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*Research Article*

# Trace Inequalities for Matrix Products and Trace Bounds for the Solution of the Algebraic Riccati Equations

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By using diagonalizable matrix decomposition and majorization inequalities, we propose new trace bounds for the product of two real square matrices in which one is diagonalizable. These bounds improve and extend the previous results. Furthermore, we give some trace bounds for the solution of the algebraic Riccati equations, which improve some of the previous results under certain conditions. Finally, numerical examples have illustrated that our results are effective and superior.



## 1. Introduction

As we all know, the Riccati equations are of great importance in both theory and practice in the analysis and design of controllers and filters for linear dynamical systems (see [1–5]). For example, consider the following linear system (see [5]):

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{1.1}$$

with the cost

$$J = \int_0^\infty \left(x^T Q x + u^T u\right) dt. \tag{1.2}$$

The optimal control rate $u^*$ the optimal cost $J^*$ of (1.1) and (1.2) are

$$u^* = Px, \qquad P = B^T K,$$
$$J^* = x_0^T K x_0, \tag{1.3}$$

where $x_0 \in R^n$ is the initial state of system (1.1) and (1.2) and $K$ is the positive semidefinite solution of the following algebraic Riccati equation (ARE):

$$A^T K + KA - KRK = -Q, \tag{1.4}$$

with $R = BB^T$ and $Q$ being positive definite and positive semidefinite matrices, respectively. To guarantee the existence of the positive definite solution to (1.4), we will make the following assumptions: the pair $(A, R)$ is stabilizable, and the pair $(Q, A)$ is observable.

In practice, it is hard to solve the ARE, and there is no general method unless the system matrices are special and there are some methods and algorithms to solve (1.4); however, the solution can be time-consuming and computationally difficult, particularly as the dimensions of the system matrices increase. Thus, a number of works have been presented by researchers to evaluate the bounds and trace bounds for the solution of the ARE (see [6–16]). Moreover, in terms of [2, 6], we know that an interpretation of $\operatorname{tr}(K)$ is that $\operatorname{tr}(K)/n$ is the average value of the optimal cost $J^*$ as $x_0$ varies over the surface of a unit sphere. Therefore, considering its applications, it is important to discuss trace bounds for the product of two matrices. In symmetric case, a number of works have been proposed for the trace of matrix products ([2, 6–8, 17–20]), and [18] is the tightest among the parallel results.

In 1995, Lasserre showed [18] the following given any matrix $A \in R^{n\times n}$, $B \in S^n$, then the following.

$$\sum_{i=1}^{n} \lambda_{[i]}\left(\overline{A}\right)\lambda_{[n-i+1]}(B) \le \operatorname{tr}(AB) \le \sum_{i=1}^{n} \lambda_{[i]}\left(\overline{A}\right)\lambda_{[i]}(B), \tag{1.5}$$

where $\overline{A} = (A + A^T)/2$.

This paper is organized as follows. In Section 2, we propose new trace bounds for the product of two general matrices. The new trace bounds improve the previous results. Then, we present some trace bounds for the solution of the algebraic Riccati equations, which improve some of the previous results under certain conditions in Section 3. In Section 4, we give numerical examples to demonstrate the effectiveness of our results. Finally, we get conclusions in Section 5.

## 2. Trace Inequalities for Matrix Products

In the following, let $R^{n\times n}$ denote the set of $n \times n$ real matrices and let $S^n$ denote the subset of $R^{n\times n}$ consisting of symmetric matrices. For $A = (a_{ij}) \in R^{n\times n}$, we assume that $\operatorname{tr}(A)$, $A^{-1}$, $A^T$, $d(A) = (d_1(A), \dots, d_n(A))$, $\sigma(A) = (\sigma_1(A), \dots, \sigma_n(A))$ denote the trace, the inverse, the transpose, the diagonal elements, the singular values of $A$, respectively, and define $(A)_{ii} = a_{ii} = d_i(A)$. If $A \in R^{n\times n}$ is an arbitrary symmetric matrix, then $\lambda(A) = (\lambda_1(A), \dots, \lambda_n(A))$ and $\operatorname{Re}\lambda(A) = (\operatorname{Re}\lambda_1(A), \dots, \operatorname{Re}\lambda_n(A))$ denote the eigenvalues and the real part of eigenvalues of $A$. Suppose $x = (x_1, x_2, \dots, x_n)$ is a real $n$-element array such as $d(A)$, $\sigma(A)$, $\lambda(A)$, $\operatorname{Re}\lambda(A)$ which is reordered, and its elements are arranged in nonincreasing order; that is, $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$. The notation $A > 0$ ($A \geq 0$) is used to denote that $A$ is a symmetric positive definite (semidefinite) matrix.

Let $\alpha, \beta$ be two real $n$-element arrays. If they satisfy

$$\sum_{i=1}^{k} \alpha_{[i]} \leq \sum_{i=1}^{k} \beta_{[i]}, \quad k = 1, 2, \dots, n, \tag{2.1}$$

then it is said that $\alpha$ is controlled weakly by $\beta$, which is signed by $\alpha \prec_w \beta$.

If $\alpha \prec_w \beta$ and

$$\sum_{i=1}^{n} \alpha_{[i]} = \sum_{i=1}^{n} \beta_{[i]}, \tag{2.2}$$

then it is said that $\alpha$ is controlled by $\beta$, which is signed by $\alpha \prec \beta$.

The following lemmas are used to prove the main results.

**Lemma 2.1** (see [21, Page 92, H.2.c]). *If $x_{[1]} \geq \cdots \geq x_{[n]}$, $y_{[1]} \geq \cdots \geq y_{[n]}$ and $x \prec y$, then for any real array $u_{[1]} \geq \cdots \geq u_{[n]}$,*

$$\sum_{i=1}^{n} x_{[i]} u_{[i]} \leq \sum_{i=1}^{n} y_{[i]} u_{[i]}. \tag{2.3}$$

**Lemma 2.2** (see [21, Page 218, B.1]). *Let $A = A^T \in R^{n\times n}$, then*

$$d(A) \prec \lambda(A). \tag{2.4}$$

**Lemma 2.3** (see [21, Page 240, F.4.a]). *Let $A \in R^{n\times n}$, then*

$$\lambda\left(\frac{A + A^T}{2}\right) \prec_w \left|\lambda\left(\frac{A + A^T}{2}\right)\right| \prec_w \sigma(A). \tag{2.5}$$

**Lemma 2.4** (see [22]). *Let $0 < m_1 \leq a_k \leq M_1$, $0 < m_2 \leq b_k \leq M_2$, $k = 1, 2, \dots, n, 1/p + 1/q = 1$, then*

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q} \leq c_{p,q} \sum_{k=1}^{n} a_k b_k, \tag{2.6}$$

*where*

$$c_{p,q} = \frac{M_1^p M_2^q - m_1^p m_2^q}{\left[p\left(M_1 m_2 M_2^q - m_1 M_2 m_2^q\right)\right]^{1/p}\left[q\left(m_1 M_2 M_1^p - M_1 m_2 m_1^p\right)\right]^{1/q}}. \tag{2.7}$$

Note that if $m_1 = 0,\ m_2 \neq 0$ or $m_2 = 0,\ m_1 \neq 0$, obviously, (2.6) holds. If $m_1 = m_2 = 0$, choose $c_{p,q} = +\infty$, then (2.6) also holds.

*Remark 2.5.* If $p = q = 2$, then we obtain Cauchy-Schwarz inequality:

$$\sum_{k=1}^n a_k b_k \leq \left(\sum_{k=1}^n a_k^2\right)^{1/2}\left(\sum_{k=1}^n b_k^2\right)^{1/2} \leq c_2 \sum_{k=1}^n a_k b_k, \tag{2.8}$$

where

$$c_2 = \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right). \tag{2.9}$$

*Remark 2.6.* Note that

$$\lim_{p\to\infty}\left(a_1^p + a_2^p + \cdots + a_n^p\right)^{1/p} = \max_{1\leq k\leq n}\{a_k\},$$

$$\begin{aligned}
\lim_{\substack{p\to\infty\\ q\to 1}} c_{p,q} &= \lim_{\substack{p\to\infty\\ q\to 1}} \frac{M_1^p M_2^q - m_1^p m_2^q}{\left[p\left(M_1 m_2 M_2^q - m_1 M_2 m_2^q\right)\right]^{1/p}\left[q\left(m_1 M_2 M_1^p - M_1 m_2 m_1^p\right)\right]^{1/q}} \\
&= \lim_{\substack{p\to\infty\\ q\to 1}} \frac{M_1^p\left[M_2^q - (m_1/M_1)^p m_2^q\right]}{M_1^{1/p}\left[p\left(m_2 M_2^q - (m_1/M_1) M_2 m_2^q\right)\right]^{1/p} M_1^{q/p}\left[q\left(m_1 M_2 - M_1 m_2 (m_1/M_1)^p\right)\right]^{1/q}} \\
&= \lim_{\substack{p\to\infty\\ q\to 1}} \frac{M_2}{M_1^{1/p+p/q-p} m_1 M_2} = \lim_{\substack{p\to\infty\\ q\to 1}} \frac{1}{M_1^{1/p-1} m_1} = \frac{M_1}{m_1}.
\end{aligned} \tag{2.10}$$

Let $p \to \infty, q \to 1$ in (2.6), then we obtain

$$m_1 \sum_{k=1}^n b_k \leq \sum_{k=1}^n a_k b_k \leq M_1 \sum_{k=1}^n b_k. \tag{2.11}$$

**Lemma 2.7.** *If* $q \geq 1$, $a_i \geq 0$ $(i = 1, 2, \ldots, n)$, *then*

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^q \leq \frac{1}{n}\sum_{i=1}^{n} a_i^q. \tag{2.12}$$

*Proof.* (1) Note that for $q = 1$, or $a_i = 0$ $(i = 1, 2, \ldots, n)$,

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^q = \frac{1}{n}\sum_{i=1}^{n} a_i^q. \tag{2.13}$$

(2) If $q > 1$, $a_i > 0$, for $x > 0$, choose $f(x) = x^q$, then $f'(x) = qx^{q-1} > 0$ and $f''(x) = q(q-1)x^{q-2} > 0$. Thus, $f(x)$ is a convex function. As $a_i > 0$ and $(1/n)\sum_{i=1}^{n} a_i > 0$, from the property of the convex function, we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^q = f\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} f(a_i) = \frac{1}{n}\sum_{i=1}^{n} a_i^q. \tag{2.14}$$

(3) If $q > 1$, without loss of generality, we may assume $a_i = 0$ $(i = 1, \ldots, r)$, $a_i > 0$ $(i = r+1, \ldots, n)$. Then from (2), we have

$$\left(\frac{1}{n-r}\right)^q \left(\sum_{i=1}^{n} a_i\right)^q = \left(\frac{1}{n-r}\sum_{i=1}^{n} a_i\right)^q \leq \frac{1}{n-r}\sum_{i=1}^{n} a_i^q. \tag{2.15}$$

Since $((n-r)/n)^q \leq (n-r)/n$, thus

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^q = \left(\frac{n-r}{n}\right)^q \left(\frac{1}{n-r}\right)^q \left(\sum_{i=1}^{n} a_i\right)^q \leq \frac{n-r}{n}\frac{1}{n-r}\sum_{i=1}^{n} a_i^q = \frac{1}{n}\sum_{i=1}^{n} a_i^q. \tag{2.16}$$

This completes the proof. □

**Theorem 2.8.** *Let* $A, B \in R^{n\times n}$, *and let* $B$ *be diagonalizable with the following decomposition:*

$$B = U\,\mathrm{diag}(\lambda_1(B), \lambda_2(B), \ldots, \lambda_n(B))U^{-1}, \tag{2.17}$$

*where* $U \in R^{n\times n}$ *is nonsingular. Then*

$$\sum_{i=1}^{n} \mathrm{Re}\,\lambda_{[i]}(B) d_{[n-i+1]}\left(\overline{U^{-1}AU}\right) \leq tr(AB) \leq \sum_{i=1}^{n} \mathrm{Re}\,\lambda_{[i]}(B) d_{[i]}\left(\overline{U^{-1}AU}\right). \tag{2.18}$$

*Proof.* Note that $(U^{-1}AU)_{ii}$ is real; by the matrix theory we have

$$\begin{aligned}
\operatorname{tr}(AB) &= \operatorname{Re}\ \operatorname{tr}(AB) = \operatorname{Re}\ \operatorname{tr}\left[AU\operatorname{diag}(\lambda_1(B),\lambda_2(B),\dots,\lambda_n(B))U^{-1}\right] \\
&= \operatorname{Re}\ \operatorname{tr}\left[U^{-1}AU\operatorname{diag}(\lambda_1(B),\lambda_2(B),\dots,\lambda_n(B))\right] \\
&= \operatorname{Re}\sum_{i=1}^{n}\lambda_i(B)\left(U^{-1}AU\right)_{ii} \\
&= \sum_{i=1}^{n}\operatorname{Re}\left[\lambda_i(B)\left(U^{-1}AU\right)_{ii}\right] \\
&= \sum_{i=1}^{n}\left(U^{-1}AU\right)_{ii}\operatorname{Re}\lambda_i(B) \\
&= \sum_{i=1}^{n}\left[\frac{U^{-1}AU+(U^{-1}AU)^T}{2}\right]_{ii}\operatorname{Re}\lambda_i(B) \\
&= \sum_{i=1}^{n}\overline{(U^{-1}AU)}_{ii}\operatorname{Re}\lambda_i(B).
\end{aligned} \tag{2.19}$$

Since $\operatorname{Re}\lambda_{[1]}(B) \ge \operatorname{Re}\lambda_{[2]}(B) \ge \cdots \ge \operatorname{Re}\lambda_{[n]}(B) \ge 0$, without loss of generality, we may assume $\operatorname{Re}\lambda(B) = (\operatorname{Re}\lambda_{[1]}(B), \operatorname{Re}\lambda_{[2]}(B), \dots, \operatorname{Re}\lambda_{[n]}(B))$. Next, we will prove the left-hand side of (2.18):

$$\sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(B)d_{[n-i+1]}\left(\overline{U^{-1}AU}\right) \le \sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(B)d_i\left(\overline{U^{-1}AU}\right). \tag{2.20}$$

If

$$d\left(\overline{U^{-1}AU}\right) = \left(d_{[n]}\left(\overline{U^{-1}AU}\right), d_{[n-1]}\left(\overline{U^{-1}AU}\right), \dots, d_{[1]}\left(\overline{U^{-1}AU}\right)\right), \tag{2.21}$$

then we obtain the conclusion. Now assume that there exists $j < k$ such that $d_j(\overline{U^{-1}AU}) > d_k(\overline{U^{-1}AU})$, then

$$\begin{aligned}
&\operatorname{Re}\lambda_{[j]}(B)d_k\left(\overline{U^{-1}AU}\right) + \operatorname{Re}\lambda_{[k]}(B)d_j\left(\overline{U^{-1}AU}\right) - \operatorname{Re}\lambda_{[j]}(B)d_j\left(\overline{U^{-1}AU}\right) \\
&\quad - \operatorname{Re}\lambda_{[k]}(B)d_k\left(\overline{U^{-1}AU}\right) = \left[\operatorname{Re}\lambda_{[j]}(B) - \operatorname{Re}\lambda_{[k]}(B)\right]\left[d_k\left(\overline{U^{-1}AU}\right) - d_j\left(\overline{U^{-1}AU}\right)\right] \le 0.
\end{aligned} \tag{2.22}$$

We use $\tilde{d}(\overline{U^{-1}AU})$ to denote the vector of $d(\overline{U^{-1}AU})$ after changing $d_j(\overline{U^{-1}AU})$ and $d_k(\overline{U^{-1}AU})$, then

$$\sum_{i=1}^{n}\sigma_{[i]}(B)\tilde{d}_i\left(\overline{U^{-1}AU}\right) \leq \sum_{i=1}^{n}\sigma_{[i]}(B)d_i\left(\overline{U^{-1}AU}\right). \tag{2.23}$$

After a limited number of steps, we obtain the left-hand side of (2.18). For the right-hand side of (2.18)

$$\sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(B)d_i\left(\overline{U^{-1}AU}\right) \leq \sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(B)d_{[i]}\left(\overline{U^{-1}AU}\right). \tag{2.24}$$

If

$$d\left(V^T AU\right) = \left(d_{[1]}\left(\overline{U^{-1}AU}\right), d_{[2]}\left(\overline{U^{-1}AU}\right), \ldots, d_{[n]}\left(\overline{U^{-1}AU}\right)\right), \tag{2.25}$$

then we obtain the conclusion. Now assume that there exists $j > k$ such that $d_j(\overline{U^{-1}AU}) < d_k(\overline{U^{-1}AU})$, then

$$\begin{aligned}
&\sigma_{[j]}(B)d_k\left(\overline{U^{-1}AU}\right) + \sigma_{[k]}(B)d_j\left(\overline{U^{-1}AU}\right) - \sigma_{[j]}(B)d_j\left(\overline{U^{-1}AU}\right) - \sigma_{[k]}(B)d_k\left(\overline{U^{-1}AU}\right) \\
&\quad = \left[\sigma_{[j]}(B) - \sigma_{[k]}(B)\right]\left[d_k\left(\overline{U^{-1}AU}\right) - d_j\left(\overline{U^{-1}AU}\right)\right] \geq 0.
\end{aligned} \tag{2.26}$$

We use $\tilde{d}(\overline{U^{-1}AU})$ to denote the vector of $d(\overline{U^{-1}AU})$ after changing $d_j(\overline{U^{-1}AU})$ and $d_k(\overline{U^{-1}AU})$, then

$$\sum_{i=1}^{n}\sigma_{[i]}(B)d_i\left(\overline{U^{-1}AU}\right) \leq \sum_{i=1}^{n}\sigma_{[i]}(B)\tilde{d}_i\left(\overline{U^{-1}AU}\right). \tag{2.27}$$

After a limited number of steps, we obtain the right-hand side of (2.18). Therefore, we have

$$\sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(B)d_{[n-i+1]}\left(\overline{U^{-1}AU}\right) \leq \operatorname{tr}(AB) \leq \sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(B)d_{[i]}\left(\overline{U^{-1}AU}\right). \tag{2.28}$$

□

Since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, applying (2.18) with $B$ in lieu of $A$, we immediately have the following corollary.

**Corollary 2.9.** *Let $A, B \in R^{n\times n}$, and let $A$ be diagonalizable with the following decomposition:*

$$A = V\,\mathrm{diag}(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))V^{-1}, \tag{2.29}$$

*where $V \in R^{n\times n}$ is nonsingular. Then*

$$\sum_{i=1}^{n} \mathrm{Re}\,\lambda_{[i]}(A) d_{[n-i+1]}\left(\overline{V^{-1}BV}\right) \leq tr(AB) \leq \sum_{i=1}^{n} \mathrm{Re}\,\lambda_{[i]}(A) d_{[i]}\left(\overline{V^{-1}BV}\right). \tag{2.30}$$

**Theorem 2.10.** *Let $A \in R^{n\times n}$, $B \in R^{n\times n}$ be normal. Then*

$$\sum_{i=1}^{n} \mathrm{Re}\,\lambda_{[i]}(B)\lambda_{[n-i+1]}\left(\overline{A}\right) \leq tr(AB) \leq \sum_{i=1}^{n} \mathrm{Re}\,\lambda_{[i]}(B)\lambda_{[i]}\left(\overline{A}\right). \tag{2.31}$$

*Proof.* Since $B$ is normal, from [23, page 101, Theorem 2.5.4], we have

$$B = U\,\mathrm{diag}(\lambda_1(B), \lambda_2(B), \ldots, \lambda_n(B))U^{-1}, \tag{2.32}$$

where $U \in R^{n\times n}$ is orthogonal. Since $U^T = U^{-1}$ and $UU^T = I$, then for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
\lambda_{[i]}\left(\overline{U^{-1}AU}\right) &= \lambda_{[i]}\left(\overline{U^{T}AU}\right) \\
&= \lambda_{[i]}\left(\frac{U^TAU + \left(U^TAU\right)^T}{2}\right) \\
&= \lambda_{[i]}\left(U^T\left(\frac{AUU^T + \left(AUU^T\right)^T}{2}\right)U\right) \\
&= \lambda_{[i]}\left(\frac{AUU^T + \left(AUU^T\right)^T}{2}\right) = \lambda_{[i]}\left(\overline{A}\right).
\end{aligned} \tag{2.33}$$

In terms of Lemmas 2.1 and 2.2, (2.18) implies

$$\begin{aligned}\sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(B)\lambda_{[n-i+1]}\left(\overline{A}\right) &= \sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(B)\lambda_{[n-i+1]}\left(\overline{U^{-1}AU}\right)\\ &\leq \sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(B)d_{[n-i+1]}\left(\overline{U^{-1}AU}\right)\\ &\leq \operatorname{tr}(AB) \leq \sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(B)d_{[i]}\left(\overline{U^{-1}AU}\right)\\ &\leq \sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(B)\lambda_{[i]}\left(\overline{U^{-1}AU}\right)\\ &= \sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(B)\lambda_{[i]}\left(\overline{A}\right).\end{aligned} \tag{2.34}$$

This completes the proof. □

Note that if $B \in S^n$, $\operatorname{Re}\lambda_{[i]}(B) = \lambda_{[i]}(B)$, then from (2.34) we obtain (1.5) immediately. This implies that (2.18) improves (1.5).

Since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, applying (2.31) with $B$ in lieu of $A$, we immediately have the following corollary.

**Corollary 2.11.** *Let $B \in R^{n\times n}$, $A \in R^{n\times n}$ be normal, then*

$$\sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(A)\lambda_{[n-i+1]}\left(\overline{B}\right) \leq tr(AB) \leq \sum_{i=1}^{n}\operatorname{Re}\lambda_{[i]}(A)\lambda_{[i]}\left(\overline{B}\right). \tag{2.35}$$

## 3. Trace Bounds for the Solution of the Algebraic Riccati Equations

Komaroff (1994) in [16] obtained the following. Let $K$ be the positive semidefinite solution of the ARE (1.4). Then the trace of $K$ has the upper bound given by

$$\operatorname{tr}(K) \leq \frac{n}{2}\lambda_{[1]}(S) + \frac{n}{2}\sqrt{\lambda_{[1]}^{2}(S) + \frac{4\operatorname{tr}\left(QR^{-1}\right)}{n}}, \tag{3.1}$$

where $S = R^{-1}A^{T} + AR^{-1}$.

In this section, by appling our new trace bounds in Section 2, we obtain some lower trace bounds for the solution of the algebraic Riccati equations. Furthermore, we obtain some upper trace bounds which improve (3.1) under certain conditions.

**Theorem 3.1.** *If $1/p+1/q=1$, and $K$ is the positive semidefinite solution of the ARE* (1.4).

(1) *There are both, upper and lower, bounds:*

$$\frac{\lambda_{[n]}(R)\lambda_{[n]}(S)+\lambda_{[n]}(R)\sqrt{\left[\lambda_{[n]}(S)\right]^2+\left(4/\lambda_{[n]}(R)\right)\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}tr\left(QR^{-1}\right)}}{2\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}}$$
$$\leq tr(K)\leq\frac{\lambda_{[1]}(S)+\sqrt{\lambda_{[1]}^{2}(S)+\left(4/c_{p,q}n^{2-1/q}\lambda_{[1]}(R)\right)\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}tr\left(QR^{-1}\right)}}{2\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}/c_{p,q}n^{2-1/q}\lambda_{[1]}(R)}. \quad (3.2)$$

(2) *If $S\geq 0$, then the trace of $K$ has the lower and upper bounds given by*

$$\frac{\left(1/c'_{p,q}n^{1-1/q}\right)\mathcal{H}+\sqrt{\left[\left(1/c'_{p,q}n^{1-1/q}\right)\mathcal{H}\right]^2+\left(4/\lambda_{[n]}(R)\right)\ell tr\left(QR^{-1}\right)}}{2\ell/\lambda_{[n]}(R)}$$
$$\leq tr(K)\leq\frac{\mathcal{H}+\sqrt{\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(S)\right]^{2/p}+\left(4/c_{p,q}n^{2-1/q}\lambda_{[1]}(R)\right)\ell tr\left(QR^{-1}\right)}}{2\ell/c_{p,q}n^{2-1/q}\lambda_{[1]}(R)}, \quad (3.3)$$

*where $\mathcal{H}$ denotes $\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(S)\right]^{1/p}$ and $\ell$ denotes $\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}$.*

(3) *If $S\leq 0$, then the trace of $K$ has the lower and upper bounds given by*

$$\frac{-\left[\sum_{i=1}^{n}\left|\lambda_{[i]}(S)\right|^{p}\right]^{1/p}+\sqrt{\left[\sum_{i=1}^{n}\left|\lambda_{[i]}(S)\right|^{p}\right]^{2/p}+4/\lambda_{[n]}(R)\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}tr\left(QR^{-1}\right)}}{2\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}/\lambda_{[n]}(R)}$$
$$\leq tr(K)\leq\frac{c_{p,q}n^{2-1/q}\lambda_{[1]}(R)}{2\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}}$$
$$\times\left\{\frac{1}{c'_{p,q}n^{1-1/q}}\left[-\sum_{i=1}^{n}\left|\lambda_{[i]}(S)\right|^{p}\right]^{1/p}\right.$$
$$\left.+\sqrt{\left[\frac{1}{c'_{p,q}n^{1-1/q}}\mathcal{N}\right]^2+\frac{4}{c_{p,q}n^{2-1/q}\lambda_{[1]}(R)}\mathcal{S}tr\left(QR^{-1}\right)}\right\}, \quad (3.4)$$

*where $\mathcal{N}$ denotes $\left[\sum_{i=1}^{n}\left|\lambda_{[i]}(S)\right|^{p}\right]^{1/p}$ and $\mathcal{S}$ denotes $\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}$,* *We have*

$$c_{p,q} = \frac{M_r^p M_k^q - m_r^p m_k^q}{\left[p\left(M_r m_k M_k^q - m_r M_k m_k^q\right)\right]^{1/p}\left[q\left(m_r M_k M_r^p - M_r m_k m_r^p\right)\right]^{1/q}},$$

$$M_r = \lambda_{[1]}(R), \qquad m_r = \lambda_{[n]}(R), \qquad M_k = \lambda_{[1]}(K), \qquad m_k = \lambda_{[n]}(K),$$

$$c'_{p,q} = \frac{M_s^p M_k^q - m_s^p m_k^q}{\left[p\left(M_s m_k M_k^q - m_s M_k m_k^q\right)\right]^{1/p}\left[q\left(m_s M_k M_s^p - M_1 m_k m_s^p\right)\right]^{1/q}},$$

$$M_s = \lambda_{[1]}(S), \qquad m_s = \lambda_{[n]}(S), \qquad S = R^{-1}A^T + AR^{-1}. \tag{3.5}$$

*Proof.* (1) Multiply (1.4) on the right and on the left by $R^{-1/2}$ to get

$$R^{-1/2}QR^{-1/2} = K_1^T K_1 - R^{-1/2}\left(A^T K + KA\right)R^{-1/2}, \tag{3.6}$$

where $K_1 = R^{1/2}KR^{-1/2}$. Take the trace of all terms in (3.6) to get

$$\operatorname{tr}\left(K_1^T K_1\right) - \operatorname{tr}\left(R^{-1}A^T K + KAR^{-1}\right) - \operatorname{tr}\left(QR^{-1}\right) = 0. \tag{3.7}$$

Since $K$ is positive semidefiniteness, $\lambda(K) = \operatorname{Re}\lambda(K)$, $\operatorname{tr}(K) = \sum_{i=1}^n \lambda_{[i]}(K) = \sum_{i=1}^n \operatorname{Re}\lambda_{[i]}(K)$, and from Lemma 2.7, we have

$$\frac{\operatorname{tr}(K)}{n^{1-1/q}} \le \left[\operatorname{tr}(K^q)\right]^{1/q} \le \operatorname{tr}(K), \tag{3.8}$$

$$\sum_{i=1}^n \lambda_{[i]}(KK) = \sum_{i=1}^n \lambda_{[i]}^2(K) \le \left[\sum_{i=1}^n \lambda_{[i]}(K)\right]^2 = [\operatorname{tr}(K)]^2. \tag{3.9}$$

By Cauchy-Schwarz inequality (2.8), it can be shown that

$$\sum_{i=1}^n \lambda_{[i]}(KK) = \sum_{i=1}^n \lambda_{[i]}^2(K) \ge \frac{\left[\sum_{i=1}^n \lambda_{[i]}(K)\right]^2}{n} = \frac{[\operatorname{tr}(K)]^2}{n}. \tag{3.10}$$

Since $K, Q$ are positive semidefiniteness, $R$ is positive definiteness, then by (1.5), note that for $i = 1, 2, \ldots, n$, $\lambda_{[i]}(R^{-1}) = \lambda_{[i]}(\overline{R^{-1}}) = 1/\lambda_{[n-i+1]}(R)$, and considering (2.6), (3.8), and (3.9), we have

$$\begin{aligned}\operatorname{tr}\left(K_1^T K_1\right) &= \operatorname{tr}\left(R^{-1}KRK\right) \le \sum_{i=1}^{n} \lambda_{[i]}\left(R^{-1}\right)\lambda_{[i]}(KRK) \\ &= \sum_{i=1}^{n} \frac{\lambda_{[i]}(KRK)}{\lambda_{[n-i+1]}(R)} \le \frac{1}{\lambda_{[n]}(R)}\operatorname{tr}(KRK) \\ &\le \frac{1}{\lambda_{[n]}(R)}\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}[\operatorname{tr}(K)]^2.\end{aligned} \tag{3.11}$$

Note that $S = R^{-1}A^T + AR^{-1}$, $\lambda_{[i]}(S) = \lambda_{[i]}(\overline{S})$, then from (1.5) we have

$$\begin{aligned}\lambda_{[n]}(S)\operatorname{tr}(K) &\le \sum_{i=1}^{n}\lambda_{[n-i+1]}\left(R^{-1}A^T + AR^{-1}\right)\lambda_{[i]}(K) \\ &\le \operatorname{tr}\left(R^{-1}A^TK + AR^{-1}K\right) = \operatorname{tr}\left(R^{-1}A^TK + KAR^{-1}\right) \\ &\le \sum_{i=1}^{n}\lambda_{[i]}\left(R^{-1}A^T + AR^{-1}\right)\lambda_{[i]}(K) \le \lambda_{[1]}(S)\operatorname{tr}(K).\end{aligned} \tag{3.12}$$

Combining (3.11) with (3.12), we obtain

$$\frac{1}{\lambda_{[n]}(R)}\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}[\operatorname{tr}(K)]^2 - \operatorname{tr}(K)\lambda_{[n]}(S) - \operatorname{tr}\left(QR^{-1}\right) \ge 0. \tag{3.13}$$

Solving (3.13) for $\operatorname{tr}(K)$ yields the left-hand side of (3.2).

Since $K, Q$ are positive semidefiniteness, $R$ is positive definiteness, then by (1.5), note that for $i = 1, 2, \ldots, n$, $\lambda_{[n-i+1]}(R^{-1}) = \lambda_{[n-i+1]}(\overline{R^{-1}}) = 1/\lambda_{[i]}(R)$, and considering (2.6), (3.8), and (3.10), we have

$$\begin{aligned}\operatorname{tr}\left(K_1^T K_1\right) &= \operatorname{tr}\left(R^{-1}KRK\right) \ge \sum_{i=1}^{n}\lambda_{[n-i+1]}\left(R^{-1}\right)\lambda_{[i]}(KRK) \\ &= \sum_{i=1}^{n}\frac{\lambda_{[i]}(KRK)}{\lambda_{[i]}(R)} \ge \frac{1}{\lambda_{[1]}(R)}\operatorname{tr}(KRK) \\ &\ge \frac{1}{c_{p,q}\lambda_{[1]}(R)}\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}\left[\sum_{i=1}^{n}\lambda_{[i]}^{q}(K^2)\right]^{1/q} \\ &\ge \frac{1}{c_{p,q}n^{2-1/q}\lambda_{[1]}(R)}\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}[\operatorname{tr}(K)]^2.\end{aligned} \tag{3.14}$$

Combining (3.12) with (3.14), we obtain

$$\frac{1}{c_{p,q}n^{2-1/q}\lambda_{[1]}(R)}\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}[\operatorname{tr}(K)]^2-\operatorname{tr}(K)\lambda_{[1]}(S)-\operatorname{tr}\left(QR^{-1}\right)\leq 0. \tag{3.15}$$

Solving (3.15) for $\operatorname{tr}(K)$ yields the right-hand side of (3.2).

(2) Note that when $S \geq 0$, by (1.5), (2.6), and (3.8), we have

$$\begin{aligned}\operatorname{tr}\left(R^{-1}A^{T}K+KAR^{-1}\right)&\geq\sum_{i=1}^{n}\lambda_{[n-i+1]}(S)\lambda_{[i]}(K)\\&\geq\frac{1}{c'_{p,q}}\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(S)\right]^{1/q}\left[\sum_{i=1}^{n}\lambda_{[i]}^{q}(K)\right]^{1/q}\\&\geq\frac{1}{c'_{p,q}n^{1-1/q}}\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(S)\right]^{1/p}\operatorname{tr}(K).\end{aligned} \tag{3.16}$$

Combining (3.11) with (3.16), we obtain

$$\frac{1}{\lambda_{[n]}(R)}\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}[\operatorname{tr}(K)]^2-\frac{1}{c'_{p,q}n^{1-1/q}}\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(S)\right]^{1/p}\operatorname{tr}(K)-\operatorname{tr}\left(QR^{-1}\right)\geq 0. \tag{3.17}$$

Solving (3.17) for $\operatorname{tr}(K)$ yields the left-hand side of (3.3).

By (1.5), (2.6), and (3.8), we have

$$\begin{aligned}\operatorname{tr}\left(R^{-1}A^{T}K+KAR^{-1}\right)&\leq\sum_{i=1}^{n}\lambda_{[i]}(S)\lambda_{[i]}(K)\\&\leq\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(S)\right]^{1/p}\left[\sum_{i=1}^{n}\lambda_{[i]}^{q}(K)\right]^{1/q}\\&\leq\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(S)\right]^{1/p}\operatorname{tr}(K).\end{aligned} \tag{3.18}$$

Combining (3.14) with (3.18), we obtain

$$\frac{1}{c_{p,q}n^{2-1/q}\lambda_{[1]}(R)}\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(R)\right]^{1/p}[\operatorname{tr}(K)]^2-\left[\sum_{i=1}^{n}\lambda_{[i]}^{p}(S)\right]^{1/p}\operatorname{tr}(K)-\operatorname{tr}\left(QR^{-1}\right)\leq 0. \tag{3.19}$$

Solving (3.19) for $\operatorname{tr}(K)$ yields the right-hand side of (3.3).

(3) Note that when $S \leq 0$, by (3.3), we obtain (3.4) immediately. This completes the proof. □

*Remark 3.2.* From Remark 2.6 and Theorem 3.1, we have the following results.

(1) Let $p \to \infty,\ q \to 1$ in (3.2), then we have

$$\begin{aligned}&\frac{\lambda_{[n]}(R)\lambda_{[n]}(S)+\lambda_{[n]}(R)\sqrt{\lambda_{[n]}^2(S)+\left(4/\lambda_{[n]}(R)\right)\lambda_{[1]}(R)\operatorname{tr}\left(QR^{-1}\right)}}{2\lambda_{[1]}(R)}\\&\quad\le \operatorname{tr}(K)\le \frac{n}{2}\lambda_{[1]}(S)+\frac{n}{2}\sqrt{\lambda_{[1]}^2(S)+\frac{4\operatorname{tr}\left(QR^{-1}\right)}{n}}.\end{aligned}\tag{3.20}$$

(2) Let $p \to \infty,\ q \to 1$ in (3.3), then we obtain (3.20).

(3) Let $p \to \infty,\ q \to 1$ in (3.4). Note that when $S \le 0$,

$$\begin{aligned}&\lim_{p\to\infty}\left[\sum_{i=1}^{n}\left|\lambda_{[i]}(S)\right|^p\right]^{1/p}=\max_{1\le i\le n}\left|\lambda_{[i]}(S)\right|=-\lambda_{[n]}(S),\\&\lim_{\substack{p\to\infty\\ q\to 1}}\frac{1}{c'_{p,q}n^{1-1/q}}\left[\sum_{i=1}^{n}\left|\lambda_{[i]}(S)\right|^p\right]^{1/p}=\min_{1\le i\le n}\left|\lambda_{[i]}(S)\right|=-\lambda_{[1]}(S).\end{aligned}\tag{3.21}$$

Then we can also obtain (3.20).

Note that the right-hand side of (3.20) is (3.1), which implies that Theorem 3.1 improves (3.1).

## 4. Numerical Examples

In this section, firstly, we will give an example to illustrate that our new trace bounds are better than those of the recent results. Then, to illustrate that the application in the algebraic Riccati equations of our results will have different superiority if we choose different $p$ and $q$, we will give two examples.

*Example 4.1.* Let

$$\begin{aligned}A&=\begin{pmatrix}0.2563 & 0.2588 & 0.1422\\ 0.2358 & 2.0451 & 0.4177\\ 0.8942 & 0.2547 & 0.9852\end{pmatrix},\\ B&=\begin{pmatrix}0.2587 & 0.5236 & 0.8541\\ 0.5236 & 1.1254 & 0.3654\\ 0.8541 & 0.3654 & 1.2541\end{pmatrix}.\end{aligned}\tag{4.1}$$

Then $\operatorname{tr}(AB) = 4.9933$ and $B$ is symmetric. Using (1.5) yields

$$0.2173 \le \operatorname{tr}(AB) \le 5.5656.\tag{4.2}$$

Using (2.18) yields

$$0.6079 \leq \operatorname{tr}(AB) \leq 5.1255, \tag{4.3}$$

where both lower and upper bounds are better than those of the main result of [18], that is, (1.5).

*Example 4.2.* Consider the system (1.1), (1.2) with

$$A = \begin{pmatrix} -15 & -23 & 27 \\ 26 & -9 & 4 \\ 35 & 72 & 18 \end{pmatrix}, \qquad BB^T = \begin{pmatrix} 6 & 1 & 3 \\ 1 & 7 & 4 \\ 3 & 4 & 8 \end{pmatrix}, \qquad Q = \begin{pmatrix} 485 & 49 & 38 \\ 49 & 64 & -92 \\ 38 & -92 & 192 \end{pmatrix} \tag{4.4}$$

and consider the corresponding ARE (1.4) with $R = BB^T$; $(A, R)$ is stabilizable and $(Q, A)$ is observable. Using (3.20) yields

$$39.0104 \leq \operatorname{tr}(K) \leq 682.1538. \tag{4.5}$$

Using (3.2), when $p = q = 2$, then we obtain

$$201.9801 \leq \operatorname{tr}(K) \leq 271.4, \tag{4.6}$$

where the upper bound is better than that of the main result of [16], that is, (3.1).

*Example 4.3.* Consider the system (1.1), (1.2) with

$$A = \begin{pmatrix} 20 & 3 & 7.5 \\ 5 & 7 & 9 \\ 2 & 0 & 4 \end{pmatrix}, \qquad BB^T = \begin{pmatrix} 9 & 1 & 3 \\ 1 & 5 & 2 \\ 3 & 2 & 6 \end{pmatrix}, \qquad Q = \begin{pmatrix} 455 & 332 & 209 \\ 332 & 304 & 127.5 \\ 209 & 127.5 & 125 \end{pmatrix} \tag{4.7}$$

and consider the corresponding ARE (1.4) with $R = BB^T$; $(A, R)$ is stabilizable and $(Q, A)$ is observable. Using (3.2), when $p = q = 2$, then we obtain

$$5.2895 \leq \operatorname{tr}(K) \leq 97.2209. \tag{4.8}$$

Using (3.20) yields

$$5.6559 \leq \operatorname{tr}(K) \leq 25.9683, \tag{4.9}$$

where the lower and upper bounds are better than those of (4.8).

## 5. Conclusions

In this paper, we have proposed lower and upper bounds for the trace of the product of two real square matrices in which one is diagonalizable. We have shown that our bounds for the trace are the tightest among the parallel trace bounds in symmetric case. Then, we have obtained some trace bounds for the solution of the algebraic Riccati equations, which improve some of the previous results under certain conditions. Finally, numerical examples have illustrated that our bounds are better than those of the previous results.

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