^{(k)}(x) = 0, \quad k = 1, 2, \ldots, n-1, \\
F^{(n)}(x) = f^{(n)}(x).
\end{gathered} \tag{3.9}$$

Using inequality (1.2) to $F(x)$ gives

$$\begin{aligned}
&\left|F(x) - \frac{1}{b-a}\int_a^b F(t)dt\right| \\
&\quad = \left|f(x) - p(x) - \frac{1}{b-a}\int_a^b (f(t) - p(t))dt\right| \\
&\quad \le \frac{\|f^{(n)}\|_\infty}{n!}\left\{\left(\left|x - \frac{a+b}{2}\right| + \frac{b-a}{2}\right)^n + \frac{(b-a)^n}{n+1}\right\}.
\end{aligned} \tag{3.10}$$

Since

$$\begin{aligned}
&\left|f(x) - p(x) - \frac{1}{b-a}\int_a^b (f(t) - p(t))dt\right| \\
&\quad = \left|\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) - \left(p(x) - \frac{1}{b-a}\int_a^b p(t)dt\right)\right| \\
&\quad \ge \left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| - \left|p(x) - \frac{1}{b-a}\int_a^b p(t)dt\right|,
\end{aligned} \tag{3.11}$$

we have

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ &\quad \le \frac{\|f^{(n)}\|_\infty}{n!}\left\{ \left(\left| x - \frac{a+b}{2} \right| + \frac{b-a}{2} \right)^n + \frac{(b-a)^n}{n+1} \right\} \\ &\qquad + \left| p(x) - \frac{1}{b-a}\int_a^b p(t)dt \right|. \end{aligned} \tag{3.12}$$

Using Ostrowski's integral inequality (1.1) one gets

$$\left| p(x) - \frac{1}{b-a}\int_a^b p(t)dt \right| \le \left\{ \frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2} \right\}(b-a)\|p'\|_\infty. \tag{3.13}$$

Notice

$$\begin{aligned} \|p'\|_\infty &= \sup_{x\in(a,b)} |p'(x)| \\ &= \sup_{x\in(a,b)} \left| \sum_{k=1}^{n-1} \frac{f^{(k)}(x_0)}{(k-1)!}(x-x_0)^{k-1} \right| \\ &\le \sum_{k=1}^{n-1} \frac{|f^{(k)}(x_0)|}{(k-1)!}\left(\left| x_0 - \frac{a+b}{2} \right| + \frac{b-a}{2} \right)^{k-1}. \end{aligned} \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12) gives (3.7). □

It is easy to see that (3.7) is the generalization of (1.2). If we let $x_0 = (a+b)/2$ in (3.7) and use (3.6), we get the following inequality.

Corollary 3.5. *With the assumptions in Theorem 3.4, we have*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ &\quad \le \frac{\|f^{(n)}\|_\infty}{n!}\left\{ \left(\left| x - \frac{a+b}{2} \right| + \frac{b-a}{2} \right)^n + \frac{(b-a)^n}{n+1} \right\} \\ &\qquad + \sum_{k=1}^{n-1} \frac{|f^{(k)}((a+b)/2)|}{(k-1)!}\left(\frac{b-a}{2} \right)^k. \end{aligned} \tag{3.15}$$

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